![Core Knowledge logo] **Core Knowledge®** **MATHEMATICS**

Pythagorean Theorem and Irrational Numbers

The spiral

Teacher Guide

Pythagorean Theorem

b

a

c

Find the area of each triangle

The Hands of a Clock

4 cm

3 cm

A

B   C

Intersecting Circles

Pythagorean Theorem and Irrational Numbers

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Pythagorean Theorem and Irrational Numbers
Teacher Guide
Core Knowledge Mathematics™

Pythagorean Theorem and Irrational Numbers

Unit Narrative

Work in this unit is designed to build on and connect students' understanding of geometry and numerical expressions. The unit begins by foreshadowing algebraic and geometric aspects of the Pythagorean Theorem and strategies for proving it. Students are shown three squares and asked to compare the area of the largest square with the sum of the areas of the other two squares. The comparison can be done by counting grid squares and comparing the counts—when the three squares are on a grid with their sides on grid lines and vertices on intersections of grid lines—using the understanding of area measurement established in grade 3. The comparison can also be done by showing that there is a shape that can be decomposed and rearranged to form the largest square or the two smallest squares. Students are provided with opportunities to use and discuss both strategies.

In the second section, students work with figures shown on grids, using the grids to estimate lengths and areas in terms of grid units, e.g., estimating the side lengths of a square, squaring their estimates, and comparing them with estimates made by counting grid squares. The term "square root" is introduced as a way to describe the relationship between the side length and area of a square (measured in units and square units, respectively), along with the notation $\sqrt{}$. Students continue to work with side lengths and areas of squares. They learn and use definitions for "rational number" and "irrational number." They plot rational numbers and square roots on the number line. They use the meaning of "square root," understanding that if a given number p is the square root of n, then $x^2 = n$. Students learn (without proof) that $\sqrt{2}$ is irrational. They understand two proofs of the Pythagorean Theorem—an algebraic proof that involves manipulation of two expressions for the same area and a geometric proof that involves decomposing and rearranging two squares. They use the Pythagorean Theorem in two and three dimensions, e.g., to determine lengths of diagonals of rectangles and right rectangular prisms and to estimate distances between points in the coordinate plane.

In the third section, students work with edge lengths and volumes of cubes and other rectangular prisms. (In this grade, all prisms are right prisms.) They are introduced to the term "cube root" and the notation $\sqrt[3]{}$. They plot square and cube roots on the number line, using the meanings of "square root" and "cube root," e.g., understanding that if a given number x is the square root of n and n is between m and p, then x^2 is between m and p and that x is between $\sqrt{m}$ and $\sqrt{p}$.

In the fourth section, students work with decimal representations of rational numbers and decimal approximations of irrational numbers. In grade 7, they used long division in order to write fractions as decimals and learned that such decimals either repeat or terminate. They build on their understanding of decimals to make successive decimal approximations of $\sqrt{2}$ and π which they plot on number lines.

Progression of Disciplinary Language

In this unit, teachers can anticipate students using language for mathematical purposes such as explaining, justifying, and comparing. Throughout the unit, students will benefit from routines designed to grow robust disciplinary language, both for their own sense-making and for building shared understanding with peers. Teachers can formatively assess how students are using language in these ways, particularly when students are using language to:

Explain

- strategies for finding area (Lesson 1)
- strategies for approximating and finding square roots (Lesson 4)
- strategies for finding triangle side lengths (Lesson 6)
- predictions about situations involving right triangles and strategies to verify (Lesson 10)
- strategies for finding distances between points on a coordinate plane (Lesson 11)
- strategies for approximating the value of cube roots (Lesson 13)

Justify

- which squares have side lengths in a given range (Lesson 1)
- ordering of irrational numbers (Lesson 5)
- ordering of hypotenuse lengths (Lesson 9)

Compare

- rational and irrational numbers (Lesson 3)
- lengths of diagonals in rectangular prisms (Lesson 10)
- strategies for approximating irrational numbers (Lesson 15)

In addition, students are expected to use language to generalize about area of squares, square roots, and approximations of side lengths and generalize about the distance between any two coordinate pairs; critique reasoning about square root approximations and critique a strategy to represent repeating decimal expansions as fractions; describe observations about the relationships between triangle side lengths and describe hypotenuses and side lengths for given triangles; interpret diagrams involving squares and right triangles; interpret equations and approximations for the value of square and cube roots; and represent relationships between side lengths and areas.

The table shows lessons where new terminology is first introduced, including when students are expected to understand the word or phrase receptively and when students are expected to produce the word or phrase in their own speaking or writing. Terms from the glossary appear bolded. Teachers should continue to support students' use of a new term in the lessons that follow the one in which it was first introduced.

Unit 8

Learning Targets

Pythagorean Theorem and Irrational Numbers

Lesson 1: The Areas of Squares and Their Side Lengths

- I can find the area of a tilted square on a grid by using methods like "decompose and rearrange" and "surround and subtract."

- I can find the area of a triangle.

Lesson 2: Side Lengths and Areas

- I can explain what a square root is.

- If I know the area of a square, I can express its side length using square root notation.

- I understand the meaning of expressions like $\sqrt{25}$ and $\sqrt{3}$.

Lesson 3: Rational and Irrational Numbers

- I know what an irrational number is and can give an example.

- I know what a rational number is and can give an example.

Lesson 4: Square Roots on the Number Line

- I can find a decimal approximation for square roots.

- I can plot square roots on the number line.

Lesson 5: Reasoning About Square Roots

- When I have a square root, I can reason about which two whole numbers it is between.

Lesson 6: Finding Side Lengths of Triangles

- I can explain what the Pythagorean Theorem says.

Lesson 7: A Proof of the Pythagorean Theorem

- I can explain why the Pythagorean Theorem is true.

Lesson 8: Finding Unknown Side Lengths

- If I know the lengths of two sides, I can find the length of the third side in a right triangle.

- When I have a right triangle, I can identify which side is the hypotenuse and which sides are the legs.

Lesson 9: The Converse

- I can explain why it is true that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$ then it must be a right triangle.

- If I know the side lengths of a triangle, I can determine if it is a right triangle or not.

Lesson 10: Applications of the Pythagorean Theorem

- I can use the Pythagorean Theorem to solve problems.

Lesson 11: Finding Distances in the Coordinate Plane

- I can find the distance between two points in the coordinate plane.

- I can find the length of a diagonal line segment in the coordinate plane.

Lesson 12: Edge Lengths and Volumes

- I can approximate cube roots.

- I know what a cube root is.

- I understand the meaning of expressions like $\sqrt[3]{5}$.

Lesson 13: Cube Roots

- When I have a cube root, I can reason about which two whole numbers it is between.

Lesson 14: Decimal Representations of Rational Numbers

- I can write a fraction as a repeating decimal.

- I understand that every number has a decimal expansion.

Lesson 15: Infinite Decimal Expansions

- I can write a repeating decimal as a fraction.

- I understand that every number has a decimal expansion.

Lesson 16: When Is the Same Size Not the Same Size?

- I can apply what I have learned about the Pythagorean Theorem to solve a more complicated problem.

- I can decide what information I need to know to be able to solve a real-world problem using the Pythagorean Theorem.

lesson	new terminology	
	receptive	productive
8.8.2	**square root**	square (of a number)
8.8.3	**irrational number** square root symbol	**rational number**
8.8.4	diagonal decimal approximation	
8.8.5		**square root** square root symbol
8.8.6	**Pythagorean Theorem** **hypotenuse** **legs**	right triangle
8.8.9	converse of the Pythagorean Theorem	**Pythagorean Theorem**
8.8.10	edge length	**hypotenuse** **legs**
8.8.12	**cube root**	
8.8.13		**cube root** edge length
8.8.14	**repeating decimal** decimal representation finite decimal expansion	
8.8.15	infinite decimal expansion	**irrational number** **repeating decimal**

Required Materials

Compasses
Copies of Instructional master
Four-function calculators
Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the Instructional master
Scientific calculators
Tracing paper
Bundles of "patty paper" are available commercially for a very low cost. These are small sheets (about 5" by 5") of transparent paper.

Section: Side Lengths and Areas of Squares

Lesson 1: The Areas of Squares and Their Side Lengths

Goals

- Calculate the area of a tilted square on a grid by using decomposition, and explain (orally) the solution method.

- Estimate the side length of a square by comparing it to squares with known areas, and explain (orally) the reasoning.

Learning Targets

- I can find the area of a tilted square on a grid by using methods like "decompose and rearrange" and "surround and subtract."

- I can find the area of a triangle.

Lesson Narrative

Students know from work in previous grades how to find the area of a square given the side length. In this lesson, we lay the groundwork for thinking in the other direction: if we know the area of the square, what is the side length? Before students define this relationship formally in the next lesson, they estimate side lengths of squares with known areas using tools such as rulers and tracing paper (MP5). They also review key strategies for finding area that they encountered in earlier grades that they will use to understand and explain informal proofs of the Pythagorean Theorem.

In the warm-up, students compare the areas of figures that can easily be determined by either composing and counting square units or decomposing the figures into simple, familiar shapes. In the next activity, students find areas of "tilted" squares by enclosing them in larger squares whose areas can be determined and then subtracting the areas of the extra triangles. The next activity reinforces the relationship between the areas of squares and their side lengths, setting the stage for the definition of a square root in the next lesson.

Alignments

Building On

- 6.EE.A.1: Write and evaluate numerical expressions involving whole-number exponents.

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Addressing

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Building Towards

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.G.B: Understand and apply the Pythagorean Theorem.

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR5: Co-Craft Questions

- MLR7: Compare and Connect

- MLR8: Discussion Supports

- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the Instructional master

Required Preparation

1 copy of the Making Squares Instructional master, pre-cut, for every 2 students. These pieces are used again in the lesson A Proof of the Pythagorean Theorem and should be saved.

Student Learning Goals

Let's investigate the squares and their side lengths.

1.1 Two Regions

Warm Up: 5 minutes

The purpose of this warm-up is for students to review how to find the area of a region on a grid by decomposing and rearranging pieces. In the following activities, students will use these techniques to check area estimates made when approximating the value of the side length of a square used to calculate the area by squaring. They will also use these techniques to understand and explain a proof of the Pythagorean Theorem.

As students work, identify students who used different strategies for finding area, including putting pieces together to make whole units in different ways and adding up the whole grid then subtracting the white space to indirectly figure out the area of the shaded region.

Building On

- 6.G.A.1

Building Towards

- 8.G.B

Launch

Tell students they are comparing the area of regions on two grids with the same unit size. Poll students on which region they predict is the larger area. Give students 2 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

Students may think that the side length of either quadrilateral is 2 units because the sides are partitioned into two sub-pieces that look about 1 unit long. If so, point out that they do not lay along grid lines, they can use a compass or tracing paper to compare the side lengths to 2 units on the grid.

Student Task Statement

Which shaded region is larger? Explain your reasoning.
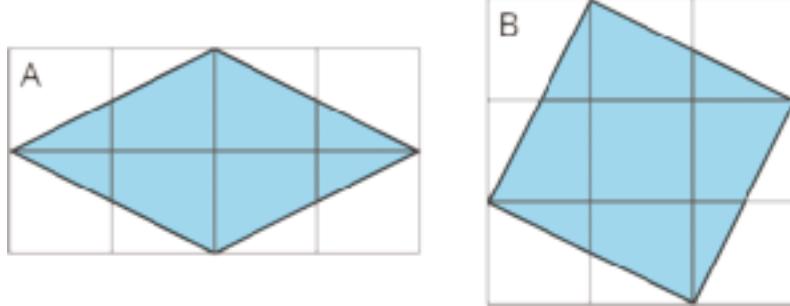
Student Response

Figure B has the larger shaded region because Figure A is 4 square units and Figure B is 5 square units.

Activity Synthesis

Poll students on which shaded region is the larger area. Select previously identified students to share their strategies for finding area. Record and display their responses for all to see.

If time allows, ask students to estimate the side lengths of each shaded region to prepare them for the work they will do in the next activity.

1.2 Decomposing to Find Area

15 minutes

The purpose of this task is for students to find the areas of squares whose side lengths are not easy to determine by inspection. The squares are represented in an increasingly abstract way: the first square shows all of the square units explicitly, making it easy to see that putting two "leftover" triangles together will make a rectangle with the same dimensions, or even piecing individual square units together that can be counted. The next only shows the grid units along the edges, providing a bridge to the third, which only provides labels for side lengths. The numbers are chosen to encourage the decomposing, rearranging, and subtracting strategy for finding the area, although decomposing and using the formulas for the area of a square and a triangle is also perfectly appropriate.

This sequence of tasks also provides practice for students in using the area and finding strategies they will need to understand and explain a proof of the Pythagorean Theorem in a later lesson.

Identify students who find the area of the extra area in the triangles by:

- composing two triangles to make a rectangle.

- using the formula for the area of a triangle and multiplying the result by four.

Be sure these strategies are made explicit for the first square to help students who are only able to count unit squares make a successful transition to the third square. While decomposing the square

and rearranging the pieces is a correct strategy, students need the subtraction strategy for more abstract work later, so it is better not to highlight the decomposing strategy that does not generalize as well.

Building On

- 6.G.A.1

Building Towards

- 8.G.B.6

- 8.NS.A.2

Instructional Routines

- MLR7: Compare and Connect

Launch

Give 2–3 minutes quiet work time and then ask students to pause and select 2–3 previously identified students to share how they found the area of the first square. Work time to finish the remaining questions.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Reference examples and displays from previous units of area formulas and methods for finding area. *Supports accessibility for: Memory; Conceptual processing*

Student Task Statement

Find the area of each shaded square (in square units).

1.

2.

3.
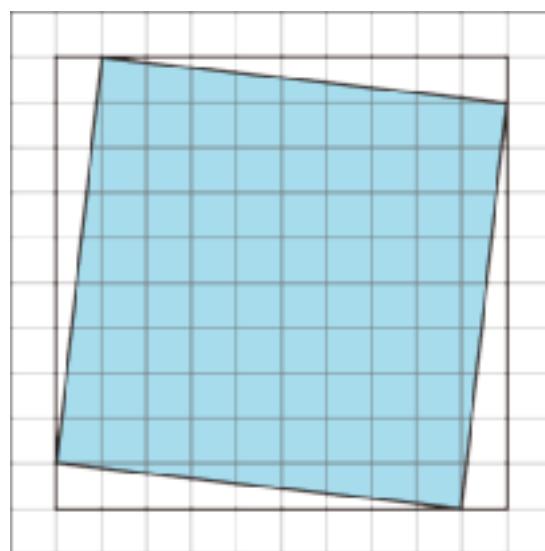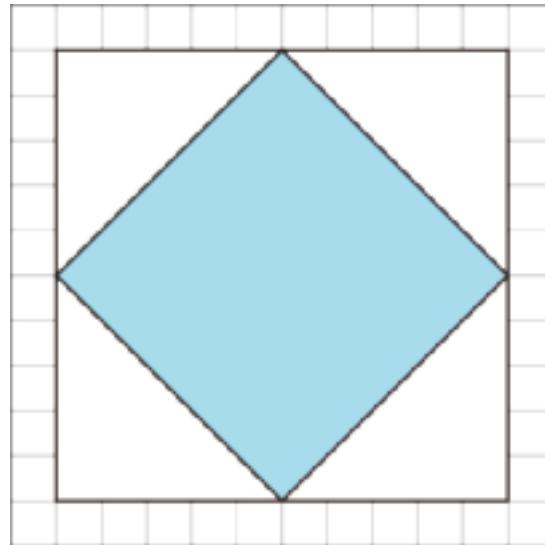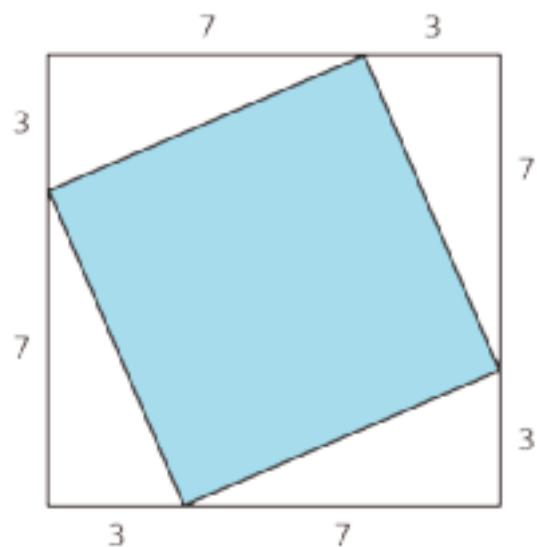
Student Response

1. 82 square units

2. 50 square units

3. 58 square units

Are You Ready for More?

Any triangle with a base of 13 and a height of 5 has an area of $\frac{65}{2}$.
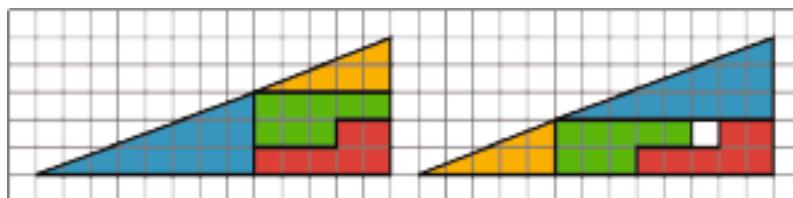
> Both shapes in the figure have been partitioned into the same four pieces. Find the area of each of the pieces, and verify the corresponding parts are the same in each picture. There appears to be one extra square unit of area in the right figure. If all of the pieces have the same area, how is this possible?

Student Response

There is at first an apparent paradox! Each of the two figures is made up of two triangles with areas 12 and 5, and two polygons with areas 8 and 7, for a total of 32. But in the right figure there seems to "magically" be room for an extra square unit of grid space!

The resolution is that, in fact, neither of the two figures are actually triangles! The slopes of the hypotenuses of the two triangles making up the figures are $\frac{2}{5}$ and $\frac{3}{8}$, which are relatively close but not equal, and so do not line up to make one straight hypotenuse for the larger apparent right triangle. So the only flaw in the argument is the expectation that the formula for the area of a triangle should apply.

Activity Synthesis

Select students to explain how they found the area of the third square. Emphasize the strategies of decomposing and either rearranging or finding the areas of the triangles using the triangle area formula.

Access for English Language Learners

Speaking, Listening: MLR7 Compare and Connect. As students prepare a visual display of their work for the last problem, look for students with different methods for calculating the area of the shaded square. As students investigate each other's work, ask them to share what worked well in a particular approach. During this discussion, listen for and amplify the language students use to describe the decomposing, rearranging, and subtracting strategy for finding the area. Then encourage students to make connections between the quantities used to calculate the area of the shaded square and the shapes in the diagram. For example, the quantity 100 represents the area, in square units, of the large square. The quantity 10.5 represents the area, in square units, of one of the triangles. This will foster students' meta-awareness and support constructive conversations as they compare strategies for calculating areas and make connections between quantities and the areas they represent.
Design Principles(s): Cultivate conversation; Maximize meta-awareness

1.3 Estimating Side Lengths from Areas

10 minutes
In this activity, students use what they know about finding the area of a square from a given side length to think about the converse problem: how do you find the side length of a square with a

given area? Students solve this problem in general in the next lesson, but this lesson sets them up for that work.

- First, they work with squares whose side lengths lie along grid lines, so both the side lengths and areas can be found simply by counting length or area units.

- They also see a "tilted" square whose area can be determined by counting square units or by decomposing and rearranging the square or a related figure. In this case, the side lengths are not apparent, so students have to reason that since the area is the same as one of the earlier squares, the side lengths must also be the same. They should check that this appears to be true by tracing one of the squares on tracing paper and seeing that it does, in fact, line up with the other.

The last question asks them to use the insights from the previous questions, tracing paper, or a ruler to estimate the side lengths of some other tilted squares (MP5). In the next lesson, they will see other techniques for estimating those side lengths as well.

Monitor for students who use these strategies for the fourth question:

- reason that the side lengths are between 5 and 6 because the areas are between 25 and 36 (MP3)

- measure with tracing paper or a ruler

Building On

- 6.EE.A.1

- 6.G.A.1

Addressing

- 8.NS.A.2

Building Towards

- 8.EE.A.2

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR5: Co-Craft Questions

- Think Pair Share

Launch

Provide access to tracing paper. Students in groups of 2. 3 minutes of quiet work time followed by a partner discussion on the first three problems. Pause students and make sure everyone agrees on the answers for the first three questions. Students continue with the fourth question, then a whole-class discussion.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "I noticed _____ so I...", "By comparing triangles I...", or "Another strategy could be _____ because...."
Supports accessibility for: Language; Social-emotional skills

Access for English Language Learners

Conversing, Writing: MLR5 Co-Craft Questions. Before revealing the questions in this activity, display the diagram of the three squares and invite students to write possible mathematical questions about the diagram. Ask students to compare the questions they generated with a partner before sharing questions with the whole class. Listen for and amplify questions about the side lengths and area of each square. If no student asks about the side lengths or area of each square, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about the side lengths and areas of squares in the coordinate plane.
Design Principle(s): Maximize meta-awareness

Student Task Statement
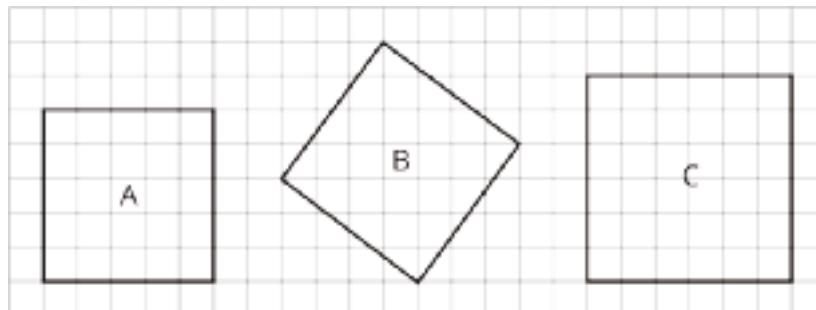
1. What is the side length of square A? What is its area?

2. What is the side length of square C? What is its area?

3. What is the area of square B? What is its side length? (Use tracing paper to check your answer to this.)

4. Find the areas of squares D, E, and F. Which of these squares must have a side length that is greater than 5 but less than 6? Explain how you know.
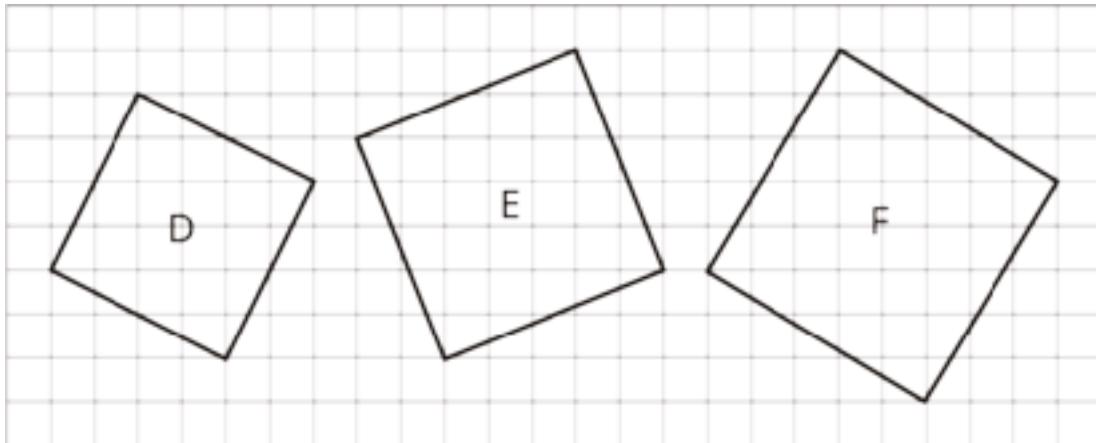
Student Response

1. 5 units. 25 square units

2. 6 units. 36 square units

3. 25 square units. 5 units

4. D: 20 square units. E: 29 square units. F: 34 square units. E and F have side lengths between 5 and 6 because their areas are between 25 and 36.

Activity Synthesis

A key take-away from this activity is that if the area of a square is in between the areas of two other squares, then the side lengths are also in between, which in the case of the squares in this task can be reinforced by estimating with tracing paper.

Select previously identified students to share their solutions. Sequence with a reasoning strategy followed by a measuring strategy using a ruler or tracing paper. Use the measuring strategy to verify the reasoning strategy.

1.4 Making Squares

Optional: 15 minutes (there is a digital version of this activity)
In this activity, students use squares with known areas to determine the total area of five shapes. How students determine the area of the shapes is left open-ended on purpose (MP1). Students may calculate each shape individually, or, with a bit of rearranging, they may "fit" the shapes into the squares. It is possible to fit all 5 shapes into the two smaller squares or into the one larger square. Identify groups who calculate the area of the shapes individually versus those fitting the shapes into squares.

While these shapes are likely to seem arbitrary to students in this lesson, they are actually part of a transformations-based proof of the Pythagorean Theorem that students will revisit in a future

lesson. The five cut-out shapes in this activity should be reserved at the end of the activity for future use.

You will need the Instructional master for this activity unless you are using the digital version.

Building On
- 6.G.A.1

Building Towards
- 8.G.B

Instructional Routines
- MLR8: Discussion Supports

Launch

Display the image of three squares for all to see. Ask students which is larger: the combined area of the two smaller squares A and B or the area of the one larger square C? Give students 30–60 seconds of quiet think time, and then poll the class for their responses. Display a tally of student responses for all to see.
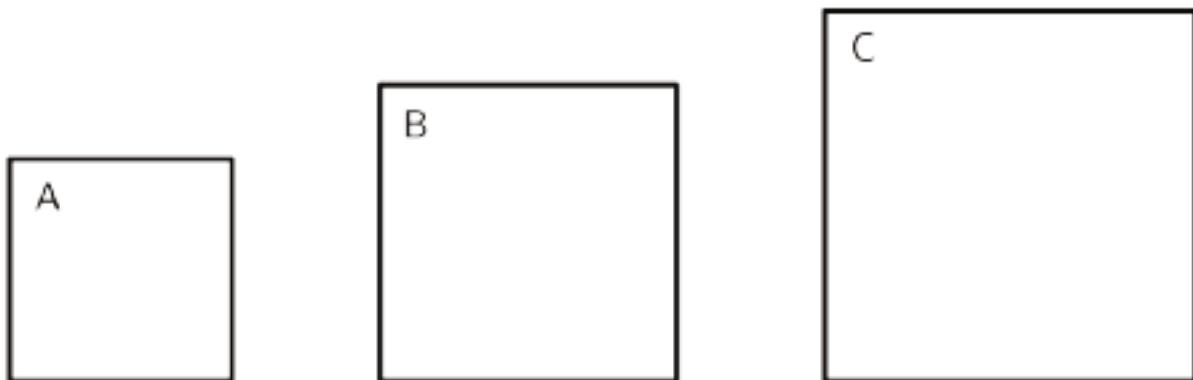
Display the image of three squares with grids for all to see. Tell students that these are the same three squares as before and repeat the previous question. Give students 30–60 seconds of quiet think time, and then select 1–2 students to explain their reasoning. (9 square units and 16 square units is the same as 25 square units, so the combined area of squares A and B is the same as the area of square C.)
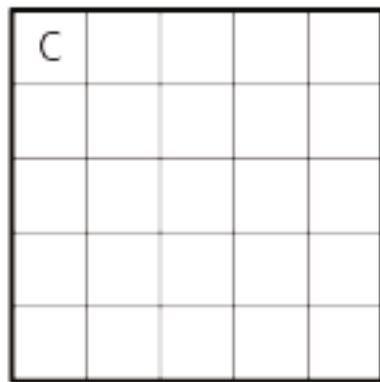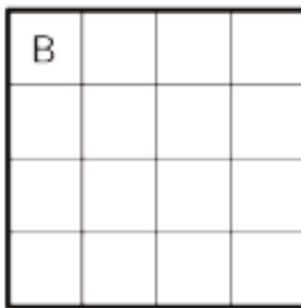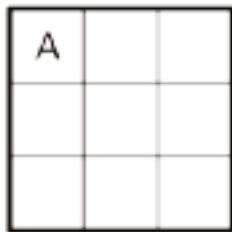
Arrange students in groups of 2 and distribute one copy of the three squares half-sheet and five pre-cut shapes from the Instructional master to each group. The five pre-cut shapes are labeled on one side to facilitate conversation.

Students using the digital version of the curriculum have an applet to use.

> ### Student Task Statement
>
> Your teacher will give your group a sheet with three squares and 5 cut out shapes labeled D, E, F, G, and H. Use the squares to find the total area of the five shapes. Assume each small square is equal to 1 square unit.

Student Response

The total area of the 5 shapes is 25 square units.

See the Instructional master for how the 5 shapes fit into the two smaller squares. Sample response:
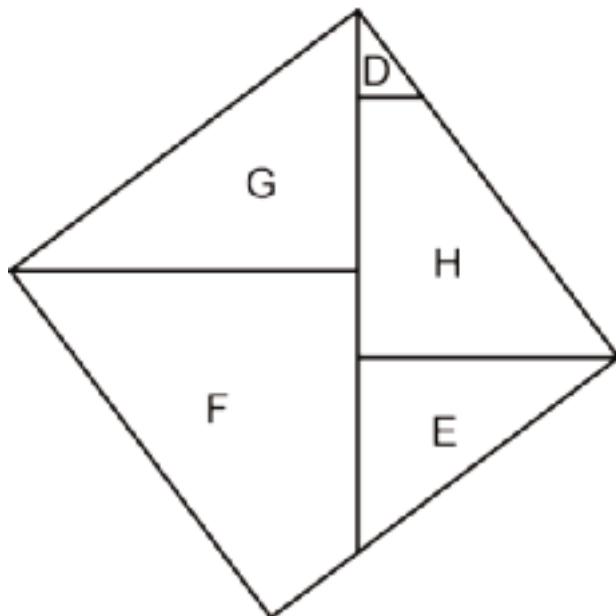
Activity Synthesis

The purpose of this discussion is for students to share the different ways they determined the total area of the shapes and to get students thinking about strategies for calculating area and rearranging figures in preparation for future lessons. Select previously identified groups to share, starting with those who calculated each shape individually. If any group fit the shapes into both the two smaller squares and then into the larger square, ask them to demonstrate how they did so for the class.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.
Supports accessibility for: Attention; Social-emotional skills

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Access for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. As students share the total area of the five shapes, press for details in students' reasoning by asking how they know the total area is 25 square units. Listen for and amplify the language students use to describe how they rearranged the five shapes to fit into either the two smaller squares or the larger square. Help students understand that if the shapes fit in a square perfectly without overlapping, then the combined area of the shapes must be the same as the area of the square. This will support rich and inclusive discussion about a strategy for calculating the total area of several shapes by arranging them into a square.
Design Principle(s): Support sense-making

Lesson Synthesis

The purpose of this synthesis is to reinforce the relationship between the side length of a square and its area.

- "If you have a square and know the side length, how can you find the area?" (Multiply the side length by itself; square the side length.)

- "How can you find the area of a square if you don't know the side lengths?" (Enclose it in a square whose side lengths you can find, find the areas of the triangles, and subtract them from the area of the big square.)

- "If you have a square and know the area, how can you find the side length?" (Find the number you multiply by itself/square that equals the area.)

1.5 It's a Square

Cool Down: 5 minutes
Building On

- 6.EE.A.1

- 6.G.A.1

Building Towards

- 8.EE.A.2

- 8.G.B.6

> **Student Task Statement**
>
> Find the area and side length of square $ACEG$.
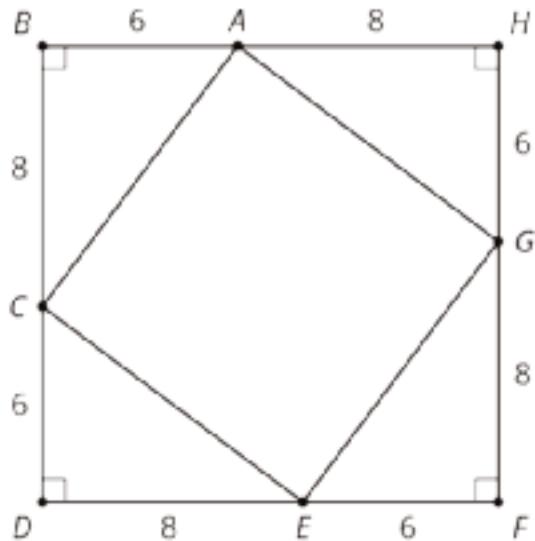
Student Response

$A = 100, s = 10$

> ## Student Lesson Summary
>
> The area of a square with side length 12 units is 12^2 or 144 units^2.
>
> The side length of a square with area 900 units^2 is 30 units because $30^2 = 900$.

Sometimes we want to find the area of a square but we don't know the side length. For example, how can we find the area of square $ABCD$?

One way is to enclose it in a square whose side lengths we do know.
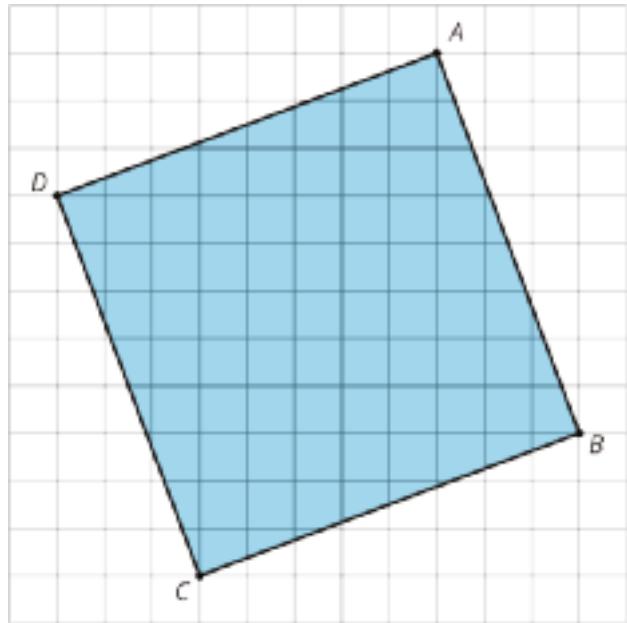
The outside square $EFGH$ has side lengths of 11 units, so its area is 121 units^2. The area of each of the four triangles is $\frac{1}{2} \cdot 8 \cdot 3 = 12$, so the area of all four together is $4 \cdot 12 = 48$ units^2. To get the area of the shaded square, we can take the area of the outside square and subtract the areas of the 4 triangles. So the area of the shaded square $ABCD$ is $121 - 48 = 73$ or 73 units^2.

Unit 8 Lesson 1
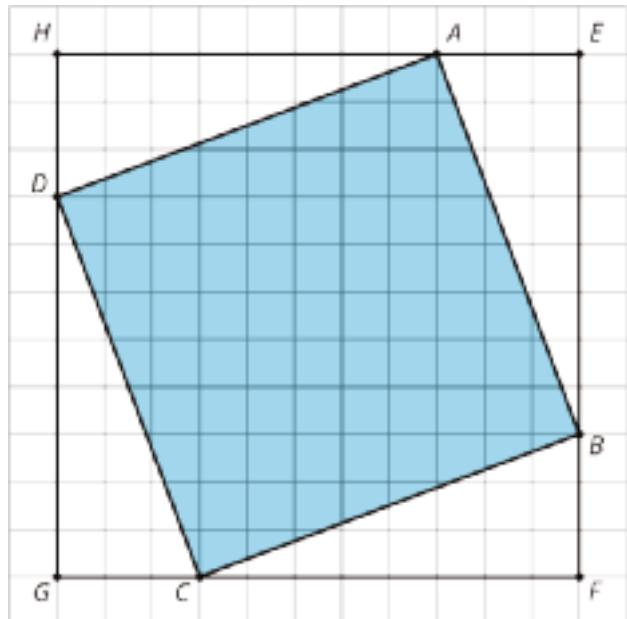
Lesson 1 Practice Problems
Problem 1
Statement
Find the area of each square. Each grid square represents 1 square unit.
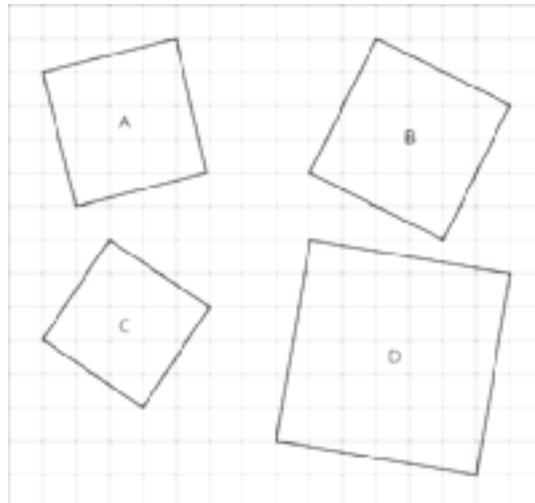
Solution

    a. 17 square units

    b. 20 square units

    c. 13 square units

    d. 37 square units

Problem 2
Statement
Find the length of a side of a square if its area is:

    a. 81 square inches

    b. $\frac{4}{25}$ cm^2

    c. 0.49 square units

    d. m^2 square units

Solution
    a. 9 inches

b. $\frac{2}{5}$ cm

c. 0.7 units

d. m units

Problem 3

Statement

Find the area of a square if its side length is:

a. 3 inches

b. 7 units

c. 100 cm

d. 40 inches

e. x units

Solution

a. 9 square inches

b. 49 square units

c. 10,000 cm^2

d. 1,600 square inches

e. x^2 square units

Problem 4

Statement

Evaluate $(3.1 \times 10^4) \cdot (2 \times 10^6)$. Choose the correct answer:

A. 5.1×10^{10}

B. 5.1×10^{24}

C. 6.2×10^{10}

D. 6.2×10^{24}

Solution

C

Problem 5

Statement

Noah reads the problem, "Evaluate each expression, giving the answer in scientific notation." The first problem part is: $5.4 \times 10^5 + 2.3 \times 10^4$.

Noah says, "I can rewrite 5.4×10^5 as 54×10^4. Now I can add the numbers: $54 \times 10^4 + 2.3 \times 10^4 = 57.3 \times 10^4$."

Do you agree with Noah's solution to the problem? Explain your reasoning.

Solution

Answers vary. Sample response: I don't agree with Noah's solution. His calculations are correct, but his final answer is not in scientific notation. To finish the problem, he should convert his answer to the form 5.73×10^5.

Problem 6

Statement

Select **all** the expressions that are equivalent to 3^8.

    A. $(3^2)^4$

    B. 8^3

    C. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

    D. $(3^4)^2$

    E. $\frac{3^6}{3^{-2}}$

    F. $3^6 \cdot 10^2$

Solution

["A", "C", "D", "E"]

Lesson 2: Side Lengths and Areas

Goals

- Comprehend the term "square root of a" (in spoken language) and the notation $\sqrt{a}$ (in written language) to mean the side length of a square whose area is a square units.

- Create a table and graph that represents the relationship between side length and area of a square, and use the graph to estimate the side lengths of squares with non-integer side lengths.

- Determine the exact side length of a square and express it (in writing) using square root notation.

Learning Targets

- I can explain what a square root is.

- If I know the area of a square, I can express its side length using square root notation.

- I understand the meaning of expressions like $\sqrt{25}$ and $\sqrt{3}$.

Lesson Narrative

In this lesson, students learn about square roots. The warm-up helps them see a single line segment as it relates to two different figures: as a side length of a triangle and as a radius of a circle. In the next activity, they use this insight to estimate the side length of a square via a geometric construction that relates the side length of the square to a point on the number line, and verify their estimate using techniques from the previous lesson. Once students locate the side length of the square as a point on the number line, they are formally introduced to **square roots** and square root notation:

$\sqrt{a}$ is the length of a side of a square whose area is a square units.

In the final activity, students use the graph of the function $y = x^2$ to estimate side lengths of squares with integer areas but non-integer side lengths.

Alignments

Building On

- 5.G.B: Classify two-dimensional figures into categories based on their properties.

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.F.B: Use functions to model relationships between quantities.

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

Building Towards

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

Instructional Routines

- MLR2: Collect and Display

- MLR8: Discussion Supports

- Notice and Wonder

- Think Pair Share

Required Materials

Four-function calculators
Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

> **Student Learning Goals**
>
> Let's investigate some more squares.

2.1 Notice and Wonder: Intersecting Circles

Warm Up: 5 minutes
The purpose of this warm-up is to get students in the habit of seeing the same line segment being a part of two different figures (MP7). In this case, the sides of the triangle are also radii of the circles. This primes them to see the sides of the square in the next activity as the radius of a circle.

Building On

- 5.G.B

Building Towards

- 8.NS.A

Instructional Routines

- Notice and Wonder

Launch

Arrange students in groups of 2. Display the diagram for all to see. Give students 1 minute of quiet work time to identify at least one thing they notice and at least one thing they wonder about the diagram. Ask students to give a signal when they have noticed or wondered about something. When the minute is up, give students 1 minute to discuss their observations and questions with their partner. Follow with a whole-class discussion.

Student Task Statement

What do you notice? What do you wonder?
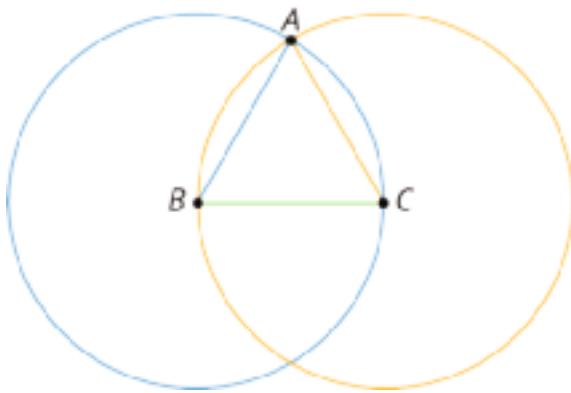
Student Response

Responses vary. Sample responses:

I notice that:

- There are two circles that intersect.
- The color of two of the sides of the triangle match the color of the circles and the third is a mixture.
- Segment AB is a radius of circle B.
- Segment AC is a radius of circle C.
- Segment BC is a radius of both circles.
- The three radii form an equilateral triangle.

I wonder:

- Why it is colored the way it is.

- If BC is the radius of one of the circles.

- If triangle ABC is equilateral.

- If this diagram is going to help me with anything.

Activity Synthesis

A segment can be a part of more than one figure. In this case, the sides of the triangle are also the radii of the circles. Being able to see parts of a figure in more than one way is helpful for solving problems.

2.2 One Square

15 minutes

The purpose of this activity is for students to estimate the side length of a square via a geometric construction that relates the side length of the square to a point on the number line, and verify their estimate using techniques from the previous lesson. Once students connect the side length to a point on the number line, they learn that this number has a name and a special notation to denote it: *square root* and the square root symbol. While this is students' first formal introduction to square roots, they will have many opportunities to deepen their understanding of square roots and practice using square root notation in later activities and lessons.

Addressing

- 8.NS.A

Building Towards

- 8.EE.A.2

Instructional Routines

- MLR2: Collect and Display

- Think Pair Share

Launch

Students in groups of 2. Give 1 minute quiet work time on the first problem and have students check in with a partner. Students continue to work. Follow with a whole-class discussion.

Student Task Statement

1. Use the circle to estimate the area of the square shown here:
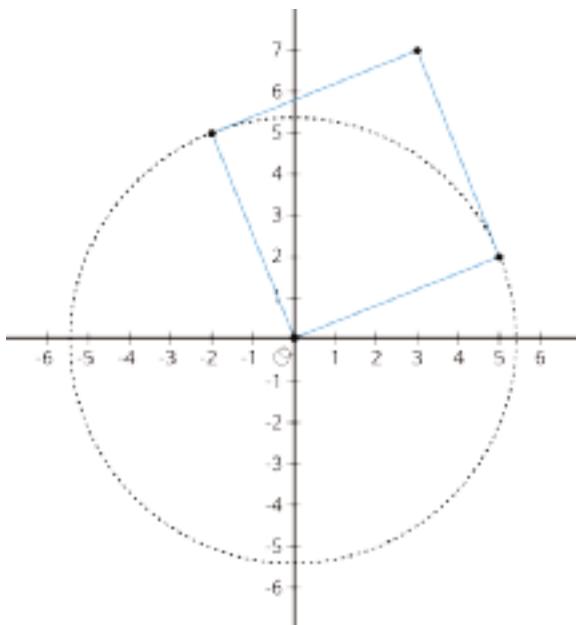
2. Use the grid to check your answer to the first problem.
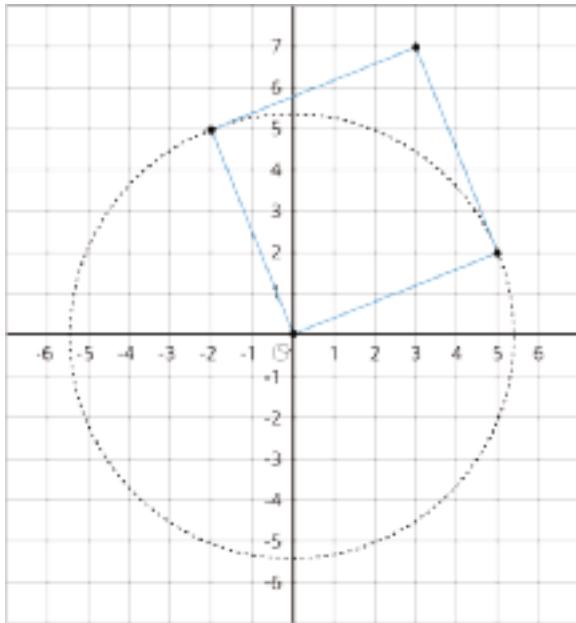
Student Response

1. Answers vary. Sample response: Approximately 28 square units. The radius of the circle is about 5.3 units, which means the area of the square is 5.3^2 units.

2. 29 square units.

Are You Ready for More?

One vertex of the equilateral triangle is in the center of the square, and one vertex of the square is in the center of the equilateral triangle. What is x?
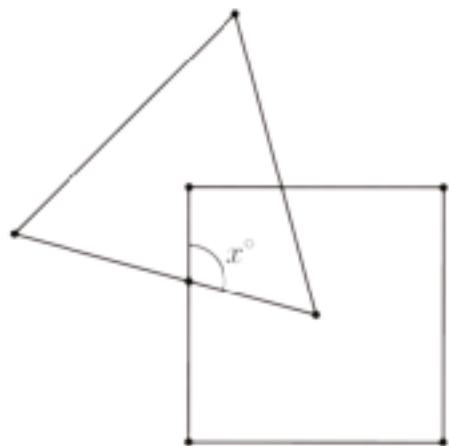
Student Response

Draw a segment to connect the center of the square to the center of the equilateral triangle. This segment cuts the 90 degree angle at the center of the equilateral triangle in half because a line from a vertex of an equilateral triangle through its center is a line of symmetry. In the same way, the segment cuts the 60 degree angle at the center of the square in half. So this segment creates a new triangle with angles 45 degrees, x degrees, and 30 degrees. Since the angles in a triangle must sum to 180 degrees, x must be equal to 105 degrees.

Activity Synthesis

Select students to share how they determined the areas in each problem and how their answers compared.

Ask, "What do you think the actual side length of the square is? That is, what is the number that when squared is equal to 29?" If not brought up in students' explanations, point out that while 5.3^2 is only 28.09, 5.35^2 is 28.6225, a number closer to 29, and ask what number they might try to square next.

Tell students that in the previous lesson and this lesson, we have seen squares that have areas that are whole numbers, but the side lengths are not whole numbers. In this activity, we saw that we can locate a point on the number line (the x-axis is a number line) that corresponds to the side length, and that this number is a *square root*, which we write like this:

$$\sqrt{29}$$

This is the way we write the exact length of the side of a square with area 29 square units. So $\left(\sqrt{29}\right)^2 = 29.$

Access for Students with Disabilities

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of: square root.
Supports accessibility for: Memory; Language

2.3 The Sides and Areas of Tilted Squares

15 minutes (there is a digital version of this activity)
In this activity, students continue to develop their understanding of square roots. Students first find the areas of three squares, estimate the side lengths using tracing paper, and then write the exact side lengths. Then they make a table of side-area pairs. They then graph the ordered pairs from the table and use the graph to estimate the values of squares with non-integer side lengths, such as the ones they drew in the previous activity.

Addressing
- 8.EE.A.2
- 8.F.B

Instructional Routines
- MLR8: Discussion Supports

- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to calculators. Remind students about the meaning and use of square root notation. Have students work together on the first problem and check each other's work. Have them make their graphs independently and then check with their partners that they look the same. Follow with a whole-class discussion.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts. Check in with students after the first 2-3 minutes of work time. Check to make sure students have attended to all parts of the original figures.
Supports accessibility for: Organization; Attention

Student Task Statement

1. Find the area of each square and estimate the side lengths using your geometry toolkit. Then write the exact lengths for the sides of each square.
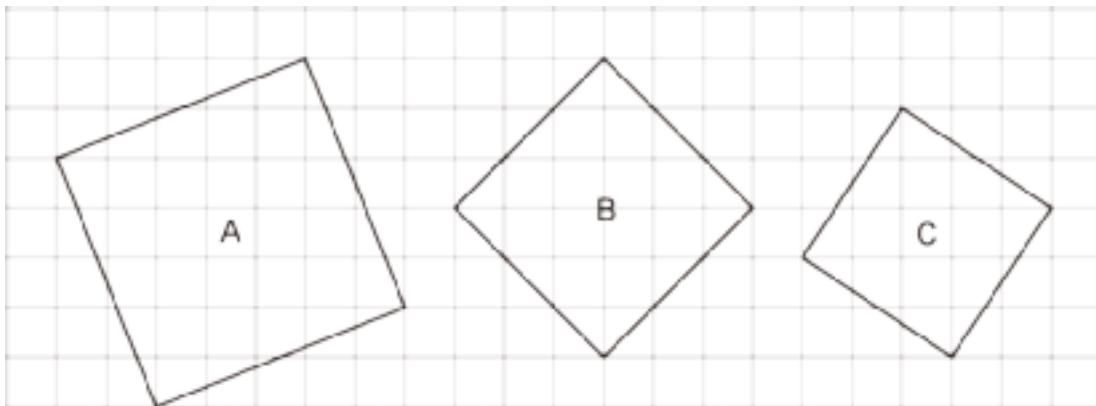
2. Complete the tables with the missing side lengths and areas.

side length, s	0.5		1.5		2.5		3.5	
area, a		1		4		9		16

side length, s	4.5		5.5		6.5		7.5	
area, a		25		36		49		64

3. Plot the points, (s, a), on the coordinate plane shown here.
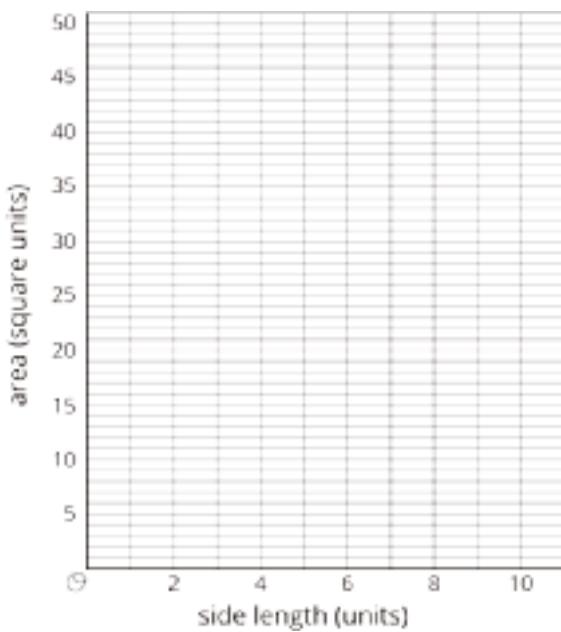
4. Use this graph to estimate the side lengths of the squares in the first question. How do your estimates from the graph compare to the estimates you made initially using your geometry toolkit?

5. Use the graph to approximate $\sqrt{45}$.

Student Response

1. Answers for estimates vary. Sample response:

   A: Area is 29 square units and side length is between 5 and 6 units. $s = \sqrt{29}$
   B: Area is 18 square units and side length is between 4 and 5 units. $s = \sqrt{18}$
   C: Area is 13 square units and side length is between 3 and 4 units. $s = \sqrt{13}$

2. Complete the table with the missing side lengths and areas.

side length, s	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8
area, a	0.25	1	2.25	4	6.25	9	12.25	16	20.25	25	30.25	36	42.25	49	56.25	64

3. All of the points (s, a) lie on the graph of the equation $y = x^2$.

4. Answers vary, but should be between the correct adjacent values in the table. Estimates should be a little more precise than when made with tracing paper.

5. $\sqrt{45}$ is between 6.5 and 7.

Activity Synthesis

Invite students to share their graphs from the second problem and display the graph for all to see. Ask: "What relationship does the graph display?" (The relationship between the side length and area

Unit 8 Lesson 2

of a square.) Select 2–3 students to share their answers to the last problem, including how they used the graph to estimate s for the tilted square they drew earlier.

Lesson Synthesis

The purpose of this discussion is to check that students understand the definition of square roots as they relate to side lengths of squares. Here are possible questions for discussion:

- "What does it mean when we write $\sqrt{100} = 10$ in terms of squares and side lengths?" (It means that a square with area 100 has side lengths of 10.)

- "If $\sqrt{17}$ is a side length of a square, what does that mean about the area?" (The area is 17 square units.)

- "Look at the graph of area as a function of side length. Should the points on the graph be connected? What would that mean in terms of squares and side lengths?" (Yes, the points should be connected because it's possible to have any positive value as a side length and then squaring that value gives the area of the square with that specific side length.)

2.4 What Is the Side Length?

Cool Down: 5 minutes
Addressing

- 8.EE.A.2

> **Student Task Statement**
>
> Write the exact value of the side length of a square with an area of
>
> 1. 100 square units.
>
> 2. 95 square units.
>
> 3. 36 square units.

4. 30 square units.

If the exact value is not a whole number, estimate the length.

Student Response

1. The side length is exactly 10 units since $10^2 = 100$. (Students might also note that $\sqrt{100} = 10$.)

2. The exact value is $\sqrt{95}$ units. It will be a little bit less than 10.

3. The side length is exactly 6 units since $6^2 = 36$. (Students might also note that $\sqrt{36} = 6$.)

4. The exact value is $\sqrt{30}$ units. It will be between 5 and 6 units.

Student Lesson Summary

We saw earlier that the area of square ABCD is 73 units^2.
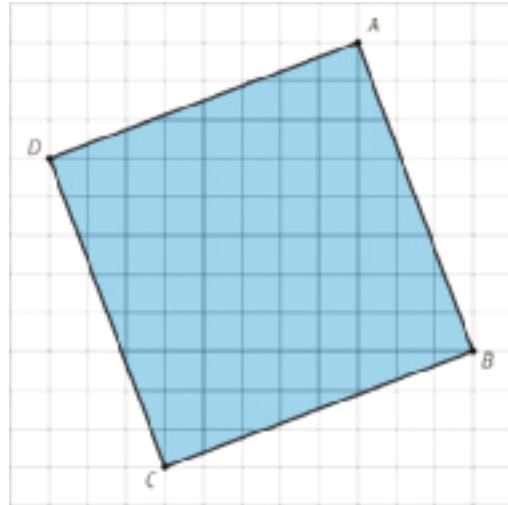
What is the side length? The area is between $8^2 = 64$ and $9^2 = 81$, so the side length must be between 8 units and 9 units. We can also use tracing paper to trace a side length and compare it to the grid, which also shows the side length is between 8 units and 9 units. But we want to be able to talk about its *exact* length. In order to write "the side length of a square whose area is 73 square units," we use the **square root** symbol. "The square root of 73" is written $\sqrt{73}$, and it means "the length of a side of a square whose area is 73 square units."

We say the side length of a square with area 73 units^2 is $\sqrt{73}$ units. This means that

$$\left(\sqrt{73}\right)^2 = 73$$

All of these statements are also true:

$\sqrt{9} = 3$ because $3^2 = 9$

$\sqrt{16} = 4$ because $4^2 = 16$

$\sqrt{10}$ units is the side length of a square whose area is 10 units^2, and $\left(\sqrt{10}\right)^2 = 10$
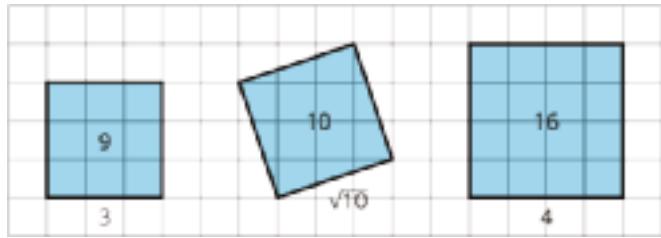
Glossary

- square root

Lesson 2 Practice Problems

Problem 1

Statement

A square has an area of 81 square feet. Select **all** the expressions that equal the side length of this square, in feet.

A. $\frac{81}{2}$

B. $\sqrt{81}$

C. 9

D. $\sqrt{9}$

E. 3

Solution

["B", "C"]

Problem 2

Statement

Write the exact value of the side length, in units, of a square whose area in square units is:

a. 36

b. 37

c. $\frac{100}{9}$

d. $\frac{2}{5}$

e. 0.0001

f. 0.11

Solution

a. 6

b. $\sqrt{37}$

c. $\frac{10}{3}$

d. $\sqrt{\frac{2}{5}}$

e. 0.01

f. $\sqrt{0.11}$

Problem 3

Statement

Square A is smaller than Square B. Square B is smaller than Square C.

The three squares' side lengths are $\sqrt{26}$, 4.2, and $\sqrt{11}$.

What is the side length of Square A? Square B? Square C? Explain how you know.

Solution

Square A: $\sqrt{11}$ units, Square B: 4.2 units, Square C: $\sqrt{26}$. I know this because $\sqrt{11}$ is between 3 and 4 and $\sqrt{26}$ is between 5 and 6, so $\sqrt{11} < 4.2 < \sqrt{26}$ and the side length of A is less than the side length of B is less than the side length of C.

Problem 4

Statement

Find the area of a square if its side length is:

a. $\frac{1}{5}$ cm

b. $\frac{3}{7}$ units

c. $\frac{11}{8}$ inches

d. 0.1 meters

e. 3.5 cm

Solution

a. $\frac{1}{25}$ cm^2

b. $\frac{9}{49}$ square units

c. $\frac{121}{64}$ square inches

d. 0.01 square meters

e. 12.25 cm^2

Problem 5

Statement

Here is a table showing the areas of the seven largest countries.

a. How much larger is Russia than Canada?

b. The Asian countries on this list are Russia, China, and India. The American countries are Canada, the United States, and Brazil. Which has the greater total area: the three Asian countries, or the three American countries?

country	area (in km^2)
Russia	1.71×10^7
Canada	9.98×10^6
China	9.60×10^6
United States	9.53×10^6
Brazil	8.52×10^6
Australia	6.79×10^6
India	3.29×10^6

Solution

a. 7.12×10^6 km^2

b. The Asian countries (2.999×10^7 vs. 2.803×10^7)

Problem 6

Statement

Select **all** the expressions that are equivalent to 10^{-6}.

A. $\frac{1}{1000000}$

B. $\frac{-1}{1000000}$

C. $\frac{1}{10^6}$

D. $10^8 \cdot 10^{-2}$

E. $\left(\frac{1}{10}\right)^6$

F. $\frac{1}{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}$

Solution

["A", "C", "E", "F"]

(From Unit 7, Lesson 5.)

Lesson 3: Rational and Irrational Numbers

Goals

- Comprehend the term "irrational number" (in spoken language) to mean a number that is not rational and that $\sqrt{2}$ is an example of an irrational number.

- Comprehend the term "rational number" (in written and spoken language) to mean a fraction or its opposite.

- Determine whether a given rational number is a solution to the equation $x^2 = 2$ and explain (orally) the reasoning.

Learning Targets

- I know what an irrational number is and can give an example.

- I know what a rational number is and can give an example.

Lesson Narrative

In previous lessons, students learned that square root notation is used to write the side length of a square given the area of the square. For example, a square whose area is 17 square units has a side length of $\sqrt{17}$ units.

In this lesson, students build on their work with square roots to learn about a new mathematical idea, *irrational numbers*. Students recall the definition of *rational numbers* (MP6) and use this definition to search for a rational number x such that $x^2 = 2$. Students should not be left with the impression that looking for and failing to find a rational number whose square is 2 is a *proof* that $\sqrt{2}$ is irrational; this exercise is simply meant to reinforce what it means to be irrational and to provide some plausibility for the claim. Students are not expected to prove that $\sqrt{2}$ is irrational in grade 8, and so ultimately must just accept it as a fact for now.

In the next lesson, students will learn strategies for finding the approximate location of an irrational number on a number line.

Alignments
Building On

- 5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

- 6.EE.A.1: Write and evaluate numerical expressions involving whole-number exponents.

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

Building Towards

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- Algebra Talk

- Anticipate, Monitor, Select, Sequence, Connect

- MLR1: Stronger and Clearer Each Time

- MLR2: Collect and Display

- MLR8: Discussion Supports

- Think Pair Share

Required Materials

Tracing paper
Bundles of "patty paper" are available
commercially for a very low cost. These are small
sheets (about 5" by 5") of transparent paper.

Required Preparation

It would be useful throughout this unit to have a list of perfect squares for easy reference. Consider hanging up a poster that shows the 20 perfect squares from 1 to 400. It is particularly handy in this lesson.

> ### Student Learning Goals
>
> Let's learn about irrational numbers.

3.1 Algebra Talk: Positive Solutions

Warm Up: 5 minutes
The purpose of this warm-up is for students to review multiplication of fractions in preparation for the main problem of this lesson: estimating solutions to the equation $x^2 = 2$. For example, $\frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}$, which is a value close to 2 so $\frac{3}{2}$ is a value close to $\sqrt{2}$. For this activity it is best if students work with fractions and do not convert any numbers to their decimal forms. Answers expressed in decimal form aren't wrong, but if students work with decimal forms, they will miss out on the purpose of this warm-up.

While four problems are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem.

Building On
- 5.NF.B.4
- 6.EE.A.1

Building Towards
- 8.NS.A.2

Instructional Routines
- Algebra Talk
- MLR8: Discussion Supports

Launch
Ask students, "Could 8 be a solution to $x^2 = 49$? Why or why not?"

Display one problem at a time. Give students 30 seconds of quiet think time for each problem, and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organization

> **Student Task Statement**
>
> Find a positive solution to each equation:
>
> $x^2 = 36$

$$x^2 = \frac{9}{4}$$

$$x^2 = \frac{1}{4}$$

$$x^2 = \frac{49}{25}$$

Student Response

- 6
- $\frac{3}{2}$
- $\frac{1}{2}$
- $\frac{7}{5}$

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their responses for all to see. At the end of the discussion, ask students to explain what they know about multiplication of fractions that helped them find the value of x^2 in each problem.

To involve more students in the conversation, consider asking:

- "Did anyone have the same strategy but would explain it differently?"
- "Did anyone solve for the value of x in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

Access for English Language Learners

Speaking: MLR8 Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimize output (for explanation)

3.2 Three Squares

10 minutes

This activity is the first of three activities in which students investigate the value of $\sqrt{2}$. In this activity, students draw three squares on small grids, building on their earlier work. Two of the three possible squares have easily countable side lengths since they line up along the grid lines. The third

likely possibility, that of a tilted square with vertices at the midpoint of each side of the grid, has sides equal to the length of a diagonal of a 1 unit square.

Monitor for students who:

- compare lengths directly by either creating a grid ruler or by tracing a segment with tracing paper and bringing it side by side with another segment. Most likely these students will say that the side length of the tilted square is around 1.5 units (or possibly, a little bit less than 1.5 units).

- who recall the square root notation from the previous lesson and expressed the side length of the tilted square as $\sqrt{2}$. Since the area of the tilted square is 2 square units, we can express its side length as $\sqrt{2}$ units.

If we use tracing paper or create a ruler scaled the same way as the grid, it can be seen that $\sqrt{2}$ is a bit less than 1.5. In the next activity, students will look for a more precise value of $\sqrt{2}$.

Addressing

- 8.EE.A.2

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

Launch

Provide access to tracing paper. Display the image from the task statement for all to see, and note that there are 9 vertices in each 2-by-2 grid. Students are asked in this activity to draw "squares of different sizes with vertices aligned to the vertices of the grid," so make sure they can interpret this correctly. It may be helpful to draw a few non-examples.

This has vertices aligned to the grid, but is not a square:
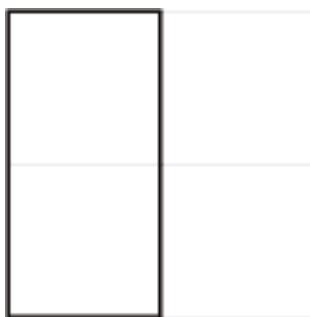
This looks like a square, but doesn't have vertices aligned to the grid:
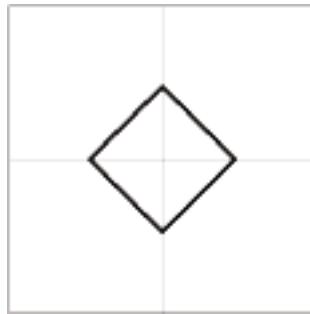
Access for Students with Disabilities

Representation: Internalize Comprehension. Provide a range of examples and counterexamples of squares with vertices that align to vertices on a grid. Consider using the provided examples in the Launch. Ask student volunteers to justify the reasoning for each.
Supports accessibility for: *Conceptual processing*

Anticipated Misconceptions

Some will say the side length of the tilted square is 1 unit, because a common misconception is that the diagonal of a square has a length of 1 unit. Ask students if the square has the same area as a grid square.

### Student Task Statement
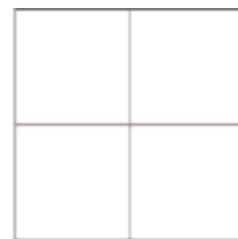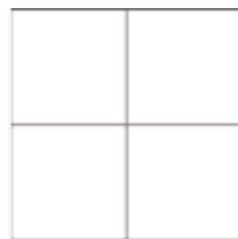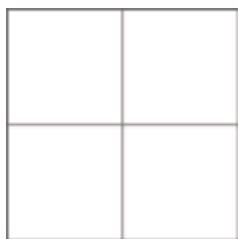
1. Draw 3 squares of different sizes with vertices aligned to the vertices of the grid.

2. For each square:

    a. Label the area.

    b. Label the side length.

    c. Write an equation that shows the relationship between the side length and the area.

Student Response

1. Answers vary. Sample response:
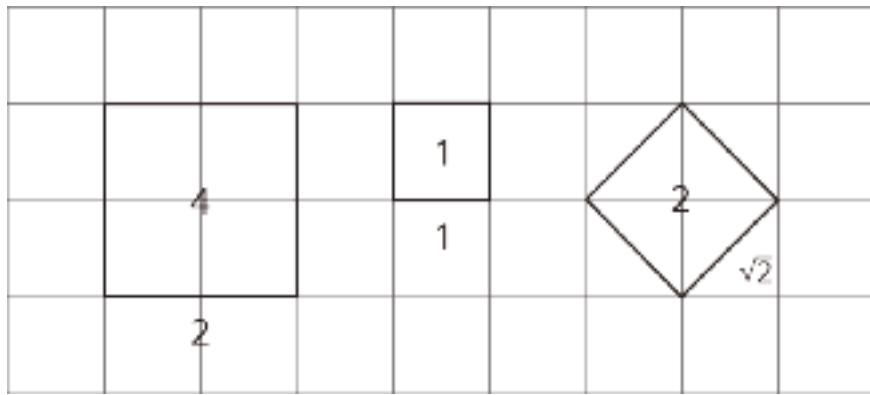
2. Sample response:

      a. The area is 2 square units.

      b. The side length is $\sqrt{2}$ units.

      c. $(\sqrt{2})^2 = 2$.

Activity Synthesis

First, select students who tried to compare the lengths directly by either creating a grid ruler or by tracing a segment with tracing paper and bringing it side by side with another segment during the discussion. Highlight the use of rotating figures to align sides in order to compare their lengths. Most likely these students will say that the side length of the tilted square is around 1.5 units (or possibly, a little bit less than 1.5 units).

Then, select students who recall the square root notation from the previous lesson and expressed the side length of the tilted square as $\sqrt{2}$. Since the area of the tilted square is 2 square units, its side length can be expressed as $\sqrt{2}$ units. This activity shows that $\sqrt{2}$ is around 1.5. In the next activity, students will look for a more precise value of $\sqrt{2}$.

Access for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. As students share their answer for the area and side length of the tilted square, press for details in students' reasoning by asking how they know the area is 2 square units. Listen for and amplify the language students use to describe either the "decompose and rearrange" or the "surround and subtract" method for finding the area of the tilted square. Then ask students to explain why the side length of the tilted square must be $\sqrt{2}$. This will support rich and inclusive discussion about strategies for finding the area and side length of a tilted square.
Design Principle(s): Support sense-making

3.3 Looking for a Solution

10 minutes

This activity is the second of three activities in which students investigate the value of $\sqrt{2}$. As a result of the previous activity, students should believe that $\sqrt{2}$ is about 1.5, maybe a little bit less.

Students should also understand that $\sqrt{2}$ is a number that we can multiply by itself and get a 2, so this is true:

$$\sqrt{2} \cdot \sqrt{2} = 2$$

The goal of this activity is to achieve a more precise value of $\sqrt{2}$ than "about 1.5, maybe a little less." In other words, an exact solution to $x^2 = 2$ is required. In this activity, students will consider whether certain candidates are solutions to this equation.

In the whole-class discussion that follows, *rational number* is defined.

Building Towards

- 8.EE.A.2

- 8.NS.A.2

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- Think Pair Share

Launch

Remind students that 1.5 is equivalent to $\frac{3}{2}$. Students in groups of 2. Give 2–3 minutes quiet work time followed by partner then a whole-class discussion.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: "First, I _____ because . . .", "I noticed _____ so I . . .", "Why did you . . .?", "I agree/disagree because . . ."
Supports accessibility for: Language; Social-emotional skills

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Student Task Statement

Are any of these numbers a solution to the equation $x^2 = 2$? Explain your reasoning.

- 1
- $\frac{1}{2}$
- $\frac{3}{2}$
- $\frac{7}{5}$

Student Response

1. No, because $1^2 = 1$, not 2.

2. No, because $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

3. No, because $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$. ($\frac{9}{4} > 2$, since $2 = \frac{8}{4}$.)

4. No, because $\left(\frac{7}{5}\right)^2 = \frac{49}{25}$. ($\frac{49}{25} < 2$, since $2 = \frac{50}{25}$.)

Activity Synthesis

Select students to share their reasoning for why each number is not a solution to $x^2 = 2$. If students do not point it out, make sure that they notice that $\frac{3}{2}$ is not a terrible approximation for $\sqrt{2}$, since $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$, and $\frac{9}{4}$ is only a bit larger than 2. $\frac{7}{5}$ is an even better approximation for $\sqrt{2}$, since $\left(\frac{7}{5}\right)^2 = \frac{49}{25}$, which is just a little bit smaller than $\frac{50}{25}$ (which equals 2). If you have posted a list of perfect squares in the room for reference, refer to this list during the discussion, because that list will come in handy when students work on the next activity.

Unit 8 Lesson 3

This is where we want to define *rational number*. A **rational number** is a fraction or its opposite. Remember that a fraction is a number on the number line that you get by dividing the unit interval into b equal parts and finding the point that is a of them from 0. We can always write a fraction in the form $\frac{a}{b}$ where a and b are whole numbers (and b is not 0), but there are other ways to write them. For example, 0.4 is a fraction because it is the point on the number line you get by dividing the unit interval into 10 equal parts and finding the point that is 4 away from 0. (You get the same point if you divide the unit interval into 5 equal parts and find the point that is 2 away from 0.) When we first learned about fractions, we didn't know about negative numbers. Rational numbers are fractions, but they can be positive or negative. So, $-\frac{2}{5}$ is also a rational number.
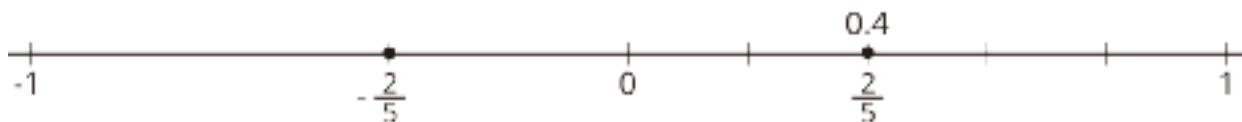
Here are some examples of rational numbers:

$$\frac{7}{4}, 0, \frac{6}{3}, 0.2, -\frac{1}{3}, -5, \sqrt{9}, -\frac{\sqrt{16}}{\sqrt{100}}$$

Can you see why they are each "a fraction or its opposite?"

Because fractions and ratios are closely related ideas, fractions and their opposites are called RATIOnal numbers.

3.4 Looking for $\sqrt{2}$

10 minutes

This activity is the third of three activities in which students investigate the value of $\sqrt{2}$. In the previous activity, students saw that $\frac{7}{5}$ is a pretty good approximation for $\sqrt{2}$. $\sqrt{2}$ is a number when multiplied by itself equals 2, and $\frac{7}{5}$ multiplied by itself is pretty close to 2, it's $\frac{49}{25}$. In the discussion of the previous activity, students also learned (or were reminded of) the definition of rational number: a fraction or its opposite.

In this activity, students have a chance to look for some more rational numbers that are close to $\sqrt{2}$. The idea is that they will look, for a while, for a fraction that can be multiplied by itself where the product is exactly 2. Of course, they won't find one, because there is no such number. In the whole-class discussion that follows, *irrational number* is defined.

Addressing

- 8.EE.A.2

- 8.NS.A

Instructional Routines

- MLR2: Collect and Display

Launch

For this activity, it is best if students do not have access to a calculator with a square root button. A calculator that shows 9 decimal places will tell you that $\sqrt{2} = 1.414213562$ (which is not true) and that $1.414213562^2 = 2$ (which is also not true). (To convince students that the calculator is lying to them, you have to make them multiply 1.414213562 by itself by hand, so better to sidestep the issue for now.) It *would* be handy for students to refer to a list of perfect squares while working on this activity, so consider posting such a list and drawing students' attention to it.

Before they get started, remind students that a rational number is a fraction or its opposite, for example, $\frac{9}{8}$. Let them know that terminating decimals are also rational, for example, $0.7 = \frac{7}{10}$.

As students work, it is possible they will focus on numbers in decimal form when searching for a rational number close to $\sqrt{2}$. If you see this, encourage students to remember their work in the previous activity and the example of $\frac{7}{5}$, whose square, $\frac{49}{25}$, was very close to 2. Are there any other values like $\frac{7}{5}$ that might be even closer to $\sqrt{2}$?

Since students could search indefinitely for a solution to the last problem with no success, ask students to stop their work in order to leave 3–4 minutes for a whole-class discussion.

Access for Students with Disabilities

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. For example, display the term rational number with its definition and examples from the previous activity.
Supports accessibility for: Conceptual processing; Memory

Access for English Language Learners

Conversing, Reading: MLR2 Collect and Display. As students work in pairs on the task, circulate and listen to as they discuss strategies for finding rational numbers that are close to $\sqrt{2}$. Record the words, phrases, and quantities students use on a visual display. Invite students to review the display, and ask questions to clarify the meaning of a word or phrase. For example, a phrase such as: "There is no number that is equal to $\sqrt{2}$" can be restated as "We could not find a rational number that is equal to $\sqrt{2}$." Encourage students to refer back to the visual display during whole-class discussions throughout the lesson and unit. This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximize meta-awareness

Unit 8 Lesson 3

Student Task Statement

A **rational number** is a fraction or its opposite (or any number equivalent to a fraction or its opposite).

1. Find some more rational numbers that are close to $\sqrt{2}$.

2. Can you find a rational number that is exactly $\sqrt{2}$?

Student Response

1. Answers vary. Sample responses:
   - $\frac{10}{7}$, because $\left(\frac{10}{7}\right)^2 = \frac{100}{49}$
   - $\frac{17}{12}$, because $\left(\frac{17}{12}\right)^2 = \frac{289}{144}$
   - $\frac{13}{9}$, because $\left(\frac{13}{9}\right)^2 = \frac{169}{81}$
   - $\frac{141}{100}$, because $\left(\frac{141}{100}\right)^2 = \frac{19,881}{10,000}$

2. No.

Are You Ready for More?

If you have an older calculator evaluate the expression

$$\left(\frac{577}{408}\right)^2$$

and it will tell you that the answer is 2, which might lead you to think that $\sqrt{2} = \frac{577}{408}$.

1. Explain why you might be suspicious of the calculator's result.

2. Find an explanation for why $408^2 \cdot 2$ could not possibly equal 577^2. How does this show that $\left(\frac{577}{408}\right)^2$ could not equal 2?

3. Repeat these questions for

$$\left(\frac{1414213562375}{10000000000000}\right)^2 \neq 2,$$

an equation that even many modern calculators and computers will get wrong.

Student Response

One reason to be suspicious is that the calculator might have rounded the answer, so it may just be that $\left(\frac{577}{408}\right)^2$ is *close* to 2. In fact, we can argue for certain that it is not. The equation $577^2 = 2 \cdot 408^2$ could not possibly be true since the left-hand side of that expression is odd, but the right-hand side is even. But now it could not possibly be that $\left(\frac{577}{408}\right)^2 \neq 2$, since after multiplying both sides by 408^2 this would mean $577^2 = 2 \cdot 408^2$. This exact argument also works for the second example, as

$1414213562375^2 \neq 2 \cdot 10000000000000^2$ for the same reason. In fact, by breaking into cases depending on which of the numerator and denominator is even or odd, this line of thinking leads to a complete proof that no fraction at all can be squared to get the number 2, proving that $\sqrt{2}$ is irrational.

Activity Synthesis

Ideally, students were given time to look for a rational number that is a solution to $x^2 = 2$. Select students with particularly close values to share their number with the class and what strategy they used to find it. Applaud students for their perseverance, but confess that no such number exists because it is an *irrational number*.

Display a number line for all to see such as the one shown here. Tell students that an **irrational number** is a number that is not rational. That is, it is a number that is not a fraction or its opposite. $\sqrt{2}$ is one example of an irrational number. It has a location on the number line, and its location can be narrowed down (it's a tiny bit to the right of $\frac{7}{5}$), but $\sqrt{2}$ cannot be found on a number line by subdividing the unit interval into b parts and taking a of them. We have to define $\sqrt{2}$ in a different way, such as the side length of a square with area 2 square units.
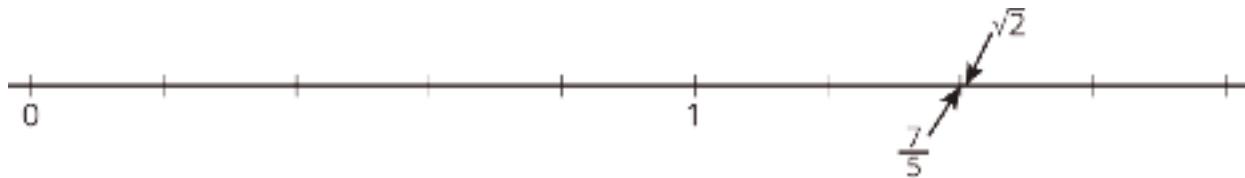
Just looking for rational solutions to $x^2 = 2$ and not finding any is not a proof that $\sqrt{2}$ is irrational. But for the purposes of this course, students are asked to take it as a fact that $\sqrt{2}$ is irrational. In students' future studies, they may have opportunities to understand or write a proof that $\sqrt{2}$ is irrational.

While the activities focus specifically on $\sqrt{2}$, there is nothing particularly special about this example. The square root of a whole number is either a whole number or irrational, so $\sqrt{10}$, $\sqrt{67}$, etc., are all irrational. Beyond grade level but worth having on hand for the sake of discussion is that rational multiples of irrational numbers are also irrational, so numbers like $5\sqrt{7}$, $-\sqrt{45}$, and $\frac{\sqrt{5}}{3}$ are also irrational.

Lesson Synthesis

The purpose of this discussion is to explicitly point out that while we have collected some evidence that supports the claim that $\sqrt{2}$ is irrational, we have not actually proved this claim. Here are some possible questions for discussion:

- "If I told you that there are no purple zebras, and you spent your whole life traveling the world and never saw a purple zebra, does it mean I was right?" (No, it is possible you just failed to find a purple zebra.)

Unit 8 Lesson 3

- "So if we spent our whole lives testing different fractions and never quite got one whose square is 2, does that mean there are no such fractions?" (No, maybe you just haven't found it yet.)

Tell students that we haven't learned enough to prove for sure that $\sqrt{2}$ is not equivalent to a fraction. For now, we just have to trust that there are numbers on the number line that are not equivalent to a fraction and that $\sqrt{2}$ is one of them. However, it is possible to get very close estimates with fractions.

3.5 Types of Solutions

Cool Down: 5 minutes
Addressing

- 8.NS.A

Student Task Statement

1. In your own words, say what a rational number is. Give at least three different examples of rational numbers.

2. In your own words, say what an irrational number is. Give at least two examples.

Student Response

Answers vary. Sample response:

1. A rational number is a fraction, like $\frac{1}{2}$, or its opposite, like $-\frac{1}{2}$. Something like 3.98 is rational too because it is equal to $\frac{398}{100}$.

2. An irrational number is one that is not rational. $\sqrt{2}$ and π are two examples.

Student Lesson Summary

In an earlier activity, we learned that square root notation is used to write the length of a side of a square given its area. For example, a square whose area is 2 square units has a side length of $\sqrt{2}$ units, which means that
$$\sqrt{2} \cdot \sqrt{2} = 2.$$
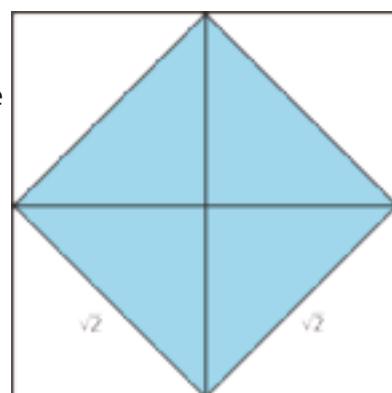
A square whose area is 25 square units has a side length of $\sqrt{25}$ units, which means that
$$\sqrt{25} \cdot \sqrt{25} = 25.$$

Since $5 \cdot 5 = 25$, we know that

$$\sqrt{25} = 5.$$

$\sqrt{25}$ is an example of a rational number. A **rational number** is a fraction or its opposite. Remember that a fraction $\frac{a}{b}$ is a point on the number line found by dividing the segment from 0 to 1 into b equal intervals and going a of those intervals to the right of 0. We can always write a fraction in the form $\frac{a}{b}$ where a and b are whole numbers (and b is not 0), but there are other ways to write them. For example, we can write $\sqrt{25} = \frac{5}{1}$. You first learned about fractions in earlier grades, and at that time, you probably didn't know about negative numbers. Rational numbers are fractions, but they can be positive or negative. So, -5 is also a rational number. Because fractions and ratios are closely related ideas, fractions and their opposites are called RATIOnal numbers.

Here are some examples of rational numbers:

$$\frac{7}{4},\ 0,\ \frac{6}{3}, 0.2, -\frac{1}{3}, -5,\ \sqrt{9},\ -\frac{\sqrt{16}}{\sqrt{100}}$$

Can you see why they are each examples of "a fraction or its opposite?"

An **irrational number** is a number that is not rational. That is, it is a number that is not a fraction or its opposite. $\sqrt{2}$ is an example of an irrational number. It has a location on the number line, and its location can be approximated by rational numbers (it's a tiny bit to the right of $\frac{7}{5}$), but $\sqrt{2}$ can not be found on a number line by dividing the segment from 0 to 1 into b equal parts and going a of those parts away from 0 (if a and b are whole numbers).
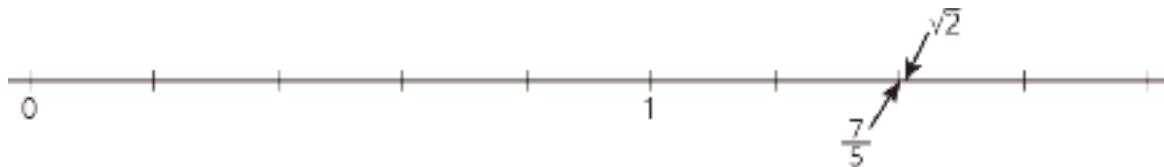
$\frac{17}{12}$ is also close to $\sqrt{2}$, because $\left(\frac{17}{12}\right)^2 = \frac{289}{144}$. $\frac{289}{144}$ is very close to 2, since $\frac{288}{144} = 2$. But we could keep looking forever for solutions to $x^2 = 2$ that are rational numbers, and we would not find any. $\sqrt{2}$ is not a rational number! It is irrational.

In your future studies, you may have opportunities to understand or write a proof that $\sqrt{2}$ is irrational, but for now, we just take it as a fact that $\sqrt{2}$ is irrational. Similarly, the square root of any whole number is either a whole number ($\sqrt{36} = 6$, $\sqrt{64} = 8$, etc.) or irrational ($\sqrt{17}$, $\sqrt{65}$, etc.). Here are some other examples of irrational numbers:

$$\sqrt{10},\ -\sqrt{3},\ \frac{\sqrt{5}}{2},\ \pi$$

Glossary

- irrational number

Unit 8 Lesson 3

- rational number

Lesson 3 Practice Problems

Problem 1

Statement

Decide whether each number in this list is *rational* or *irrational*.

$$\frac{-13}{3}, \ 0.1234, \ \sqrt{37}, \ -77, \ -\sqrt{100}, \ -\sqrt{12}$$

Solution

Rational: $\frac{-13}{3}, 0.1234, -77, -\sqrt{100}$; Irrational: $\sqrt{37}, -\sqrt{12}$

Problem 2

Statement

Which value is an exact solution of the equation $m^2 = 14$?

A. 7

B. $\sqrt{14}$

C. 3.74

D. $\sqrt{3.74}$

Solution

B

Problem 3

Statement

A square has vertices $(0, 0), (5, 2), (3, 7)$, and $(-2, 5)$. Which of these statements is true?

A. The square's side length is 5.

B. The square's side length is between 5 and 6.

C. The square's side length is between 6 and 7.

D. The square's side length is 7.

Solution

B
(From Unit 8, Lesson 2.)

Problem 4

Statement

Rewrite each expression in an equivalent form that uses a single exponent.

    a. $(10^2)^{-3}$

    b. $(3^{-3})^2$

    c. $3^{-5} \cdot 4^{-5}$

    d. $2^5 \cdot 3^{-5}$

Solution

    a. 10^{-6} (or equivalent)

    b. 3^{-6} (or equivalent)

   c. 12^{-5} (or equivalent)

    d. $\left(\frac{2}{3}\right)^5$ (or equivalent)

(From Unit 7, Lesson 8.)

Problem 5

Statement

The graph represents the area of arctic sea ice in square kilometers as a function of the day of the year in 2016.
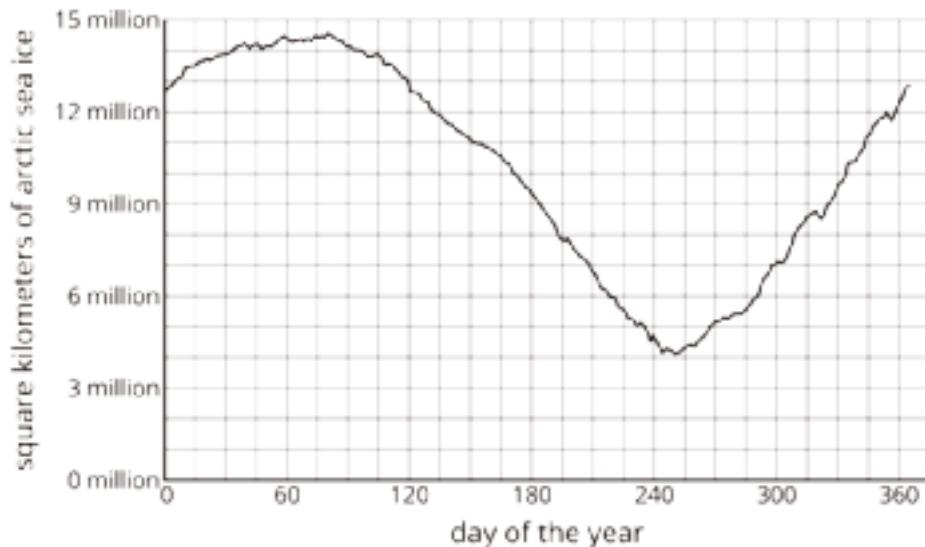
a. Give an approximate interval of days when the area of arctic sea ice was decreasing.

■　　　b. On which days was the area of arctic sea ice 12 million square kilometers?

Solution

a. Answers vary. Correct responses should be close to "day 75 to day 250."

b. Days 135, 350, and 360

(From Unit 5, Lesson 5.)

Problem 6

Statement

The high school is hosting an event for seniors but will also allow some juniors to attend. The principal approved the event for 200 students and decided the number of juniors should be 25% of the number of seniors. How many juniors will be allowed to attend? If you get stuck, try writing two equations that each represent the number of juniors and seniors at the event.

Solution

40 juniors. Sample reasoning: Solve the system $s + j = 200$, $j = 0.25s$ (or equivalent), where s represents the number of seniors and j represents the number of juniors.

(From Unit 4, Lesson 14.)

Lesson 4: Square Roots on the Number Line

Goals

- Calculate an approximate value of a square root to the nearest tenth, and represent the square root as a point on the number line.

- Determine the exact length of a line segment on a coordinate grid and express the length (in writing) using square root notation.

- Explain (orally) how to verify that a value is a close approximation of a square root.

Learning Targets

- I can find a decimal approximation for square roots.

- I can plot square roots on the number line.

Lesson Narrative

In this lesson, students begin to transition from understanding square roots simply as side lengths to recognizing that all square roots are specific points on the number line. This understanding takes time to develop because students have previously only worked with rational numbers, which can be found by dividing the segment between two numbers into equal intervals. In the first activity, they still find $\sqrt{10}$ by relating it to the side length of a square of area 10 square units, but then are asked to approximate the value of $\sqrt{10}$ to the nearest tenth. In the second activity, students find a decimal approximation for $\sqrt{3}$ by looking at areas and also computing squares of numbers. This lesson shows students that irrational numbers are *numbers*—specific points on the number line—and we can find them by rotating a tilted square until it is sitting "flat." This is the conceptual foundation for the approximation work in the next lesson.

Alignments

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR1: Stronger and Clearer Each Time

- MLR8: Discussion Supports

- Notice and Wonder

- Think Pair Share

Required Materials

Compasses
Four-function calculators

Tracing paper
Bundles of "patty paper" are available commercially for a very low cost. These are small sheets (about 5" by 5") of transparent paper.

> **Student Learning Goals**
>
> Let's explore square roots.

4.1 Notice and Wonder: Diagonals

Warm Up: 5 minutes
This warm-up transitions from work in previous lessons and prepares students to locate square roots on a number line in this lesson. Students must use the structure of the circle to relate the length of the segment to a point on the number line (MP7).

Addressing

- 8.EE.A.2

- 8.NS.A.2

Instructional Routines

- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

What do you notice? What do you wonder?
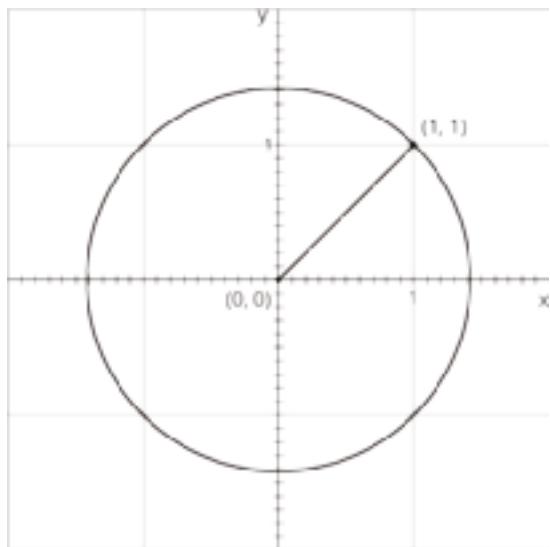
Student Response

Things students may notice:

- The center of the circle is at $(0, 0)$.
- There is a point labeled at $(1, 1)$ on the circle.
- There are many tick lines between 0 and 1.

Things students may wonder:

- How to find the distance across the circle?
- Where exactly does the circle land on the x and y axis?

Activity Synthesis

Ask students what the exact length is (it should be familiar to them from earlier lessons). The focus of the discussion is how you can see the decimal approximation from the diagram by looking at where the circle intersects an axis.

4.2 Squaring Lines

10 minutes
In this activity, students determine the length of a "diagonal" line segment on a grid. Students can give an exact value for the length of the line segment by finding the area of a square and writing the side length using square root notation. The goal of this activity is for students to connect values expressed using square roots with values expressed in decimal form—a form they are more familiar with.

Monitor for students who draw a tilted square for the first problem during the first two minutes of work time. Then monitor for students who use the following strategies to find the length of the segment:

- drawing a square and finding the area
- using tracing paper
- using a compass to make a circle

Addressing

- 8.EE.A.2
- 8.NS.A.2

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

Launch

For this activity, it is best if students do not have access to a calculator with a square root button. If student calculators do have a square root button that students are familiar with, tell students that their explanations about their answers to the second problem need to dig deeper than pressing a button. In later lessons, however, they will be able to use it.

Before students begin, remind students that "exact length" means it can't just be an approximation, so if it is not a rational number, we should write it with square root notation. For example, a square with area 17 has a side length of exactly $\sqrt{17}$, which is a little larger than 4, since $4^2 = 16$.

Begin by displaying the diagram for all to see. Ask students how this diagram is similar and how it is different from the diagram in the warm-up. Then 2–3 minutes after students begin working, pause the class and select a previously identified student who drew a square on the grid to share what they did and why. Give 2–3 minutes work time to finish the problems followed by a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to tracing paper, straight edges, and compasses to support information processing in estimating the length of the line segment.
Supports accessibility for: Visual-spatial processing; Organization

---

Student Task Statement
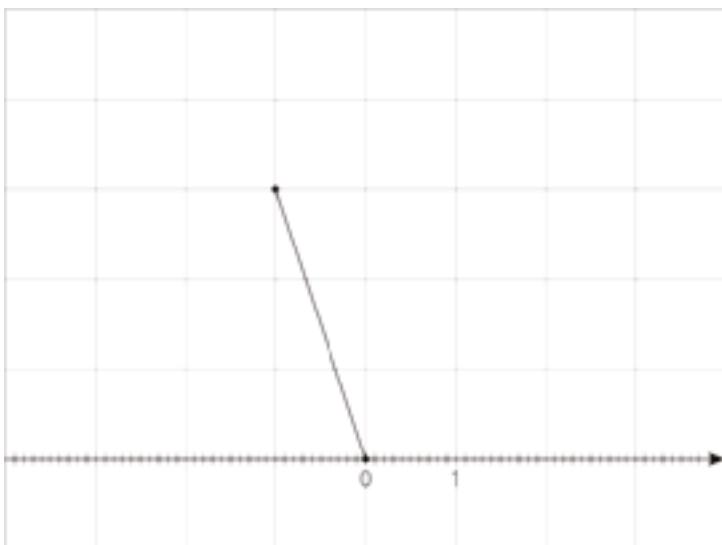
1. Estimate the length of the line segment to the nearest tenth of a unit (each grid square is 1 square unit).

2. Find the exact length of the segment.

Student Response

1. 3.1 units

2. $\sqrt{10}$ units

Activity Synthesis

Select students to present in this sequence:

- Someone who drew a square and used the area to find the exact side length.

- Someone who used tracing paper. This is essentially like the number line as a ruler. Ask students what this tells us about the exact value we found with the square. ($\sqrt{17}$ is about 3.1.)

- Someone who used a compass to find the approximate side length. This is a more formal geometric construction, but it is just another way to use the number line as a ruler.

Access for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. As students share their methods for finding the exact length of the line segment, press for details in students' reasoning by asking how they know the figure they drew in the coordinate plane is a square. Listen for and amplify the language students use to describe the important features of the square (e.g., opposite sides are equal, opposite sides are parallel, each angle is 90°). Then ask students to explain why the side length of the tilted square must be $\sqrt{10}$. This will support rich and inclusive discussion about strategies for finding the exact length of a line segment in the coordinate plane. *Design Principle(s): Support sense-making*

4.3 Square Root of 3

10 minutes

In previous activities and lessons, students found the exact area of a square in order to find an approximation for the square root of an integer. In this activity, students start with a square root of an integer, and draw a square to verify that a given approximation of the square root is reasonable. This is the first time students have seen or drawn squares that do not have vertices at the intersection of grid lines, so it may take them a few minutes to make sense of the new orientation.

Addressing

- 8.EE.A.2
- 8.NS.A.2

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Think Pair Share

Launch

Give students access to four-function calculators. Display the diagram for all to see. Ask students what is the same and what is different about this diagram and diagrams they have seen in earlier activities. Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by partner and whole-class discussions.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer for data collection and organizing information.
*Supports accessibility for: Language; Organization*

Student Task Statement

Diego said that he thinks that $\sqrt{3} \approx 2.5$.
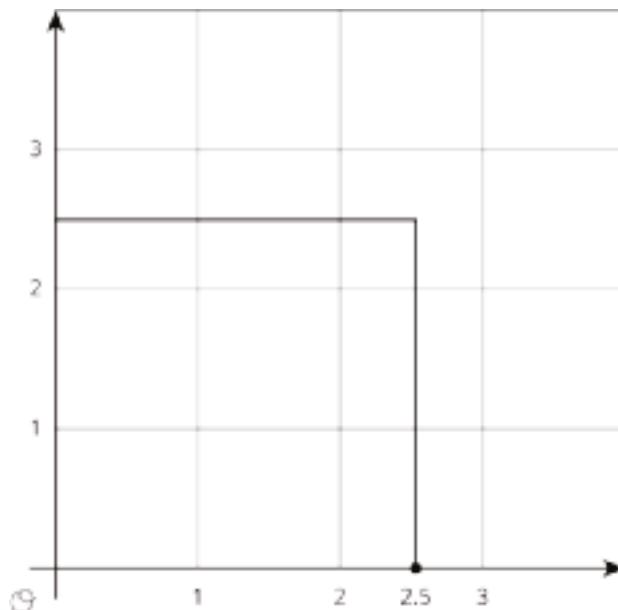
1. Use the square to explain why 2.5 is not a very good approximation for $\sqrt{3}$. Find a point on the number line that is closer to $\sqrt{3}$. Draw a new square on the axes and use it to explain how you know the point you plotted is a good approximation for $\sqrt{3}$.

2. Use the fact that $\sqrt{3}$ is a solution to the equation $x^2 = 3$ to find a decimal approximation of $\sqrt{3}$ whose square is between 2.9 and 3.1.

Student Response

Answers vary. Sample response:

1. Any number between 1.5 and 2 will give a better approximation, and the corresponding square will have an area between 2.25 and 4. Because you can see the area of the square, you can know that the approximation is better.

2. 1.73

Are You Ready for More?

A farmer has a grassy patch of land enclosed by a fence in the shape of a square with a side length of 4 meters. To make it a suitable home for some animals, the farmer would like to carve out a smaller square to be filled with water, as in the figure.

What should the side length of the smaller square be so that half of the area is grass and half is water?
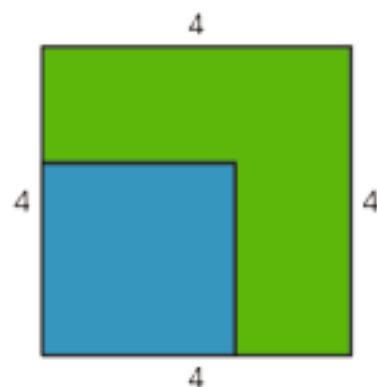
Student Response

The area enclosed by the fence is 16 square meters, so we want the area of both the grassy region and the water region to be 8 square meters. For the blue square in the figure to have an area of 8 square meters, the side length needs to be $\sqrt{8}$ meters, or about 2.8 meters.

Activity Synthesis

Invite one or two students to share their squares. Then tell students, "The square of a point on the number line can be visualized as the area of a literal square. This can help us estimate the value of a square root. Simply squaring the number can as well. Let's check the squares of some numbers that are potential approximations of $\sqrt{3}$."

Then ask students to suggest decimal approximations, and check together as a class by finding their squares. Students should be using each guess to make a better guess next. For example, if they try 1.5, then the square is 2.25, which is too low. This suggests trying bigger. Because we know that 2 is too big (because $2^2 = 4$, it should be somewhere in between 1.5 and 2.) For example, students might suggest this order:

$1^2 = 1$ and $2^2 = 4$

$1.5^2 = 2.25$

$1.8^2 = 3.24$

$1.7^2 = 2.89$

$1.72^2 = 2.9584$

$1.73^2 = 2.9929$

So 1.73 is a pretty good approximation of $\sqrt{3}$.

Lesson Synthesis

The goal of this discussion is to check that students know how to approximate square roots. In the previous lesson, students learned that some square roots, $\sqrt{2}$ in particular, are not rational. But they are still numbers, and we can reason about their approximate value using more familiar numbers.

- "How can you approximate the value of $\sqrt{130}$?" ($\sqrt{130}$ is somewhere between 11 and 12 because $11^2 = 121$ and $12^2 = 144$.)

- "So we know $\sqrt{130}$ is somewhere between 11 and 12. Can we get more accurate than that? How?" (We could try squaring numbers from 11 to 12 like 11.1, 11.2, etc. . . to find the one closest to 130.)

4.4 Approximating $\sqrt{18}$

Cool Down: 5 minutes
Addressing

- 8.EE.A.2

- 8.NS.A.2

> **Student Task Statement**
>
> Plot $\sqrt{18}$ on the x-axis. Consider using the grid to help.
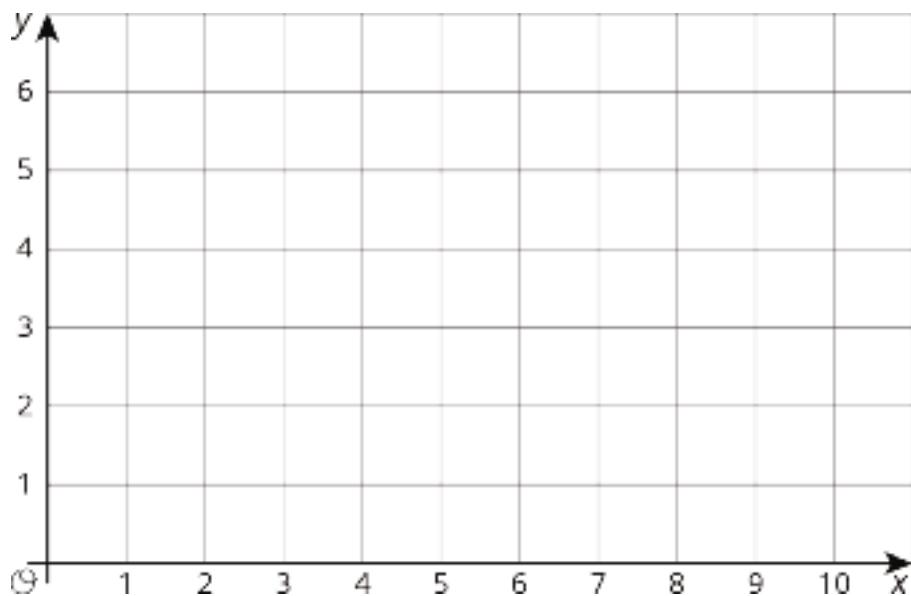
Student Response

About 4.2.

## Student Lesson Summary

Here is a line segment on a grid. What is the length of this line segment?
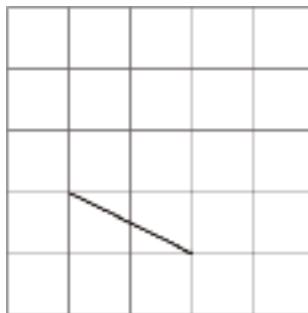
By drawing some circles, we can tell that it's longer than 2 units, but shorter than 3 units.
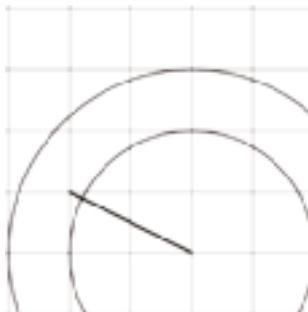
To find an exact value for the length of the segment, we can build a square on it, using the segment as one of the sides of the square.
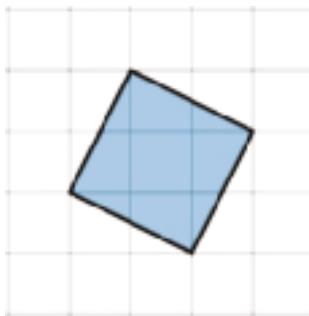
The area of this square is 5 square units. (Can you see why?) That means the exact value of the length of its side is $\sqrt{5}$ units.

Notice that 5 is greater than 4, but less than 9. That means that $\sqrt{5}$ is greater than 2, but less than 3. This makes sense because we already saw that the length of the segment is in between 2 and 3.

With some arithmetic, we can get an even more precise idea of where $\sqrt{5}$ is on the number line. The image with the circles shows that $\sqrt{5}$ is closer to 2 than 3, so let's find the value of 2.1^2 and 2.2^2 and see how close they are to 5. It turns out that $2.1^2 = 4.41$ and $2.2^2 = 4.84$, so we need to try a larger number. If we increase our search by a tenth, we find that $2.3^2 = 5.29$. This means that $\sqrt{5}$ is greater than 2.2, but less than 2.3. If we wanted to keep going, we could try 2.25^2 and eventually narrow the value of $\sqrt{5}$ to the hundredths place. Calculators do this same process to many decimal places, giving an approximation like $\sqrt{5} \approx 2.2360679775$. Even though this is a lot of decimal places, it is still not exact because $\sqrt{5}$ is irrational.

Lesson 4 Practice Problems

Problem 1

Statement

a. Find the exact length of each line segment.
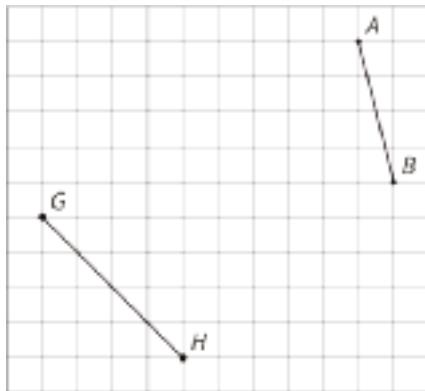
b. Estimate the length of each line segment to the nearest tenth of a unit. Explain your reasoning.

Solution

a. $AB = \sqrt{17}, GH = \sqrt{32}$

b. $AB \approx 4.1$, because $4.1^2 = 16.81$ and $4.2^2 = 17.64$. $GH \approx 5.7$, because $5.6^2 = 31.36$ and $5.7^2 = 32.49$.

Problem 2

Statement

Plot each number on the x-axis: $\sqrt{16}, \ \sqrt{35}, \ \sqrt{66}$. Consider using the grid to help.
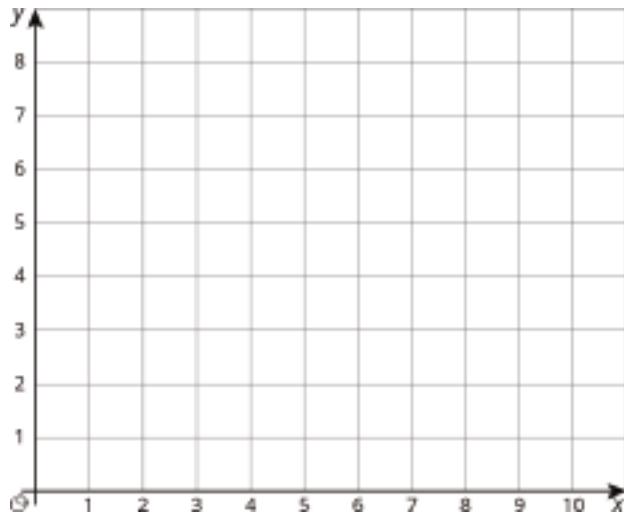
Solution

$\sqrt{16}$ at 4, $\sqrt{35}$ a little less than 6, $\sqrt{66}$ a little more than 8

Problem 3

Statement

Use the fact that $\sqrt{7}$ is a solution to the equation $x^2 = 7$ to find a decimal approximation of $\sqrt{7}$ whose square is between 6.9 and 7.1.

Solution

Answers vary. Sample responses: 2.63, 2.64, 2.65, 2.66

Problem 4

Statement

Graphite is made up of layers of graphene. Each layer of graphene is about 200 picometers, or 200×10^{-12} meters, thick. How many layers of graphene are there in a 1.6-mm-thick piece of graphite? Express your answer in scientific notation.

Solution

About 8×10^6. The thickness of the graphite is 1.6×10^{-3} meters. The number of layers of graphene is given by $\frac{1.6 \times 10^{-3}}{200 \cdot 10^{-12}} = 0.008 \times 10^9$. This number, in scientific notation, is 8×10^6, or about 8 million.

(From Unit 7, Lesson 14.)

Problem 5

Statement

Here is a scatter plot that shows the number of assists and points for a group of hockey players. The model, represented by $y = 1.5x + 1.2$, is graphed with the scatter plot.
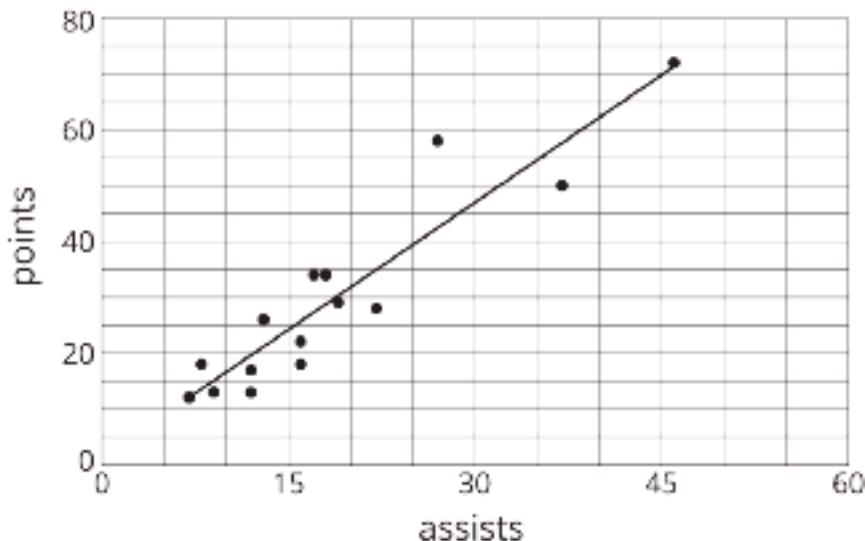
a. What does the slope mean in this situation?

b. Based on the model, how many points will a player have if he has 30 assists?

Solution

a. For every assist, a player's points have gone up by 1.5.

b. Approximately 46.2 points

(From Unit 6, Lesson 6.)

Problem 6

Statement

The points $(12, 23)$ and $(14, 45)$ lie on a line. What is the slope of the line?

Solution

$\frac{22}{2}$ (or 11)

(From Unit 3, Lesson 5.)

Lesson 5: Reasoning About Square Roots

Goals

- Comprehend that $-\sqrt{a}$ represents the opposite of $\sqrt{a}$.

- Determine a solution to an equation of the form $x^2 = a$ and represent the solution as a point on the number line.

- Identify the two whole number values that a square root is between and explain (orally) the reasoning.

Learning Targets

- When I have a square root, I can reason about which two whole numbers it is between.

Lesson Narrative

The purpose of this lesson is to encourage students to reason about square roots and reinforce the idea that they are numbers on a number line. This lesson continues students' move from geometric to algebraic characterizations of square roots.

Alignments

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- MLR3: Clarify, Critique, Correct

- Think Pair Share

- True or False

> **Student Learning Goals**
>
> Let's approximate square roots.

5.1 True or False: Squared

Warm Up: 5 minutes
The purpose of this warm-up is for students to analyze symbolic statements about square roots and decide if they are true or not based on the meaning of the square root symbol.

Addressing

- 8.EE.A.2

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

> **Student Task Statement**
>
> Decide if each statement is true or false.
>
> $\left(\sqrt{5}\right)^2 = 5$
> $\left(\sqrt{10}\right)^2 = 100$
>
> $\left(\sqrt{9}\right)^2 = 3$
> $\left(\sqrt{16}\right) = 2^2$
>
> $7 = \left(\sqrt{7}\right)^2$

Student Response

true, false, true, false, true

Activity Synthesis

Poll students on their responses for each problem. Record and display their responses for all to see. If all students agree, ask 1 or 2 students to share their reasoning. If there is disagreement, ask students to share their reasoning until an agreement is reached.

5.2 Square Root Values

10 minutes
The purpose of this activity is for students to think about square roots in relation to the two whole number values they are closest to. Students are encouraged to use numerical approaches, especially the fact that $\sqrt{a}$ is a solution to the equation $x^2 = a$, rather than less efficient geometric methods (which may not even work). Students can draw a number line if that helps them reason

about the magnitude of the given square roots, but this is not required. However the reason, students must construct a viable argument (MP3).

Addressing

- 8.EE.A.2

- 8.NS.A.2

Instructional Routines

- MLR3: Clarify, Critique, Correct

- Think Pair Share

Launch

Do not give students access to calculators. Students in groups of 2. 2 minutes of quiet work time followed by a partner then a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to a number line that includes rational numbers to support information processing.
Supports accessibility for: Visual-spatial processing; Organization

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Student Task Statement

What two whole numbers does each square root lie between? Be prepared to explain your reasoning.

1. $\sqrt{7}$

2. $\sqrt{23}$

3. $\sqrt{50}$

4. $\sqrt{98}$

Student Response

1. 2 and 3. 2^2 is 4 and 3^2 is 9, so $\sqrt{7}$ is between 2 and 3.

2. 4 and 5. 4^2 is 16 and 5^2 is 24, so $\sqrt{23}$ is between 4 and 5.

3. 7 and 8. 7^2 is 49 and 8^2 is 64, so $\sqrt{50}$ is between 7 and 8.

4. 9 and 10. 9^2 is 81 and 10^2 is 100, so $\sqrt{98}$ is between 9 and 10.

> **Are You Ready for More?**
>
> Can we do any better than "between 3 and 4" for $\sqrt{12}$? Explain a way to figure out if the value is closer to 3.1 or closer to 3.9.

Student Response

Answers vary. Sample response: Since $3.5^2 = 12.25$, we know that it is somewhere between 3 and 3.5. That tells us that it is closer to 3.1 than 3.9.

Activity Synthesis

Discuss:

- "What strategy did you use to figure out the two whole numbers?" (I made a list of perfect squares and then found which two the number was between.)

- "Did anyone use inequality symbols when writing their answers?" (Yes, for the first problem, I wrote $2 < \sqrt{5} < 3$.)

Once the class is satisfied with which two whole numbers the square roots lie between, ask students to think more deeply about their relationship. Give 1–2 minutes for students to pick one of the last two square roots and figure out which whole number the square root is closest to and to be ready to explain how they know. One possible misconception that could be covered here is that if a number is exactly halfway between two perfect squares, then the square root of that number is also halfway between the square root of the perfect squares. For example, students may think that $\sqrt{26}$ is halfway between 4 and 6 since 26 is halfway between 16 and 36. It's close, since $\sqrt{26} \approx 5.099$, but it's slightly larger than "halfway."

This is a good opportunity to remind students of the graph they made earlier showing the relationship between the side length and area of a square. The graph showed a non-proportional relationship, so making proportional assumptions about relative sizes will not be accurate.

Reading, Writing, Speaking: MLR3 Clarify, Critique, Correct. Before students share their strategies for figuring out the two whole numbers that each square root lies between, present an incorrect solution based on a misconception about the definition of exponents. For example, "$\sqrt{7}$ is in between 2 and 4, because 2^2 is 4, and 4^2 is 8"; or "$\sqrt{23}$ is in between 11 and 12, because 11^2 is 22, and 12^2 is 24." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who clarify the meaning of a number raised to the power of 2. This routine will engage students in meta-awareness as they critique and correct the language used to relate square roots to the two whole number values they are closest to.
Design Principles(s): Cultivate conversation; Maximize meta-awareness

5.3 Solutions on a Number Line

10 minutes
The purpose of this activity is for students to use rational approximations of irrational numbers to place both rational and irrational numbers on a number line and to reinforce the definition of a square root as a solution to the equation of the form $x^2 = a$. This is also the first time that students have thought about negative square roots.

Addressing

- 8.EE.A.2

- 8.NS.A.2

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- Think Pair Share

Launch

Do not provide students with access to calculators. Students in groups of 2. 2 minutes of quiet work time followed by a partner, then a whole-class discussion.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example, use a kinesthetic representation of the number line on a clothesline. Students can place and adjust numbers on folder paper or cardstock on the clothesline in a hands-on manner.
*Supports accessibility for: Conceptual processing*

Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time. After students have had time to plot x, y, and z on the number line, ask them to write a brief explanation of their reasoning for each number on their paper. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "How do you know that $z = \sqrt{30}$?" and "How do you know that $\sqrt{30}$ is between 5 and 6?", etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine both their ideas and their verbal and written output.
Design Principles(s): Optimize output (for explanation); Maximize meta-awareness

Student Task Statement

The numbers x, y, and z are positive, and $x^2 = 3$, $y^2 = 16$, and $z^2 = 30$.
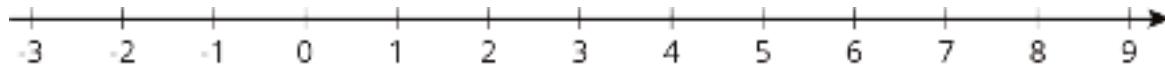
1. Plot x, y, and z on the number line. Be prepared to share your reasoning with the class.

2. Plot $-\sqrt{2}$ on the number line.
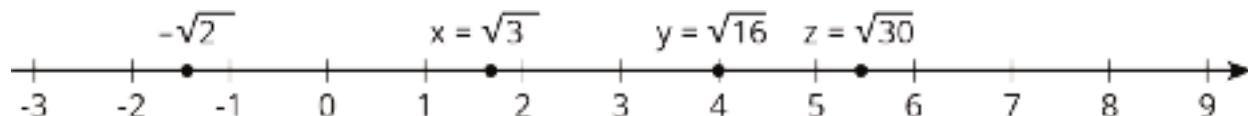
Student Response

Activity Synthesis

Display the number line from the activity for all to see. Select groups to share how they chose to place values onto the number line. Place the values on the displayed number line as groups share, and after each placement poll the class to ask if students used the same reasoning or different reasoning. If any students used different reasoning, invite them to share with the class.

Conclude the discussion by asking students to share how they placed $-\sqrt{2}$ and why.

Unit 8 Lesson 5

Lesson Synthesis

To approximate a square root, start by finding the whole numbers it lies between, and then try to get more accurate approximations.

- "How can we find the whole numbers that a square root lies between?" (Look at the squares of whole numbers whose squares are greater than and less than the number inside the square root symbol, like 121 and 144 for $\sqrt{130}$.)

- "How can we get a better approximation?" (Test values between those two whole numbers.)

5.4 Betweens

Cool Down: 5 minutes
Addressing

- 8.EE.A.2

- 8.NS.A.2

> **Student Task Statement**
>
> Which of the following numbers are greater than 6 and less than 8? Explain how you know.
>
> - $\sqrt{7}$
>
> - $\sqrt{60}$
>
> - $\sqrt{80}$

Student Response

Only $\sqrt{60}$.

Since $6^2 = 36$ and $8^2 = 64$, the number inside the square root must be between 36 and 64.

> # Student Lesson Summary
>
> In general, we can approximate the values of square roots by observing the whole numbers around it, and remembering the relationship between square roots and squares. Here are some examples:
>
> - $\sqrt{65}$ is a little more than 8, because $\sqrt{65}$ is a little more than $\sqrt{64}$ and $\sqrt{64} = 8$.
>
> - $\sqrt{80}$ is a little less than 9, because $\sqrt{80}$ is a little less than $\sqrt{81}$ and $\sqrt{81} = 9$.
>
> - $\sqrt{75}$ is between 8 and 9 (it's 8 point something), because 75 is between 64 and 81.
>
> - $\sqrt{75}$ is approximately 8.67, because $8.67^2 = 75.1689$.
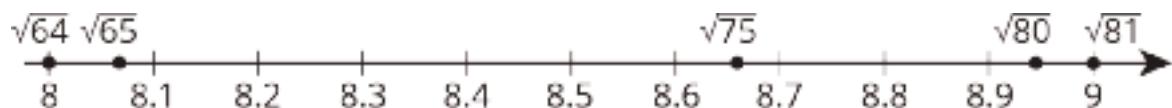
If we want to find a square root between two whole numbers, we can work in the other direction. For example, since $22^2 = 484$ and $23^2 = 529$, then we know that $\sqrt{500}$ (to pick one possibility) is between 22 and 23.

Many calculators have a square root command, which makes it simple to find an approximate value of a square root.

Lesson 5 Practice Problems
Problem 1
Statement

a. Explain how you know that $\sqrt{37}$ is a little more than 6.

b. Explain how you know that $\sqrt{95}$ is a little less than 10.

c. Explain how you know that $\sqrt{30}$ is between 5 and 6.

Solution

a. $\sqrt{36}$ is exactly 6, and $\sqrt{37}$ is a little more than that.

b. $\sqrt{100}$ is exactly 10, and $\sqrt{95}$ is a little less than that.

c. $\sqrt{25} = 5$, $\sqrt{36} = 6$, and $\sqrt{30}$ is in between.

Problem 2
Statement

Plot each number on the number line:
$$6, \sqrt{83}, \sqrt{40}, \sqrt{64}, 7.5$$

Solution
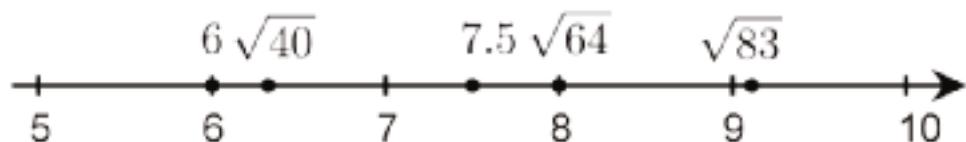
Problem 3
Statement

The equation $x^2 = 25$ has *two* solutions. This is because both $5 \cdot 5 = 25$, and also $-5 \cdot -5 = 25$. So, 5 is a solution, and also -5 is a solution.

Select **all** the equations that have a solution of -4:

A. $10 + x = 6$

B. $10 - x = 6$

C. $-3x = -12$

D. $-3x = 12$

E. $8 = x^2$

F. $x^2 = 16$

Solution

["A", "D", "F"]

Problem 4

Statement

Find all the solutions to each equation.

a. $x^2 = 81$

b. $x^2 = 100$

c. $\sqrt{x} = 12$

Solution

a. 9 and -9

b. 10 and -10

c. 144

Problem 5

Statement

Select all the irrational numbers in the list.

$$\frac{2}{3}, \frac{-123}{45}, \sqrt{14}, \sqrt{64}, \sqrt{\frac{9}{1}}, -\sqrt{99}, -\sqrt{100}$$

Solution

$\sqrt{14}, -\sqrt{99}$

(From Unit 8, Lesson 3.)

Problem 6

Statement

Each grid square represents 1 square unit. What is the exact side length of the shaded square?
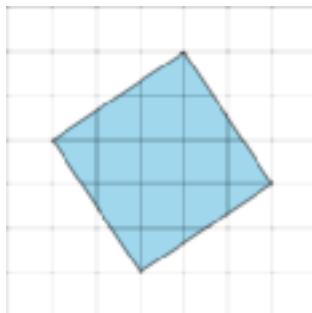
Solution

$\sqrt{13}$ units

(From Unit 8, Lesson 2.)

Problem 7

Statement

For each pair of numbers, which of the two numbers is larger? Estimate how many times larger.

    a. $0.37 \cdot 10^6$ and $700 \cdot 10^4$

    b. $4.87 \cdot 10^4$ and $15 \cdot 10^5$

    c. $500,000$ and $2.3 \cdot 10^8$

Solution

    a. $700 \cdot 10^4$, about 20 times larger

   b. $15 \cdot 10^5$, about 30 times larger

    c. $2.3 \cdot 10^8$, about 500 times larger

(From Unit 7, Lesson 10.)

Problem 8

Statement

The scatter plot shows the heights (in inches) and three-point percentages for different basketball players last season.
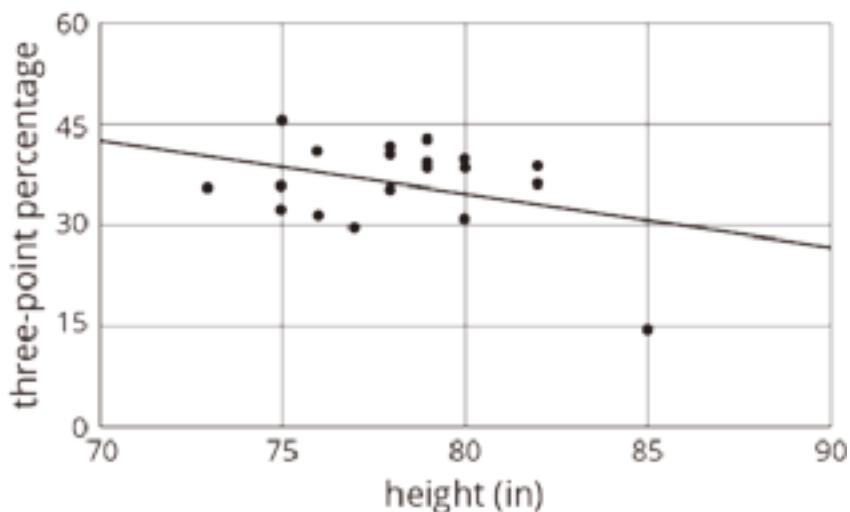
a. Circle any data points that appear to be outliers.

b. Compare any outliers to the values predicted by the model.

Solution

a. The point at (85, 14) is an outlier.

b. This point represents a player who had a significantly worse (by about 15% of the attempts) three-point percentage than the model predicts for his height.

(From Unit 6, Lesson 4.)

Section: The Pythagorean Theorem

Lesson 6: Finding Side Lengths of Triangles

Goals

- Comprehend the term "Pythagorean Theorem" (in written and spoken language) as the equation $a^2 + b^2 = c^2$ where a and b are the lengths of the legs and c is the length of the hypotenuse of a right triangle.

- Describe (orally) patterns in the relationships between the side lengths of triangles.

- Determine the exact side lengths of a triangle in a coordinate grid and express them (in writing) using square root notation.

Learning Targets

- I can explain what the Pythagorean Theorem says.

Lesson Narrative

This is the first of three lessons in which students investigate relationships between the side lengths of right and non-right triangles leading to the **Pythagorean Theorem**.

In the warm-up for this lesson, students notice and wonder about 4 triangles. While there is a lot to notice, one important aspect is whether the triangle is a right triangle or not. This primes them to notice patterns of right and non-right triangles in the other activities in the lesson. In the next two activities, students systematically look at the side lengths of right and non-right triangles for patterns (MP8). By the end of this lesson, they see that for right triangles with **legs** a and b and **hypotenuse** c, the side lengths are related by $a^2 + b^2 = c^2$. In the next lesson they will prove the Pythagorean Theorem.

Alignments

Building On

- 5.G.B.4: Classify two-dimensional figures in a hierarchy based on properties.

- 7.G.A: Draw, construct, and describe geometrical figures and describe the relationships between them.

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Addressing

- 8.G.B: Understand and apply the Pythagorean Theorem.

- 8.G.B.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Building Towards

- 8.G.B: Understand and apply the Pythagorean Theorem.

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

Instructional Routines

- MLR2: Collect and Display

- MLR7: Compare and Connect

- Which One Doesn't Belong?

Student Learning Goals

Let's find triangle side lengths.

6.1 Which One Doesn't Belong: Triangles

Warm Up: 5 minutes
In this warm-up, students compare four triangles. To give all students access the activity, each triangle has one obvious reason it does not belong. One key thing for them to notice is whether the triangle is a right triangle or not.

Building On

- 5.G.B.4

- 7.G.A

- 8.EE.A.2

Building Towards

- 8.G.B.6

Instructional Routines

- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4. Display the image of the four triangles for all to see. Ask students to indicate when they have noticed one triangle that does not belong and can explain why. Give students 1 minute of quiet think time and then time to share their thinking with their group. After everyone has conferred in groups, ask the group to offer at least one reason *each* figure doesn't belong.

Student Task Statement

Which triangle doesn't belong?
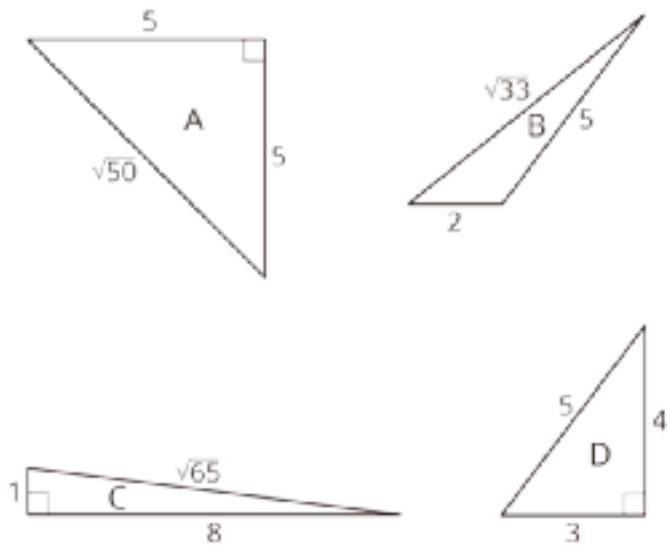
Student Response

Answers vary. Sample responses:

A is the only one that is upside down (the base is not on the bottom). Or is the only one that is isosceles.

B is the only one that isn't a right triangle.

C is the only long skinny one. Or it is the only one that doesn't have a side length of 5.

D is the only one where all three side lengths are whole numbers.

Activity Synthesis

Ask each group to share one reason why a particular triangle does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given make sense.

If no student brings up the fact that Triangle B is the only one that is not a right triangle, be sure to point that out.

6.2 A Table of Triangles

15 minutes

In this activity, students calculate the side lengths of the triangles by both drawing in tilted squares and reasoning about segments that must be congruent to segments whose lengths are known.

Students then record both the side length and the area of the squares in tables and look for patterns. The purpose of this task is for students to think about the relationships between the squares of the side lengths of triangles as a lead up to the Pythagorean Theorem at the end of this lesson.

Note that students do not have to draw squares to find every side length. Some squares are intentionally positioned so that students won't be able to draw squares and must find other ways to find the side lengths. Some segments are congruent to others whose lengths are already known.

Building Towards

- 8.G.B

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 2–3. Display the image of the triangle on a grid for all to see and ask students to consider how they would find the value of each of the side lengths of the triangle.
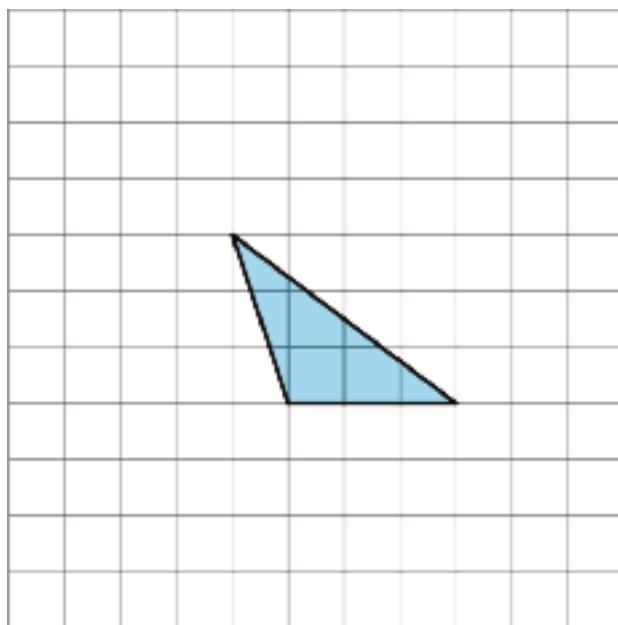
After 1–2 minutes of quiet think time, ask partners to discuss their strategies and then calculate the values. Select 2–3 groups to share their strategies and the values for the side lengths they found ($\sqrt{9} = 3$, $\sqrt{10}$, $\sqrt{25} = 5$). Next, show the same image but with three squares drawn in, each using one of the sides of the triangle as a side length.
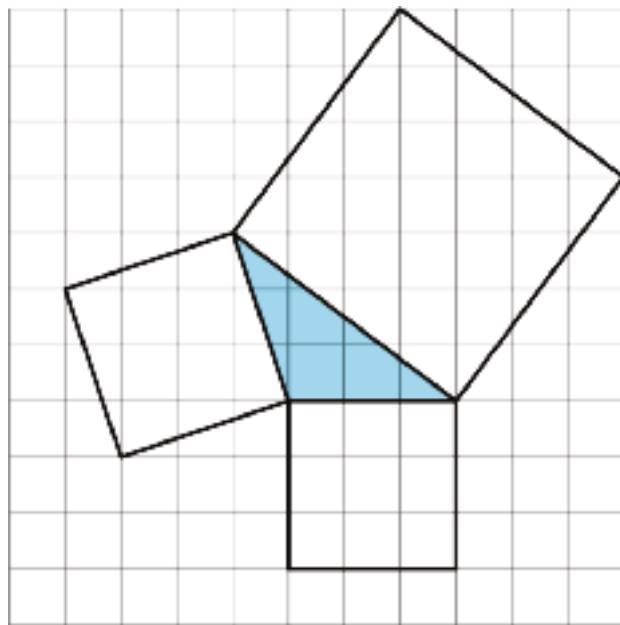
This directly reflects work students have done previously for finding the length of a diagonal on a grid. Students may point out that for the side that is not diagonal, the square is not needed. This is true, but, if no student points it out, note that $3 = \sqrt{9}$, and so the strategy of drawing in a square still works.

Tell students they will use their strategies to determine the side lengths of several triangles in the activity. Alert them to the fact that it's possible to figure out some of the side lengths without having to draw a square. Encourage groups to divide up the work completing the tables and discuss strategies to find the rest of the unknown side lengths.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, reference examples from the previous lessons on finding the length of a diagonal of a grid by drawing squares to provide an entry point into this activity.
Supports accessibility for: Social-emotional skills; Conceptual processing

Conversing, Reading: MLR2 Collect and Display. As students work in groups on the task, circulate and listen to as they discuss what they notice about the values in the table for Triangles E and Q that does not apply to the other triangles. Write down the words and phrases students use on a visual display. As students review the language collected, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as: "The values of a and b add up to c." can be restated as "The sum of a^2 and b^2 is c^2. Encourage students to refer back to the visual display during whole-class discussions throughout the lesson and unit. This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
*Design Principle(s): Support sense-making; Maximize meta-awareness*

Anticipated Misconceptions

Some students may use the language hypotenuse and legs for all of the triangles in the activity. If you hear this, remind students that those words only apply to right triangles.

Student Task Statement

1. Complete the tables for these three triangles:
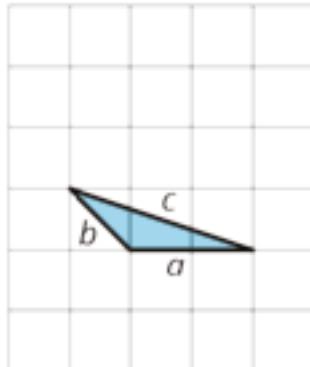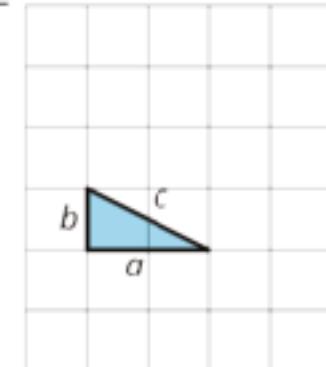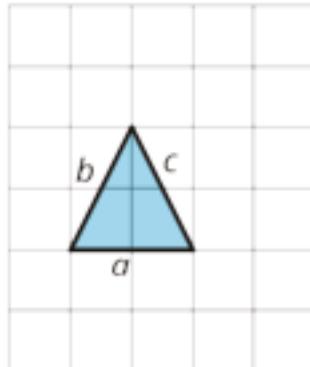
triangle	a	b	c
D			
E			
F			

| triangle	a^2	b^2	c^2
D			
E			
F			

2. What do you notice about the values in the table for Triangle E but not for Triangles D and F?

3. Complete the tables for these three more triangles:

P

Q

R
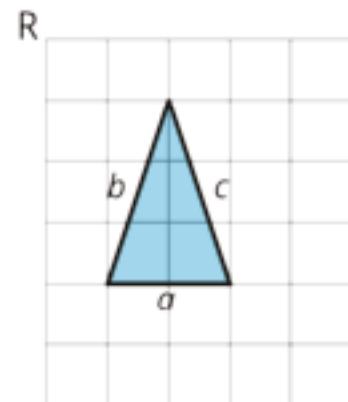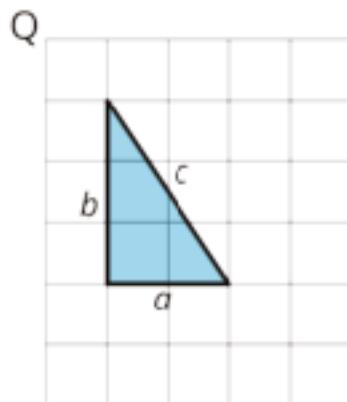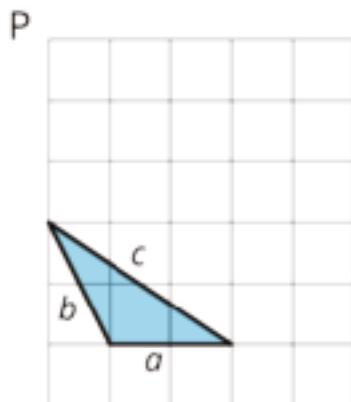
triangle	a	b	c
P			
Q			
R			

triangle	a^2	b^2	c^2
P			
Q			
R			

4. What do you notice about the values in the table for Triangle Q but not for Triangles P and R?

5. What do Triangle E and Triangle Q have in common?

Student Response

Answers vary. Sample responses:

1.

	a	b	c
triangle D	2	$\sqrt{2}$	$\sqrt{10}$
triangle E	2	1	$\sqrt{5}$
triangle F	2	$\sqrt{5}$	$\sqrt{5}$

	a^2	b^2	c^2
triangle D	4	2	10
triangle E	4	1	5
triangle F	4	5	5

2. The sum of $a^2 = 4$ and $b^2 = 1$ equals $c^2 = 5$.

3.

	a	b	c
triangle P	2	$\sqrt{5}$	$\sqrt{13}$
triangle Q	2	3	$\sqrt{13}$
triangle R	2	$\sqrt{10}$	$\sqrt{10}$

	a^2	b^2	c^2
triangle P	4	5	13
triangle Q	4	9	13
triangle R	4	10	10

4. The sum of $a^2 = 4$ and $b^2 = 9$ equals $c^2 = 13$.

5. Triangle E and Triangle Q are both right triangles.

Unit 8 Lesson 6

Activity Synthesis

Invite groups to share their responses to the activity and what they noticed about the relationships between specific triangles. Hopefully, someone noticed that $a^2 + b^2 = c^2$ for triangles E and Q and someone else noticed they are right triangles. If so, ask students if any of the other triangles are right triangles (they are not). If students do not see these patterns, don't give it away. Instead, tell students that we are going to look at more triangles to find a pattern.

6.3 Meet the Pythagorean Theorem

10 minutes

In this task, students can use squares or count grid units to find side lengths and check whether the Pythagorean identity $a^2 + b^2 = c^2$ holds or not. If students don't make the connection that it works for the two right triangles but not the other one, this should be brought to their attention. In the synthesis of this activity or the lesson synthesis, the teacher formally states the Pythagorean Theorem and lets students know they will prove it in the next lesson.

Addressing

- 8.G.B

**Instructional Routines**

- MLR7: Compare and Connect

Launch

Arrange students in groups of 2. Give students 4 minutes of quiet work time followed by partner and then whole-class discussions.

> **Student Task Statement**
>
> 1. Find the missing side lengths. Be prepared to explain your reasoning.
>
> 2. For which triangles does $a^2 + b^2 = c^2$?
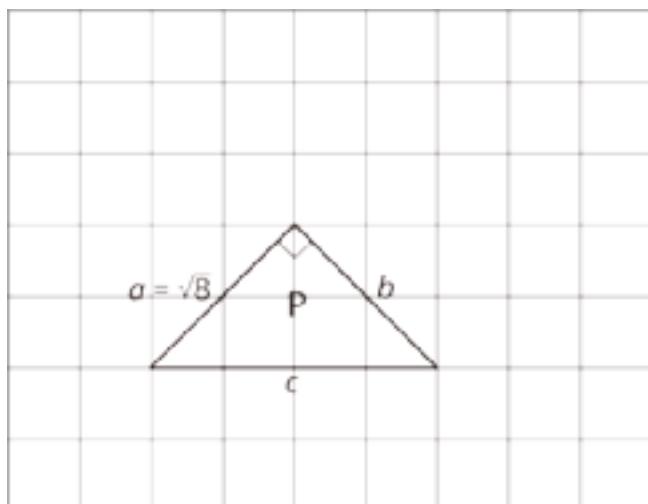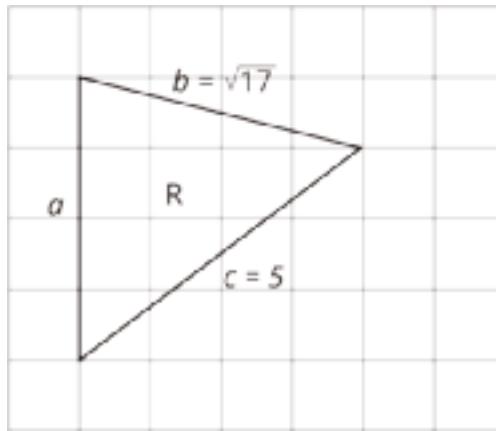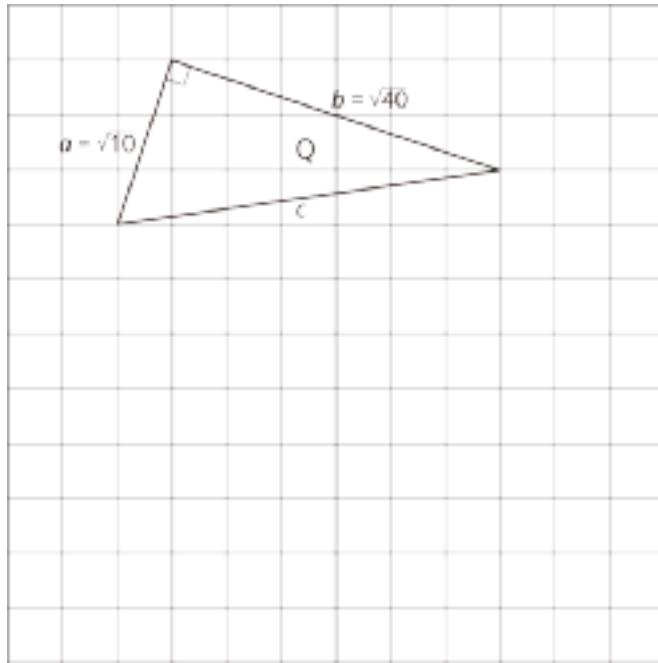
Student Response

1. For triangle P: $c = 4$, which we can see by counting. $b = \sqrt{8}$, which we can see by the fact that it is an isosceles triangle or by drawing a square on the side and finding its area.

   For triangle Q: $c = \sqrt{50}$ which we can see by drawing a square on the side and finding its area.

   For triangle R: $a = 4$ which we can see by counting.

2. $a^2 + b^2 = c^2$ for triangles P and Q, but not triangle R.

If the four shaded triangles in the figure are congruent right triangles, does the inner quadrilateral have to be a square? Explain how you know.
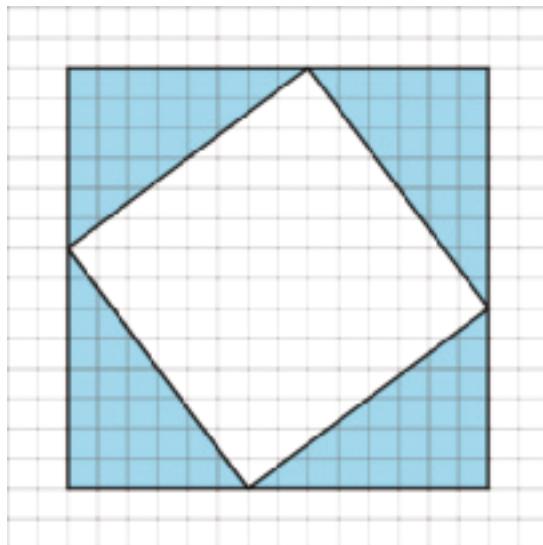
Student Response

Yes. To prove the inner quadrilateral is a square, we need to show that it has 4 sides of equal length and 4 right angles.

Equal length sides: Since the four triangles are congruent, all four hypotenuses are the same length, so all four sides of the inner quadrilateral have equal length.

Right angles: Each triangle has a 90 degree angle and two others. Call the other angles x and y and label them on all of the congruent triangles. Since the angles in a triangle sum to 180 degrees, we know $x + y + 90 = 180$. There is also straight angle consisting of x, y, and any angle from the inner quadrilateral, which we'll call z. Now $x + y + z = 180$ and $x + y + 90 = 180$, so $z = 90$.

Activity Synthesis

Ask selected students to share their reasoning. Make sure the class comes to an agreement. Then tell students that the **Pythagorean Theorem** says:

If a, b, and c are the sides of a right triangle, where c is the **hypotenuse**, then

$$a^2 + b^2 = c^2$$

It is important for students to understand that it only works for *right* triangles. Tell them we will prove that this is always true in the next lesson.

Lesson Synthesis

In this lesson we looked at the relationship between the side lengths of different triangles. We saw a pattern for right triangles that did not hold for non-right triangles. Ask students:

- "What was the relationship we saw for the right triangles we looked at?" (The sum of the squares of the legs was equal to the square of the hypotenuse.)

If time allows, draw a few right triangles with labeled side lengths marked a, b, and c and display for all to see. Ask students to check that the Pythagorean Theorem is true for these triangles. As students work, check to make sure they understand that when $a^2 + b^2$, a and b need to be squared first, and then added. Some students may confuse exponents with multiplying by 2, and assume they can "factor" the expression.

In the next lesson, we will actually prove that what we saw in these examples is *always* true for right triangles.

6.4 Does a Squared Plus b Squared Equal c Squared?

Cool Down: 5 minutes
Addressing

- 8.G.B.7

Student Task Statement

For each of the following triangles, determine if $a^2 + b^2 = c^2$, where a, b, and c are side lengths of the triangle. Explain how you know.
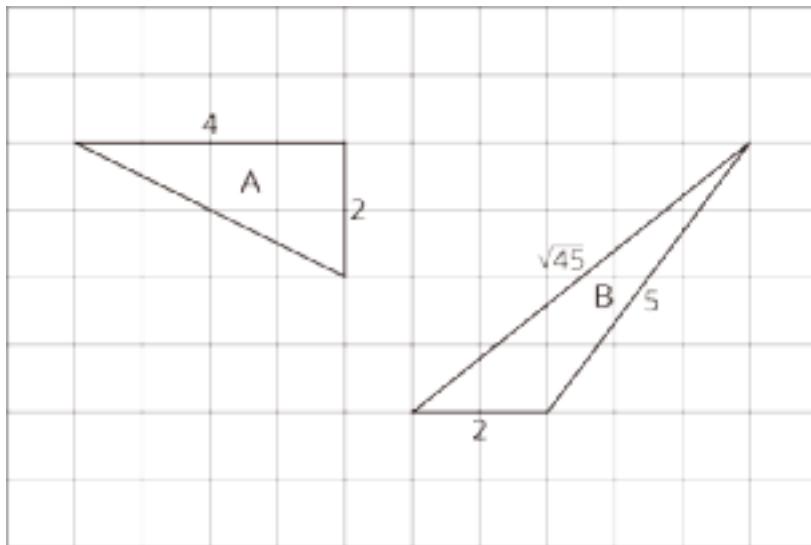
Student Response

It is true for A because it is a right triangle. You can also find the third side length by constructing a square on it and checking.

It is not true for B. You can see this by squaring the side lengths.

Student Lesson Summary

A *right triangle* is a triangle with a right angle. In a right triangle, the side opposite the right angle is called the **hypotenuse**, and the two other sides are called its **legs**. Here are some right triangles with the hypotenuse and legs labeled:
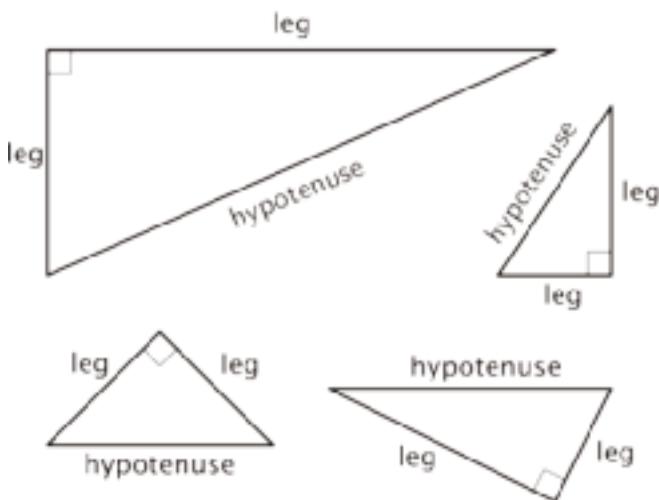
We often use the letters a and b to represent the lengths of the shorter sides of a triangle and c to represent the length of the longest side of a right triangle. If the triangle is a right triangle, then a and b are used to represent the lengths of the legs, and c is used to represent the length of the hypotenuse (since the hypotenuse is always the longest side of a right triangle). For example, in this right triangle, $a = \sqrt{20}$, $b = \sqrt{5}$, and $c = 5$.
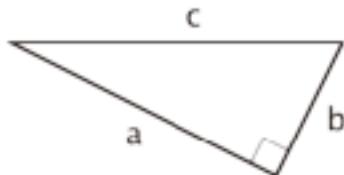
Here are some right triangles:
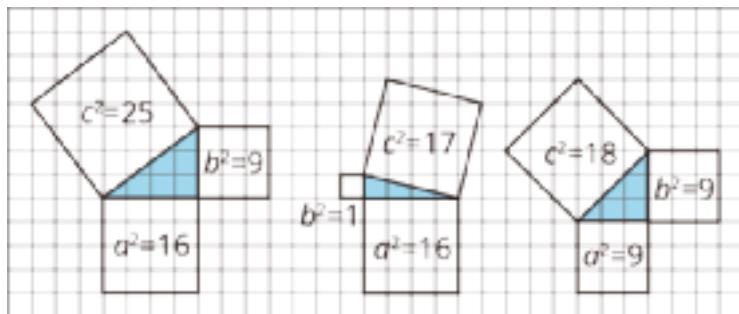
Notice that for these examples of right triangles, the square of the hypotenuse is equal to the sum of the squares of the legs. In the first right triangle in the diagram, $9 + 16 = 25$, in the second, $1 + 16 = 17$, and in the third, $9 + 9 = 18$. Expressed another way, we have
$$a^2 + b^2 = c^2$$
This is a property of all right triangles, not just these examples, and is often known as the **Pythagorean Theorem**. The name comes from a mathematician named Pythagoras who lived in ancient Greece around 2,500 BCE, but this property of right triangles was also discovered independently by mathematicians in other ancient cultures including Babylon, India, and China. In China, a name for the same relationship is the Shang Gao Theorem. In future

Unit 8 Lesson 6

lessons, you will learn some ways to explain why the Pythagorean Theorem is true for *any* right triangle.

It is important to note that this relationship does not hold for *all* triangles. Here are some triangles that are not right triangles, and notice that the lengths of their sides do not have the special relationship $a^2 + b^2 = c^2$. That is, $16 + 10$ does not equal 18, and $2 + 10$ does not equal 16.
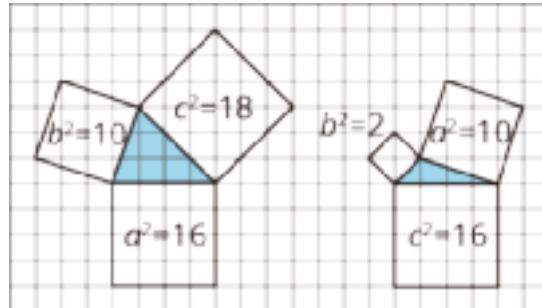
Glossary

- hypotenuse

- legs

- Pythagorean Theorem

Lesson 6 Practice Problems

Problem 1

Statement

Here is a diagram of an acute triangle and three squares.
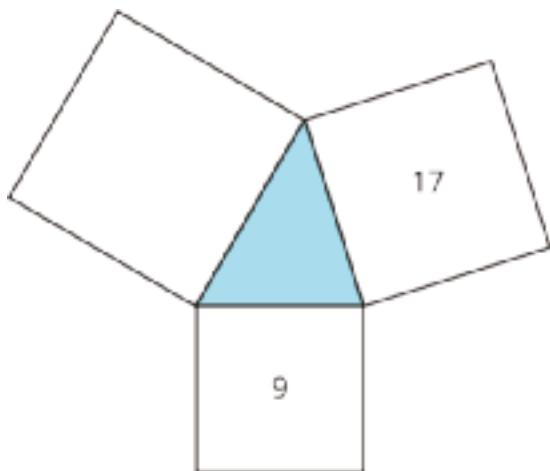
Priya says the area of the large unmarked square is 26 square units because $9 + 17 = 26$. Do you agree? Explain your reasoning.

Solution

No, I disagree. Priya's pattern only works for right triangles, and this is an acute triangle.

Problem 2

Statement

m, p, and z represent the lengths of the three sides of this right triangle.
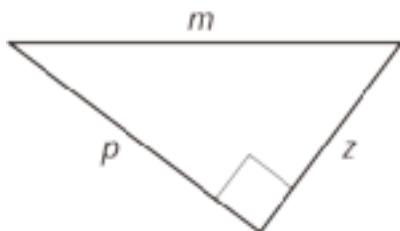
Select **all** the equations that represent the relationship between m, p, and z.

A. $m^2 + p^2 = z^2$

B. $m^2 = p^2 + z^2$

C. $m^2 = z^2 + p^2$

D. $p^2 + m^2 = z^2$

E. $z^2 + p^2 = m^2$

F. $p^2 + z^2 = m^2$

Solution

["B", "C", "E", "F"]

Problem 3

Statement

The lengths of the three sides are given for several right triangles. For each, write an equation that expresses the relationship between the lengths of the three sides.

a. 10, 6, 8

b. $\sqrt{5}, \sqrt{3}, \sqrt{8}$

c. 5, $\sqrt{5}, \sqrt{30}$

d. 1, $\sqrt{37}, 6$

e. 3, $\sqrt{2}, \sqrt{7}$

Solution

a. $6^2 + 8^2 = 10^2$

b. $\sqrt{5}^2 + \sqrt{3}^2 = \sqrt{8}^2$

c. $5^2 + \sqrt{5}^2 = \sqrt{30}^2$

d. $1^2 + 6^2 = \sqrt{37}^2$

e. $\sqrt{2}^2 + \sqrt{7}^2 = 3^2$

Problem 4

Statement

Order the following expressions from least to greatest.

$25 \div 10$ $\qquad$ $250{,}000 \div 1{,}000$ $\qquad$ $2.5 \div 1{,}000$ $\qquad$ $0.025 \div 1$

Solution

- $2.5 \div 1{,}000$
- $0.025 \div 1$
- $25 \div 10$
- $250{,}000 \div 1{,}000$

(From Unit 4, Lesson 1.)

Problem 5

Statement

Which is the best explanation for why $-\sqrt{10}$ is irrational?

A. $-\sqrt{10}$ is irrational because it is not rational.

B. $-\sqrt{10}$ is irrational because it is less than zero.

C. $-\sqrt{10}$ is irrational because it is not a whole number.

D. $-\sqrt{10}$ is irrational because if I put $-\sqrt{10}$ into a calculator, I get -3.16227766, which does not make a repeating pattern.

Solution

D
(From Unit 8, Lesson 3.)

Problem 6

Statement

A teacher tells her students she is just over 1 and $\frac{1}{2}$ billion seconds old.

a. Write her age in seconds using scientific notation.

b. What is a more reasonable unit of measurement for this situation?

    c. How old is she when you use a more reasonable unit of measurement?

Solution

a. 1.5×10^9

b. Years

c. She is about 48 years old. There are 31,536,000 seconds in a year. $1.5 \times 10^9 \div 31{,}536{,}000$ is about 47.6.

(From Unit 7, Lesson 15.)

Lesson 7: A Proof of the Pythagorean Theorem

Goals

- Calculate an unknown side length of a right triangle using the Pythagorean Theorem, and explain (orally) the reasoning.

- Explain (orally) an area-based algebraic proof of the Pythagorean Theorem.

Learning Targets

- I can explain why the Pythagorean Theorem is true.

Lesson Narrative

In the warm-up of this lesson, students study a diagram they will use to prove the Pythagorean Theorem. In the first activity they prove the Pythagorean Theorem using the diagram. Then they apply the Pythagorean Theorem in the next activity. The final activity before the cool down is an optional look at a transformational proof of the Pythagorean Theorem.

Alignments

Addressing

- 8.G.B: Understand and apply the Pythagorean Theorem.

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

- 8.G.B.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Building Towards

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

Instructional Routines

- MLR2: Collect and Display

- MLR3: Clarify, Critique, Correct

- MLR5: Co-Craft Questions

- Notice and Wonder

- Think Pair Share

Required Materials

Copies of Instructional

master

Required Preparation

If you choose to do the optional activity, you will need the 5 cut-out shapes from the Making Squares Instructional master used in the first lesson of this unit—1 set of 5 for every 2 students. You will also need copies of the A Transformational Proof Instructional master—1 copy for every 2 students.

Student Learning Goals

Let's prove the Pythagorean Theorem.

7.1 Notice and Wonder: A Square and Four Triangles

Warm Up: 5 minutes

The purpose of this warm-up is to give students a chance to study a diagram that they will need to understand for an upcoming proof of the Pythagorean Theorem. The construction depends on the triangles being right triangles, so students get to contrast it with a similarly constructed figure with non-right triangles. In that case, the composite figure is not a square.

Building Towards

- 8.G.B.6

Instructional Routines

- Notice and Wonder

Launch

Arrange students in groups of 2. Display the diagram for all to see. Give students 1 minute of quiet work time to identify at least one thing they notice and at least one thing they wonder about the diagram. Ask students to give a signal when they have noticed or wondered about something. When the minute is up, give students 1 minute to discuss their observations and questions with their partner. Follow with a whole-class discussion.

Student Task Statement
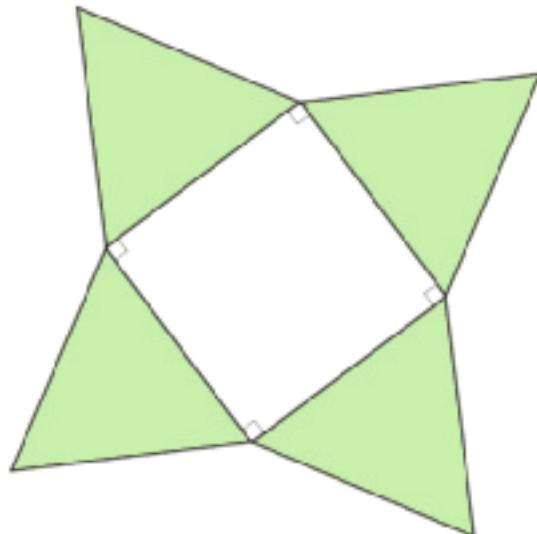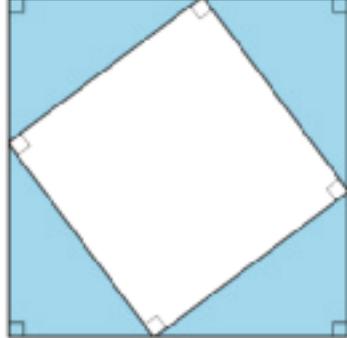
■ What do you notice? What do you wonder?

Student Response

Answers vary. Sample response:

I notice that:

- There are two figures both made up of a square and four triangles.

- The triangles in the left-hand figure are right triangles and the other triangles are not.

- The figure on the left is a square.

- The figure on the right is not a square.

- The small squares in the middle look the same size.

I wonder:

- Are the squares in the middle the same size?

- Are the blue triangles all the same size?

- Are the green triangles all the same size?

- Does it matter if the triangles are right triangles?

- Does it matter if they are equilateral triangles?

- What are these figures going to be used for?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class whether they agree or disagree and to explain alternative ways of thinking, referring back to the images each time.

Tell students that when you take a square and put a congruent right triangle on each side as shown on the left, they form a larger square (they will be able to prove this in high school). But it doesn't work if the triangles are not right triangles. We will use this construction in the next activity.

7.2 Adding Up Areas

15 minutes
The purpose of this activity is for students to work through an area-based algebraic proof of the Pythagorean Theorem (MP1). One of the figures used in this particular proof, G, was first encountered by students at the start of the year during a unit on transformations and again in a recent lesson where they reasoned about finding the area of the triangles.

While there are many proofs of the Pythagorean Theorem similar to the one in this activity, they often rely on $(a + b)^2 = a^2 + 2ab + b^2$, which is material beyond the scope of grade 8. For this proof, students reason about the areas of the two squares with the same dimensions. Each square is divided into smaller regions in different ways and it is by using the equality of the total area of each square that they uncover the Pythagorean Theorem. The extension uses this same division to solve a challenging area composition and decomposition problem.

Addressing

- 8.G.B.6

Instructional Routines

- MLR5: Co-Craft Questions

- Think Pair Share

Launch

Begin by explaining to students how the two figures are constructed. Each figure starts with a square with side length $a + b$.

- Figure F partitions the square into two squares and two rectangles.

- Figure G takes a right triangle with legs a and b and puts one identical copy of it in each corner of the square. The copies touch each other because the short leg of one and the long leg of the one next to it add up to $a + b$, so they fit exactly into a side. So they form a quadrilateral in the middle. We know the quadrilateral is a square because
  - The corners must be 90 degree angles:
    - The two acute angles in each triangle must sum to 90 degrees because the sum of the angles in a triangle is 180 degrees, and the third angle is 90 degrees.

    - The two smaller angles along with one of the corners of the quadrilateral form a straight angle with a measure of 180 degrees, that means that the angle at the corner must also be 90 degrees.

  - All four sides are the same length: they all correspond to a hypotenuse of one of the congruent right triangles.

Arrange students in groups of 2. Give 3 minutes of quiet work time for the first two problems. Ask partners to share their work and come to an agreement on the area of each figure and region before moving on to the last problem. Follow with a whole-class discussion.

Student Task Statement

Both figures shown here are squares with a side length of $a + b$. Notice that the first figure is divided into two squares and two rectangles. The second figure is divided into a square and four right triangles with legs of lengths a and b. Let's call the hypotenuse of these triangles c.
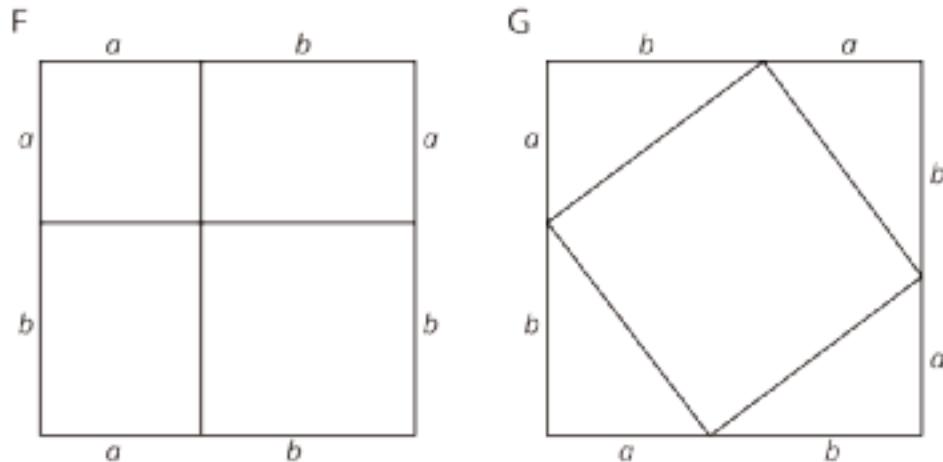
1. What is the total area of each figure?

2. Find the area of each of the 9 smaller regions shown the figures and label them.

3. Add up the area of the four regions in Figure F and set this expression equal to the sum of the areas of the five regions in Figure G. If you rewrite this equation using as few terms as possible, what do you have?

Student Response

1. $(a + b)^2$

2. Figure F: a^2, ab, ab, b^2. Figure G: $\frac{1}{2}ab$, $\frac{1}{2}ab$, $\frac{1}{2}ab$, $\frac{1}{2}ab$, c^2.

3. $a^2 + b^2 = c^2$. The sum of the area of regions in F is $a^2 + 2ab + b^2$, and the sum of the area of regions in G is $4(\frac{1}{2}ab) + c^2 = 2ab + c^2$. Then $a^2 + 2ab + b^2 = 2ab + c^2$ implies $a^2 + b^2 = c^2$.

Are You Ready for More?

Take a 3-4-5 right triangle, add on the squares of the side lengths, and form a hexagon by connecting vertices of the squares as in the image. What is the area of this hexagon?
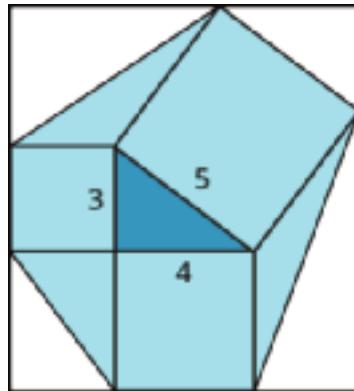
Student Response

We can find the area of the shaded region if we can find the area of the surrounding rectangle and subtract the area of the four unshaded triangles. To find the dimensions of the rectangle, add in three more copies of the right triangle as in our proof of the Pythagorean Theorem. By writing down all the side lengths (the image on the left), we can see that the width of the rectangle is 10 and its height is 11, and so has an area of 110.
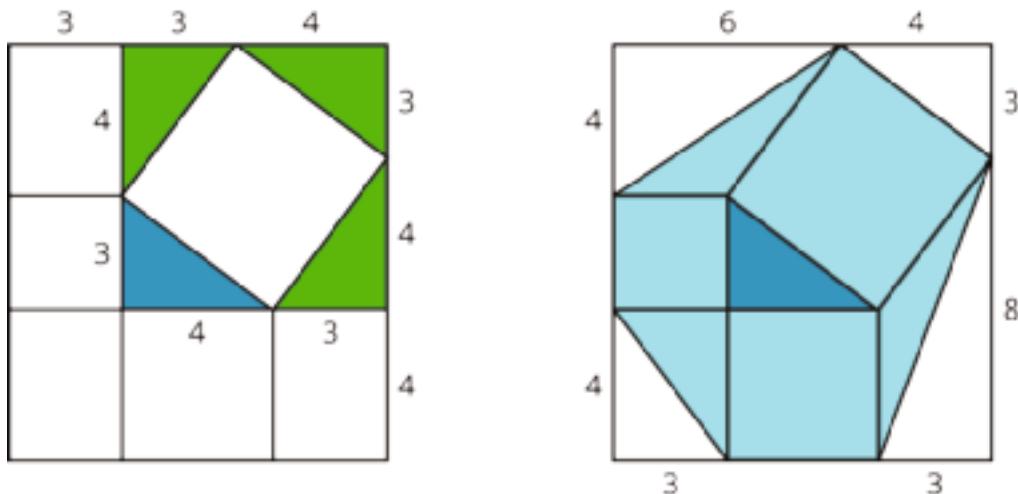
Returning to the original image (the image on the right), we see that the four triangles we have to subtract have areas 12, 6, 6, and 12. The area of the shaded region is $110 - 12 - 6 - 6 - 12 = 74$.

Activity Synthesis

Begin the discussion by selecting 2–3 groups to share their work and conclusion for the third question. Make sure the last group presenting concludes with $a^2 + b^2 = c^2$ or something close enough that the class can get there with a little prompting. For example, if groups are stuck with the equation looking something like $a^2 + ab + b^2 + ab = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$, encourage them to try and combine like terms and remove any quantities both sides have in common on each side.

After groups have shared, ask students how they see the regions in each figure matching the regions in the other figure. For example, since the two small squares in Figure F match the one large square in Figure G, how do the rectangles and triangles match? After some quiet think time, select 1–2 students to explain how they see it. (The area of the two rectangles is the same as the area of two of the triangles since, if I put two of the triangles together, I get a rectangle that is a wide and b long.) Show students an image with the diagonals added in, such as the one shown here, to help make the connection between the two figures clearer.
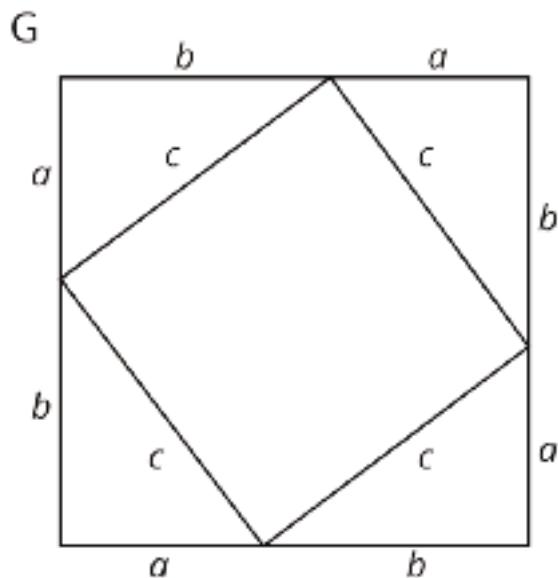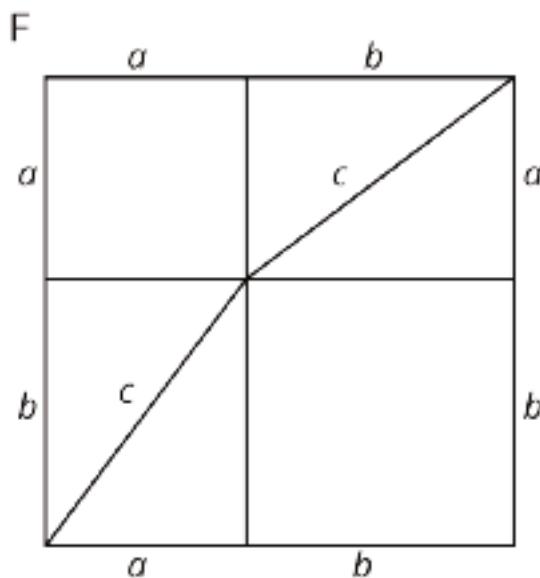
Note how these figures can be made for any right triangle with legs a and b and hypotenuse c.

7.3 Let's Take it for a Spin

10 minutes
Before this lesson, students could only find the length of a segment between the intersection of grid lines in a square grid by computing the area of a related square. The Pythagorean Theorem makes it possible to find the length of any segment that is a side of a right triangle.

Addressing

- 8.G.B.7

Instructional Routines

- MLR3: Clarify, Critique, Correct

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 3 minutes of quiet work time followed by partner and then whole-class discussions.

Student Task Statement

Find the unknown side lengths in these right triangles.

Student Response

$$x = \sqrt{29}$$

$$y = \sqrt{8}$$

Activity Synthesis

Invite a few students to share their reasoning with the class for each unknown side length. As students share, record their steps for all to see, showing clearly the initial setup with $a^2 + b^2 = c^2$.

Reading, Writing, Speaking: MLR3 Clarify, Critique, Correct. Before students share their methods for finding the side length of a right triangle, present an incorrect solution based on a common error you observe in the class. For example, "I know that $a = \sqrt{8}$, $b = 4$, and $c = y$, so when I use the Pythagorean Theorem, I get the equation $(\sqrt{8})^2 + 4^2 = y^2$. This equation simplifies to $8 + 16 = y^2$. When I solve for y, I get $y = \sqrt{24}$." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who clarify the meaning of each term in the equation $a^2 + b^2 = c^2$. In the Pythagorean Theorem, a^2 and b^2 represent the square of the legs of the right triangle, whereas c^2 represents the square of the hypotenuse. This routine will engage students in meta-awareness as they critique and correct a common error when applying the Pythagorean Theorem.
Design Principles(s): Cultivate conversation; Maximize meta-awareness

7.4 A Transformational Proof

Optional: 15 minutes (there is a digital version of this activity)
In this activity, students explore a transformations-based proof of the Pythagorean Theorem, calling back to their work with transformations earlier in the year. Since this proof is not one students are expected to derive on their own, the focus of this activity is on understanding the steps and why they are possible from a transformations perspective.

Listen for students using precise language when describing their transformations.

The digital activity demonstrates the same proof in a slightly different way. Students have the opportunity to explore three different right triangles in the applets.

Addressing

- 8.G.B.6

Instructional Routines

- MLR2: Collect and Display

Launch

Tell students that today we are going to think about how to use transformations to show the relationship between the sides of right triangles. Briefly review the language of rigid transformations (translation, rotation, reflection) with students using the 5 pre-cut pieces from the Making Squares Instructional master from the first lesson in this unit.

Arrange students in groups of 2. Before students begin, remind them that if a problem asks them to explain, then they are expected to use precise language when describing the transformation of the shapes. Distribute pre-cut shapes from the Making Squares Instructional master and the A Transformational Proof Instructional master to each group. Leave 3–4 minutes for a whole-class discussion.

For students using the digital activity, there are no paper copies needed. Have students work in groups of two with the digital applet to explore the relationship between the squares and the Pythagorean Theorem.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organization; Attention

Access for English Language Learners

Conversing, Reading: MLR2 Collect and Display. As students work in pairs on the task, circulate and listen as they discuss their observations about the relationship between the squares and the Pythagorean Theorem. Write down the words and phrases students use on a visual display. As students review the language collected in the visual display, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as: "The small squares add up to the big square." can be restated as "The sum of the areas of the small squares, $a^2 + b^2$, is equal to the area of the large square, c^2.". Encourage students to refer back to the visual display during whole-class discussions throughout the lesson and unit. This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximize meta-awareness

Student Task Statement

Your teacher will give your group a sheet with 4 figures and a set of 5 cut out shapes labeled D, E, F, G, and H.

1. Arrange the 5 cut out shapes to fit inside Figure 1. Check to see that the pieces also fit in the two smaller squares in Figure 4.

2. Explain how you can transform the pieces arranged in Figure 1 to make an exact copy of Figure 2.

3. Explain how you can transform the pieces arranged in Figure 2 to make an exact copy of Figure 3.

4. Check to see that Figure 3 is congruent to the large square in Figure 4.

5. If the right triangle in Figure 4 has legs a and b and hypotenuse c, what have you just demonstrated to be true?

Student Response

1. No response necessary.

2. Rotate Triangle G 90 degrees counterclockwise about it's uppermost point.

3. Rotate the triangle created by shapes G and H 90 degrees clockwise about the uppermost point of H.

4. No response necessary.

5. Since these shapes can be arranged to be either the area $a^2 + b^2$ of the two smaller squares or the area c^2 of the larger square, we have demonstrated that for a right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$ is true.

Activity Synthesis

Select previously identified groups to share their explanations for each transformation and their conclusion to the last problem. If possible, have them show each transformation for all students to see.

An important takeaway for this activity is that this proof can be generalized to any right triangle. To help students see why, ask them to consider how the diagonal lines in Figure 1 were created. Give 1–2 minutes for partners to discuss and then select 2–3 groups to share their ideas. Hopefully, at least one group noticed that the diagonals create two congruent right triangles with sides of length a, b, and c—the same as the original right triangle. This means that this process could be duplicated to show that for any right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$ is true.

Lesson Synthesis

Review the proof of the Pythagorean Theorem.

7.5 When is it True?

Cool Down: 5 minutes
Addressing

- 8.G.B

> **Student Task Statement**
>
> The Pythagorean Theorem is

1. True for all triangles

2. True for all right triangles

3. True for some right triangles

4. Never true

Student Response

True for all right triangles

Student Lesson Summary

The figures shown here can be used to see why the Pythagorean Theorem is true. Both large squares have the same area, but they are broken up in different ways. (Can you see where the triangles in Square G are located in Square F? What does that mean about the smaller squares in F and H?) When the sum of the four areas in Square F are set equal to the sum of the 5 areas in Square G, the result is $a^2 + b^2 = c^2$, where c is the hypotenuse of the triangles in Square G and also the side length of the square in the middle. Give it a try!
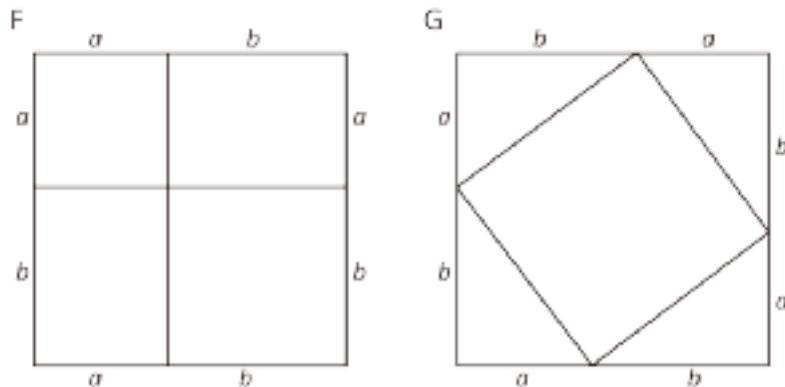
This is true for any right triangle. If the legs are a and b and the hypotenuse is c, then $a^2 + b^2 = c^2$. This property can be used any time we can make a right triangle. For example, to find the length of this line segment:
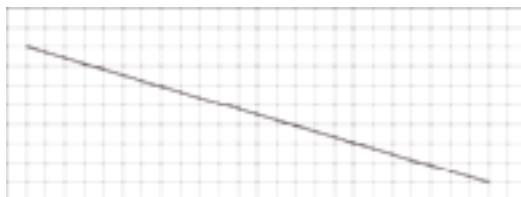
The grid can be used to create a right triangle, where the line segment is the hypotenuse and the legs measure 24 units and 7 units:
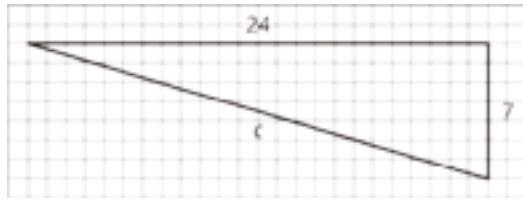
Since this is a right triangle, $24^2 + 7^2 = c^2$. The solution to this equation (and the length of the line segment) is $c = 25$.

Lesson 7 Practice Problems
Problem 1
Statement
a. Find the lengths of the unlabeled sides.
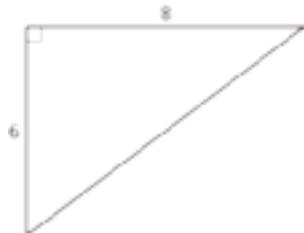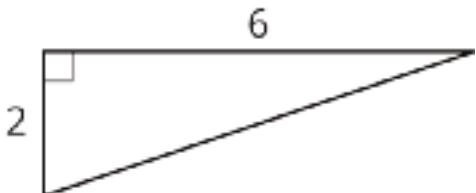
b. One segment is n units long and the other is p units long. Find the value of n and p. (Each small grid square is 1 square unit.)
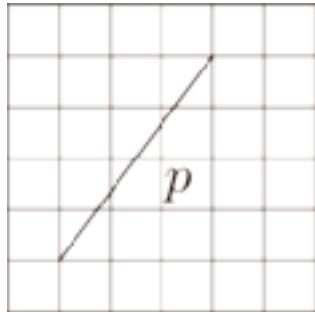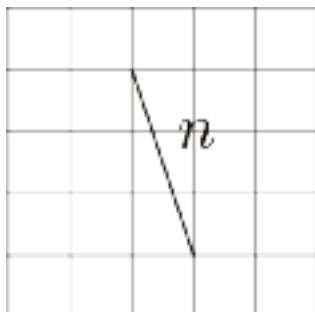
Solution
a.

    i. $\sqrt{40}$, approximately 6.3

    ii. $\sqrt{100}$, exactly 10

b.

    i. $\sqrt{10}$ because $1^2 + 3^2 = 10$

    ii. $\sqrt{25}$ (or 5) because $3^2 + 4^2 = 25$

Problem 2

Statement

Use the areas of the two identical squares to explain why $5^2 + 12^2 = 13^2$ without doing any calculations.
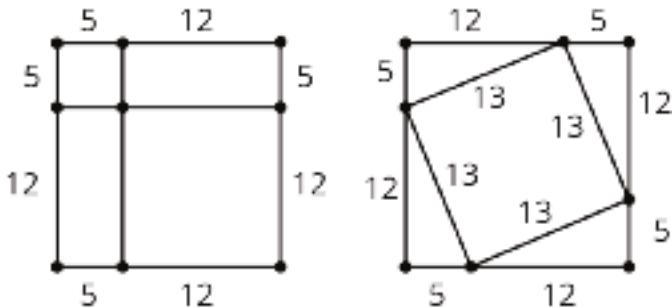
Solution

Answers vary. Sample explanation: The areas of the two large squares are the same since they are both 17 by 17 units. The area of the two rectangles on the left square are the same as the area of the 4 triangles in the right square (each pair of triangles makes a rectangle). So the area of the two smaller squares on the left must be the same as the area of the smaller square on the right. This means $5^2 + 12^2 = 13^2$.

Problem 3

Statement

Each number is between which two consecutive integers?

   a. $\sqrt{10}$

   b. $\sqrt{54}$

   c. $\sqrt{18}$

   d. $\sqrt{99}$

   e. $\sqrt{41}$

Solution

   a. 3 and 4

   b. 7 and 8

   c. 4 and 5

d. 9 and 10

e. 6 and 7

Problem 4
Statement

a. Give an example of a rational number, and explain how you know it is rational.

b. Give three examples of irrational numbers.

Solution

a. Answers vary. Sample response: $\frac{2}{3}$ is a rational number because rational numbers are fractions and their opposites and $\frac{2}{3}$ is a fraction.

b. Answers vary. Sample response: $\sqrt{2}$, $-\sqrt{12}$, $\sqrt{1.5}$

Problem 5
Statement

Write each expression as a single power of 10.

a. $10^5 \cdot 10^0$

b. $\frac{10^9}{10^0}$

Solution

a. 10^5

b. 10^9

Problem 6
Statement

Andre is ordering ribbon to make decorations for a school event. He needs a total of exactly 50.25 meters of blue and green ribbon. Andre needs 50% more blue ribbon than green ribbon for the basic design, plus an extra 6.5 meters of blue ribbon for accents. How much of each color of ribbon does Andre need to order?

Solution

Andre needs to order 17.5 meters of green ribbon and 32.5 meters of blue ribbon. Strategies vary. Sample strategy: Let b represent the length of blue ribbon and g represent the length of green ribbon. Then $b = 1.5g + 6.25$ and $b + g = 50.25$. Substituting $1.5g + 6.25$ in for b in the second equation gives $(1.5g + 6.25) + g = 50.25$. Solving for g gives $g = 17.5$. Since the two kinds of ribbon must combine to make 50.25 meters, then $b = 50.25 - 17.5$, so $b = 32.5$ meters.

(From Unit 4, Lesson 15.)

Lesson 8: Finding Unknown Side Lengths

Goals

- Calculate unknown side lengths of a right triangle by using the Pythagorean Theorem, and explain (orally) the solution method.

- Label the "legs" and "hypotenuse" on a diagram of a right triangle.

Learning Targets

- If I know the lengths of two sides, I can find the length of the third side in a right triangle.

- When I have a right triangle, I can identify which side is the hypotenuse and which sides are the legs.

Lesson Narrative

The purpose of this lesson is to use the Pythagorean Theorem to find unknown side lengths of a right triangle.

Alignments
Addressing

- 8.G.B.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Building Towards

- 8.G.B: Understand and apply the Pythagorean Theorem.

Instructional Routines

- MLR3: Clarify, Critique, Correct

- MLR8: Discussion Supports

- Think Pair Share

- Which One Doesn't Belong?

> **Student Learning Goals**
>
> Let's find missing side lengths of right triangles.

8.1 Which One Doesn't Belong: Equations

Warm Up: 5 minutes
The purpose of this warm-up is to prime students for solving equations that arise while using the Pythagorean Theorem.

Building Towards

- 8.G.B

Instructional Routines

- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4. Display the equations for all to see. Ask students to indicate when they have noticed one that does not belong and can explain why. Give students 1 minute of quiet think time and then time to share their thinking with their small group. In their small groups, tell each student to share their reason why a particular equation does not belong and together find at least one reason each question doesn't belong.

> **Student Task Statement**
>
> Which one doesn't belong?
>
> $3^2 + b^2 = 5^2$
>
> $b^2 = 5^2 - 3^2$
>
> $3^2 + 5^2 = b^2$
>
> $3^2 + 4^2 = 5^2$

Student Response

Answers vary. Sample responses:

$3^2 + b^2 = 5^2$: the only one where b is not isolated

$b^2 = 5^2 - 3^2$: the only one with one term on the left and two terms on the right, the only one with subtraction

$3^2 + 5^2 = b^2$: the only one that is not based on a 3-4-5 Pythagorean triple

$3^2 + 4^2 = 5^2$: the only one with all numbers

Activity Synthesis

Ask each group to share one reason why a particular equation does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given make sense.

8.2 Which One Is the Hypotenuse?

5 minutes
This activity helps students identify the hypotenuse in right triangles in different orientations.

Building Towards

- 8.G.B

Instructional Routines

- MLR8: Discussion Supports

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 1 minute of quiet work time and then have them compare with a partner. Follow with a whole-class discussion.

> **Student Task Statement**
>
> Label all the hypotenuses with c.
>
>
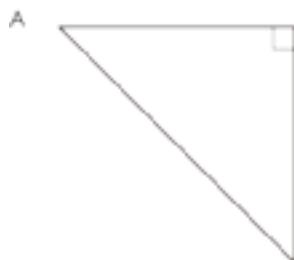
Student Response

All triangles except Triangle B should have a c on the side opposite the right angle. Triangle B is not a right triangle, therefore it does not have a hypotenuse.

Activity Synthesis

Ask students which triangles are right triangles, and then ask them which side is the hypotenuse for each one. Ask, "In a right triangle, does it matter which is a and which is b?" (No.)

8.3 Find the Missing Side Lengths

20 minutes
The purpose of this activity is to give students practice finding missing side lengths in a right triangle using the Pythagorean Theorem.

Addressing

- 8.G.B.7

Instructional Routines

- MLR3: Clarify, Critique, Correct

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 10 minutes of quiet work time and then have them compare with a partner. If partners disagree about any of their answers, ask them to explain their reasoning to one another until they reach agreement. Follow with a whole-class discussion.

Student Task Statement

1. Find c.
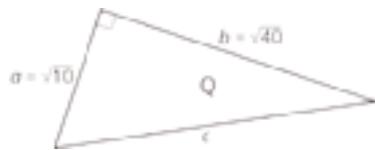
2. Find b.
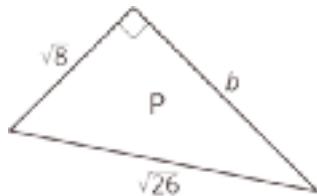
3. A right triangle has sides of length 2.4 cm and 6.5 cm. What is the length of the hypotenuse?

4. A right triangle has a side of length $\frac{1}{4}$ and a hypotenuse of length $\frac{1}{3}$. What is the length of the other side?

5. Find the value of x in the figure.
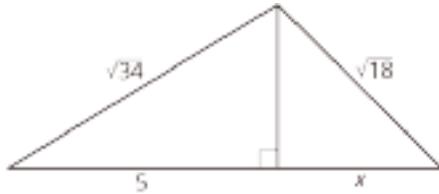
Student Response

1. $\sqrt{50}$

2. $\sqrt{18}$

3. $\sqrt{48.01}$

4. $\sqrt{\frac{7}{144}}$

5. $x = 3$

Are You Ready for More?

The spiral in the figure is made by starting with a right triangle with both legs measuring one unit each. Then a second right triangle is built with one leg measuring one unit, and the other

leg being the hypotenuse of the first triangle. A third right triangle is built on the second triangle's hypotenuse, again with the other leg measuring one unit, and so on.
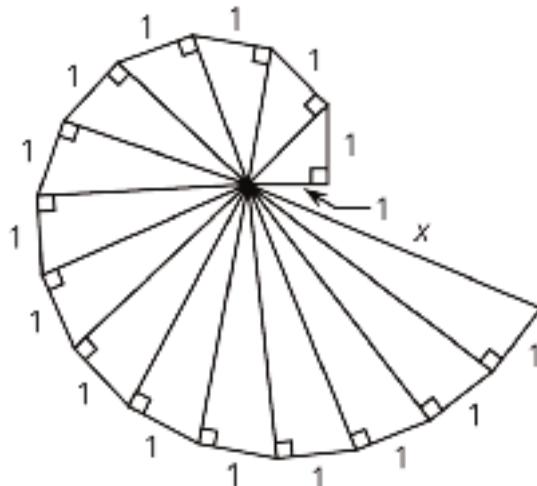
Find the length, x, of the hypotenuse of the last triangle constructed in the figure.

Student Response

We can repeatedly apply the Pythagorean Theorem. The first hypotenuse equals $\sqrt{2}$, since $(\sqrt{2})^2 = 1^2 + 1^2$. The second right triangle has legs 1 and $\sqrt{2}$, so has a hypotenuse of $\sqrt{3}$, since $(\sqrt{3})^2 = (\sqrt{2})^2 + 1^2$. This pattern continues, with the next hypotenuses having length $\sqrt{4}$, the $\sqrt{5}$, etc. By counting until the end, we find that the 15th and last hypotenuse has a length x equal to $\sqrt{16}$, so $x = 4$.

Activity Synthesis

Ask students to share how they found the missing side lengths. If students drew triangles for the two questions that did not have an image, display a few of these for all to see, noting any differences between them. For example, students may have drawn triangles with different orientations or labeled different sides as a and b.

For the last question, ask students to say what they did first to try and solve for x. For example, while many students may have found the length of the unknown altitude first and then used that value to find x, others may have set up the equation $34 - 5^2 = 18 - x^2$.

Point out that when you know two sides of a right triangle, you can always find the third by using the Pythagorean identity $a^2 + b^2 = c^2$. Remind them that it is important to keep track of which side is the hypotenuse.

Reading, Writing, Speaking: MLR3 Clarify, Critique, Correct. Before students share their method for the questions that did not have an image, present an incorrect solution based on a common error related to labeling the sides of a right triangle. For the right triangle with a side of length $\frac{1}{4}$ and a hypotenuse of length $\frac{1}{3}$, draw a right triangle with the legs labeled $\frac{1}{4}$ and $\frac{1}{3}$. Provide an incorrect explanation such as: "I know that $a = \frac{1}{4}$ and $b = \frac{1}{3}$, so when I use the Pythagorean Theorem, I get the equation $(\frac{1}{4})^2 + (\frac{1}{3})^2 = c^2$." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who clarify the meaning of the hypotenuse and identify c as the length of the hypotenuse in the Pythagorean Theorem. This routine will engage students in meta-awareness as they critique and correct a common error when labeling the sides of a right triangle.
Design Principles(s): Cultivate conversation; Maximize meta-awareness

Lesson Synthesis

The purpose of this discussion is to check that students understand the Pythagorean Theorem and how it can be used to determine information about triangles. Ask students to draw a right triangle and label 2 of the 3 sides. Tell them to swap triangles with another student, solve for the missing length, then swap back to check the other person's work. Select a few groups to share their triangles and, if possible, display them for all to see while sharing how they solved for the unknown length.

8.4 Could be the Hypotenuse, Could be a Leg

Cool Down: 5 minutes
Addressing

- 8.G.B.7

> **Student Task Statement**
>
> A right triangle has sides of length 3, 4, and x.
>
> 1. Find x if it is the hypotenuse.
>
> 2. Find x if it is one of the legs.

Student Response

1. $x = \sqrt{25}$ or $x = 5$

2. $x = \sqrt{7}$

Student Lesson Summary

There are many examples where the lengths of two legs of a right triangle are known and can be used to find the length of the hypotenuse with the Pythagorean Theorem. The Pythagorean Theorem can also be used if the length of the hypotenuse and one leg is known, and we want to find the length of the other leg. Here is a right triangle, where one leg has a length of 5 units, the hypotenuse has a length of 10 units, and the length of the other leg is represented by g.
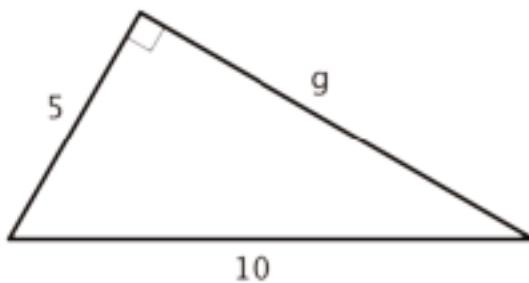
Start with $a^2 + b^2 = c^2$, make substitutions, and solve for the unknown value. Remember that c represents the hypotenuse: the side opposite the right angle. For this triangle, the hypotenuse is 10.

$$a^2 + b^2 = c^2$$
$$5^2 + g^2 = 10^2$$
$$g^2 = 10^2 - 5^2$$
$$g^2 = 100 - 25$$
$$g^2 = 75$$
$$g = \sqrt{75}$$

Use estimation strategies to know that the length of the other leg is between 8 and 9 units, since 75 is between 64 and 81. A calculator with a square root function gives $\sqrt{75} \approx 8.66$.

Lesson 8 Practice Problems
Problem 1
Statement
Find the exact value of each variable that represents a side length in a right triangle.
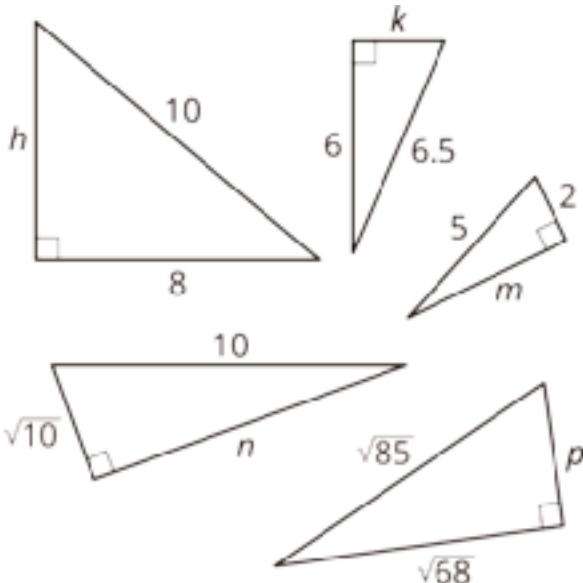
Solution

$h = 6$ (because $100 - 64 = 36$ and $\sqrt{36} = 6$)

$k = 2.5$ (because $42.25 - 36 = 6.25$ and $\sqrt{6.25} = 2.5$)

$m = \sqrt{21}$ because $25 - 4 = 21$

$n = \sqrt{90}$ because $100 - 10 = 90$

$p = \sqrt{17}$ because $85 - 68 = 17$

Problem 2
Statement
A right triangle has side lengths of a, b, and c units. The longest side has a length of c units. Complete each equation to show three relations among a, b, and c.

- $c^2 =$
- $a^2 =$
- $b^2 =$

Solution

- $c^2 = a^2 + b^2$ or $c^2 = b^2 + a^2$
- $a^2 = c^2 - b^2$
- $b^2 = c^2 - a^2$

(From Unit 8, Lesson 7.)

Problem 3

Statement

What is the exact length of each line segment? Explain or show your reasoning. (Each grid square represents 1 square unit.)
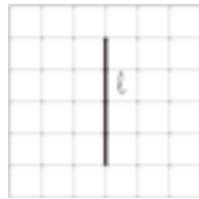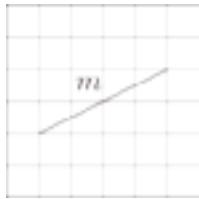
a.   b.   c.
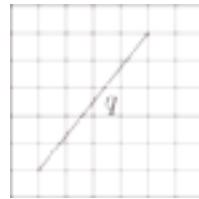
Solution

a. 4 units. The segment is along the grid lines, so count the squares.

b. $\sqrt{20}$ because $4^2 + 2^2 = 20$

c. $\sqrt{41}$ because $4^2 + 5^2 = 41$

(From Unit 8, Lesson 7.)

Problem 4

Statement

In 2015, there were roughly 1×10^6 high school football players and 2×10^3 professional football players in the United States. About how many times more high school football players are there? Explain how you know.

Solution

There are approximately 500 times more high school football players. $\frac{1 \times 10^6}{2 \times 10^3} = 0.5 \times 10^3 = 5 \times 10^2$

Problem 5

Statement

Evaluate:

a. $\left(\frac{1}{2}\right)^3$

b. $\left(\frac{1}{2}\right)^{-3}$

Solution

a. $\frac{1}{8}$

b. 8

Problem 6

Statement

Here is a scatter plot of weight vs. age for different Dobermans. The model, represented by $y = 2.45x + 1.22$, is graphed with the scatter plot. Here, x represents age in weeks, and y represents weight in pounds.
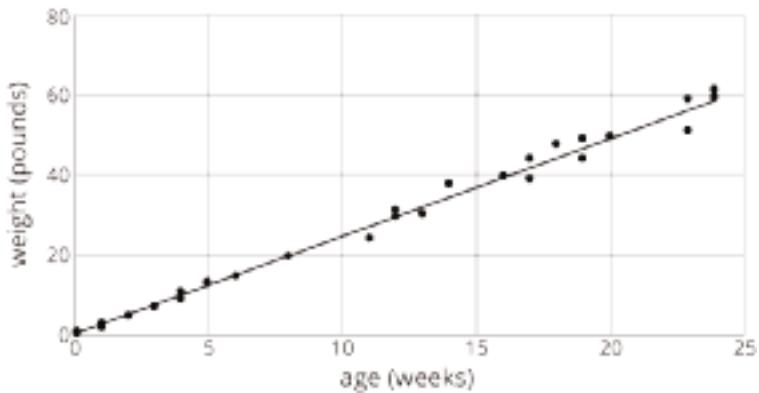
a. What does the slope mean in this situation?

b. Based on this model, how heavy would you expect a newborn Doberman to be?

Solution

a. The slope means that a doberman can be expected to gain 2.45 pounds per week.

b. 1.22 pounds (the y-intercept of the function).

Lesson 9: The Converse

Goals

- Determine whether a triangle with given side lengths is a right triangle using the converse of the Pythagorean Theorem.

- Generalize (orally) that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$ then the triangle must be a right triangle.

- Justify (orally) that a triangle with side lengths 3, 4, and 5 must be a right triangle.

Learning Targets

- I can explain why it is true that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$ then it must be a right triangle.

- If I know the side lengths of a triangle, I can determine if it is a right triangle or not.

Lesson Narrative

This lesson guides students through a proof of the converse of the Pythagorean Theorem. Then students have an opportunity to decide if a triangle with three given side lengths is or is not a right triangle.

Alignments

Addressing

- 8.G.B: Understand and apply the Pythagorean Theorem.

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

Building Towards

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

Instructional Routines

- MLR5: Co-Craft Questions

- MLR7: Compare and Connect

> **Student Learning Goals**
>
> Let's figure out if a triangle is a right triangle.

9.1 The Hands of a Clock

Warm Up: 5 minutes
This warm-up is preparation for the argument of the converse of the Pythagorean Theorem that we will construct in the next activity. The warm-up relies on the Pythagorean Theorem and

geometrically intuitive facts about how close or far apart the two hands of a clock can get from one another.

Building Towards

- 8.G.B.6

Launch

Arrange students in groups of 2. Give students 1 minute of quiet think time followed by partner and then whole-class discussions.

> ### Student Task Statement
>
> Consider the tips of the hands of an analog clock that has an hour hand that is 3 centimeters long and a minute hand that is 4 centimeters long.
>
>
>
> Over the course of a day:
>
> 1. What is the farthest apart the two tips get?
>
> 2. What is the closest the two tips get?
>
> 3. Are the two tips ever exactly five centimeters apart?

Student Response

1. If the two hands are pointing in opposite directions, the tips will be 7 centimeters apart.

2. If the two hands are pointing in the same direction (for example, at noon), the tips will be 1 centimeter apart.

3. Yes. Whenever the two hands make a right angle (for example, at 3:00), then by the Pythagorean Theorem, the two tips will be 5 centimeters apart, since $3^2 + 4^2 = 5^2$.
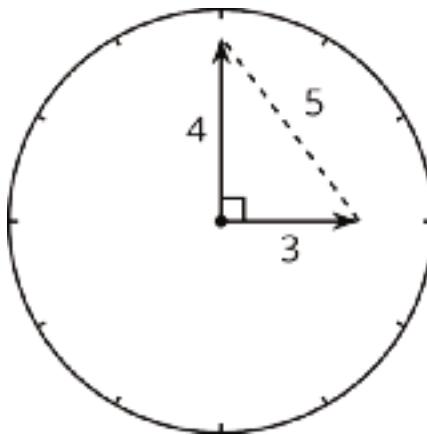
Activity Synthesis

Invite students share their solutions.

Make the following line of reasoning explicit:

Imagine two hands starting together, where one hand stays put and the other hand rotates around the face of the clock. As it rotates, the distance between its tip and the tip of the other hand continually increases until they are pointing in opposite directions. So from one moment to the next, the tips get farther apart.

(Proving this requires mathematics beyond grade 8, so for now we will just accept the results of the thought experiment as true.)

9.2 Proving the Converse

15 minutes
This activity introduces students to the *converse* of the Pythagorean Theorem: In a triangle with side lengths a, b, and c, if we have $a^2 + b^2 = c^2$, then the triangle *must* be a right triangle, and c must be its hypotenuse. Since up until this unit we rarely phrase things as a formal theorem, this may be the first time students have directly considered the idea that a theorem might work one way but not the other. For example, it is not clear at first glance that there is no such thing as an obtuse triangle with side lengths 3, 4, and 5, as in the image. But since $3^2 + 4^2 = 5^2$, the converse of the Pythagorean Theorem will say that the triangle *must* be a right triangle.
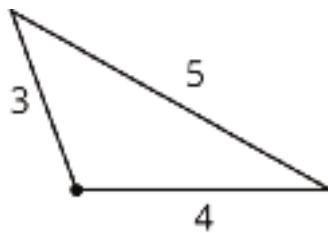
The argument this activity presents for this result is based on the thought experiment that as you rotate the sides of length 3 and 4 farther apart, the distance between their endpoints also grows,

from a distance of 1 when they are pointing in the same direction, to a distance of 7 when pointing in opposite direction. There is then one and only one angle along the way where the distance between them is 5, and by the Pythagorean Theorem, this happens when the angle between them is a right angle. This argument generalizes to an arbitrary right triangle, proving that the one and only angle that gives $a^2 + b^2 = c^2$ is precisely the right angle.

The next activity will more heavily play up the distinction between the Pythagorean Theorem and its converse, but it is worth emphasizing here as well.

Addressing

- 8.G.B.6

Instructional Routines

- MLR5: Co-Craft Questions

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time. Ask partners to share their work and come to an agreement. Follow with a whole-class discussion.

Access for Students with Disabilities

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example, use geogebra or hands-on manipulatives to demonstrate how increasing or decreasing the angle between 2 sides affects the opposite side length. Invite students to make conjectures and generalizations for different cases.
Supports accessibility for: Conceptual processing

Student Task Statement

Here are three triangles with two side lengths measuring 3 and 4 units, and the third side of unknown length.
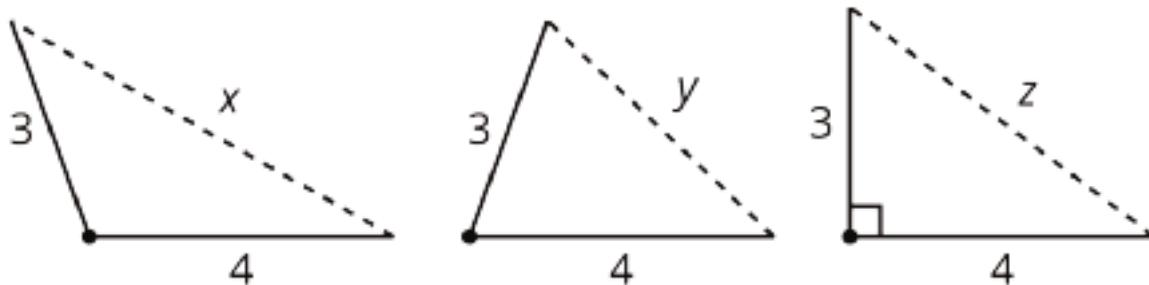
Sort the following six numbers from smallest to largest. Put an equal sign between any you know to be equal. Be ready to explain your reasoning.

$$1 \quad 5 \quad 7 \quad x \quad y \quad z$$

Student Response

$1 < y < 5 = z < x < 7$.

As in the warm-up, the distance between the ends of the two sides has to be between 1 and 7 in all three cases. Since the third triangle is a right triangle, we can apply the Pythagorean Theorem to see that $z = 5$, since $3^2 + 4^2 = 5^2$. Finally, it must be that $x > z > y$, since as you rotate the side of length 3 away from the bottom side, the distance between them gets farther.

Are You Ready for More?

A related argument also lets us distinguish acute from obtuse triangles using only their side lengths.

Decide if triangles with the following side lengths are acute, right, or obtuse. In right or obtuse triangles, identify which side length is opposite the right or obtuse angle.

- $x = 15$, $y = 20$, $z = 8$

- $x = 8$, $y = 15$, $z = 13$

- $x = 17$, $y = 8$, $z = 15$

Student Response

Take the two smaller sides and call them a and b. Call the longest side c. Compute $a^2 + b^2$. If this equals c^2, it's a right triangle. If c^2 is too big, the triangle is obtuse. If c^2 is too small, the triangle is acute.

- obtuse, since $8^2 + 15^2$ is 17^2. 20 is too big, so the triangle is obtuse, and the side of length 20 is opposite the obtuse angle.

- acute, since $8^2 + 13^2$ is 233. 15^2 is too small (it's 225), so the triangle is acute.

- right, since $8^2 + 15^2 = 17^2$. The side of length 17 is opposite the right angle.

Activity Synthesis

Select some groups to share the reasoning. It is important to get out the following sequence of ideas:

- Just as we saw with the clock problem, the length of the third side continually increases from 1 to 7 as the angles increases between $0°$ and $180°$.

- Because of this, there is one and only one angle along the way that gives a third side length of 5.

- By the Pythagorean Theorem, if the angle is a right angle, the third side length is 5.

Combining these, we see that the one and only angle that gives a third side length of 5 is the right angle. That is, *every* triangle with side lengths 3, 4, and 5 is a right triangle with hypotenuse 5. Triangles like the one displayed here are thus impossible.
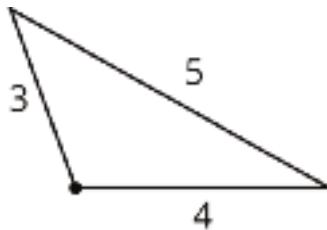
As a class, discuss how the argument could have been run with any two starting side lengths a and b instead of specifically 3 and 4.
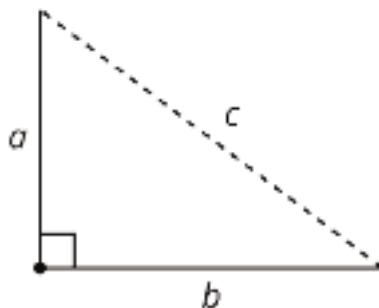
Though we do not need to write it as formally, the length of the third side continually increases from $a - b$ (or $b - a$) up to $a + b$ as the angle increases between $0°$ and $180°$. When the angle is a right angle, the third side has length equal to the value of c that makes $a^2 + b^2 = c^2$, and as in the previous discussion, that is the *only* angle that gives this length.

We conclude that the *only* way we could have $a^2 + b^2 = c^2$ is if the triangle is a right triangle, with hypotenuse c. This result is called the *converse of the Pythagorean Theorem*. Together, the Pythagorean Theorem and its converse provide an amazing relationship between algebra and geometry: The algebraic statement $a^2 + b^2 = c^2$ is completely equivalent to the geometric statement that the triangle with side lengths a, b, and c is a right triangle.

9.3 Calculating Legs of Right Triangles

10 minutes

The purpose of this activity is for students to apply the Pythagorean Theorem and its converse in mathematical contexts. In the first problem, students apply the Pythagorean Theorem to find unknown side lengths. In the second problem, students use the fact that by changing side lengths so that they satisfy $a^2 + b^2 = c^2$, the resulting triangle is guaranteed to be a right triangle.

Addressing

- 8.G.B

Instructional Routines

- MLR7: Compare and Connect

Launch

Provide students with access to calculators.

> **Student Task Statement**
>
> 1. Given the information provided for the right triangles shown here, find the unknown leg lengths to the nearest tenth.
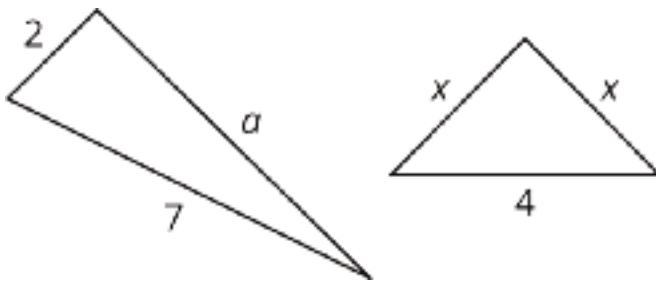
2. The triangle shown here is not a right triangle. What are two different ways you change *one* of the values so it would be a right triangle? Sketch these new right triangles, and clearly label the right angle.
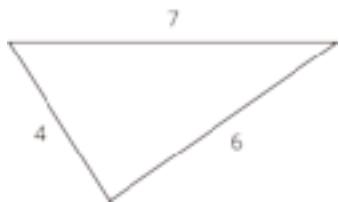
Student Response

1. $a = \sqrt{45}$, $x = \sqrt{8}$. For the first triangle, $a^2 + 2^2 = 7^2$, which means $a^2 = 49 - 4 = 45$ and $a = \sqrt{45}$. For the second triangle, $x^2 + x^2 = 4^2$, which means $2x^2 = 16$ and $x = \sqrt{8}$.

2. Answers vary. Sample response: If 4 and 6 were legs of a right triangle, then the hypotenuse would be the value of c in the equation $4^2 + 6^2 = c^2$. This means $c^2 = 52$ and $c = \sqrt{52} \approx 7.2$.

Activity Synthesis

The goal of this discussion is for students to use the Pythagorean Theorem to reason about the side lengths of right triangles. Select 2–3 students to share their work for the two problems.

For the second problem, make a list of all possible changes students figured out that would make the triangle a right triangle by changing the length of just one side. Depending on where students decided the right angle is located and which side they selected to change, there are nine possible solutions. Can students find them all?

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.
Supports accessibility for: Attention; Social-emotional skills

Unit 8 Lesson 9

Speaking, Listening: MLR7 Compare and Connect. Ask students to prepare a visual display to show two different ways to change the triangle so that it would be a right triangle. As students investigate each other's work. Listen for and amplify the language students use to explain how they used the converse of the Pythagorean Theorem to justify that the triangle is a right triangle. Encourage students to explain how the various equations of the form $a^2 + b^2 = c^2$ informed how they sketched the right triangles. For example, the equation $4^2 + b^2 = 6^2$ implies that the length of the hypotenuse is 6, whereas the equation $4^2 + b^2 = 7^2$ implies that the length of the hypotenuse is 7. This will foster students' meta-awareness and support constructive conversations as they compare strategies for creating right triangles and make connections between equations of the form $a^2 + b^2 = c^2$ and the right triangles they represent. *Design Principles(s): Cultivate conversation; Maximize meta-awareness*

Lesson Synthesis

The purpose of this discussion is to reinforce the takeaways the Pythagorean Theorem and its converse.

- "What has to be true in order to be sure a triangle is a right triangle?" (The sum of the squares of the two shorter sides must equal the square of the longest side.)

Remind students that this is a result of the *converse* of the Pythagorean Theorem. The Pythagorean Theorem is an example of a theorem where the converse is also true. That is, the Pythagorean Theorem states that for a right triangle with sides a, b, and c, with c the length of the hypotenuse, the relationship $a^2 + b^2 = c^2$ is always true. The converse of the Pythagorean Theorem states that if a, b, and c are side lengths of a triangle and $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

To illustrate this, display the image shown here of a triangle with sides 6, 8, and 10.
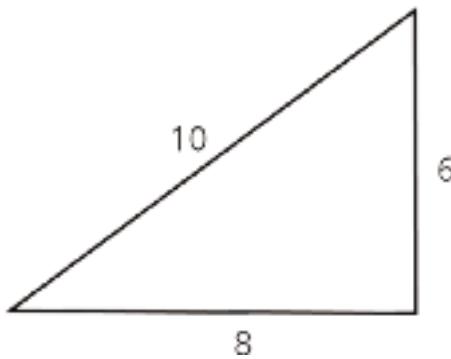
Ask students to decide if this is a right triangle or not. After some quiet work time, poll the class for which type of triangle they think it is. Remind students that while the angle across from the side of length 10 looks like a right angle, we can't be sure it is just from the image. However, since

$6^2 + 8^2 = 10^2$ is true, We know, by the converse of the Pythagorean Theorem, that the triangle is a right triangle and that the angle across from the side of length 10 is a right angle.

9.4 Is It a Right Triangle?

Cool Down: 5 minutes
Addressing

- 8.G.B

> **Student Task Statement**
>
> The triangle has side lengths 7, 10, and 12. Is it a right triangle? Explain your reasoning.

Student Response

No. If this was a right triangle, then $7^2 + 10^2$ would equal 12^2, which it does not.

> **Student Lesson Summary**
>
> What if it isn't clear whether a triangle is a right triangle or not? Here is a triangle:
>
>
>
> Is it a right triangle? It's hard to tell just by looking, and it may be that the sides aren't drawn to scale.
>
> If we have a triangle with side lengths a, b, and c, with c being the longest of the three, then the *converse* of the Pythagorean Theorem tells us that any time we have $a^2 + b^2 = c^2$, we *must* have a right triangle. Since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, any triangle with side lengths 8, 15, and 17 *must* be a right triangle.
>
> Together, the Pythagorean Theorem and its converse provide a one-step test for checking to see if a triangle is a right triangle just using its side lengths. If $a^2 + b^2 = c^2$, it is a right triangle. If $a^2 + b^2 \neq c^2$, it is not a right triangle.
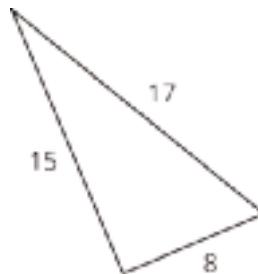
Lesson 9 Practice Problems
Problem 1
Statement
Which of these triangles are definitely right triangles? Explain how you know. (Note that not all triangles are drawn to scale.)
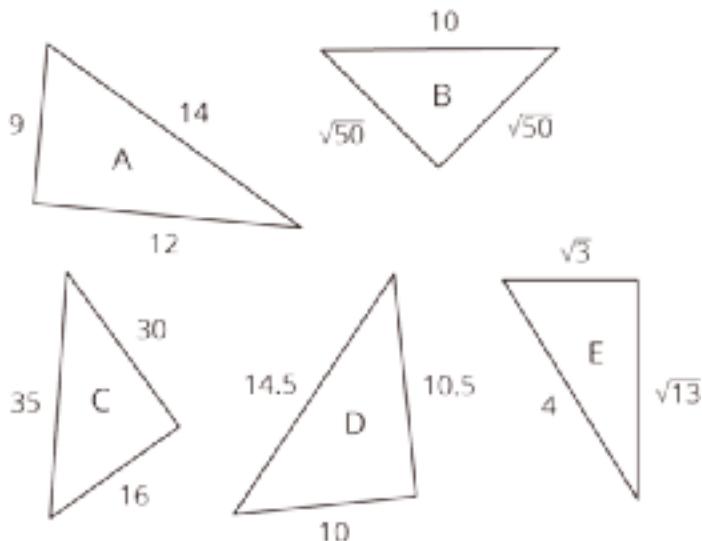
Solution
B, D, and E are right triangles. A and C are not.

- A: $9^2 + 12^2 = 14^2$ is false
- B: $\sqrt{50}^2 + \sqrt{50}^2 = 10^2$ is true
- C: $16^2 + 30^2 = 35^2$ is false
- D: $10^2 + 10.5^2 = 14.5^2$ is true
- E: $\sqrt{3}^2 + \sqrt{13}^2 = 4^2$ is true

Problem 2
Statement
A right triangle has a hypotenuse of 15 cm. What are possible lengths for the two legs of the triangle? Explain your reasoning.
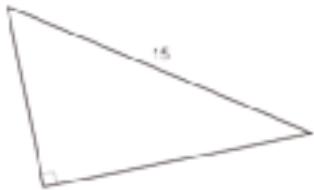
Solution

Answers vary. Sample responses: $\sqrt{200}$ and 5; $\sqrt{125}$ and 10. If the legs of the triangle are a and b, then we can set up the equation $a^2 + b^2 = 15^2$. This means a^2 and b^2 must sum to 225. If $a^2 = 25$, then $b = 200$. If $a^2 = 100$, then $b^2 = 125$.

Problem 3

Statement

In each part, a and b represent the length of a leg of a right triangle, and c represents the length of its hypotenuse. Find the missing length, given the other two lengths.

   a. $a = 12, b = 5, c = ?$

   b. $a = ?, b = 21, c = 29$

Solution

   a. $c = 13$. If $a = 12$ and $b = 5$ then $12^2 + 5^2 = c^2$.

   b. $a = 20$. If $b = 21$ and $c = 29$ then $a^2 + 21^2 = 29^2$.

(From Unit 8, Lesson 8.)

Problem 4

Statement

For which triangle does the Pythagorean Theorem express the relationship between the lengths of its three sides?

Solution

B
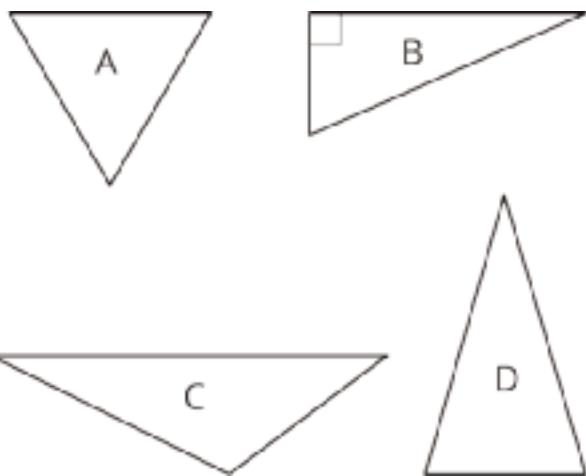
(From Unit 8, Lesson 6.)

Problem 5

Statement

Andre makes a trip to Mexico. He exchanges some dollars for pesos at a rate of 20 pesos per dollar. While in Mexico, he spends 9000 pesos. When he returns, he exchanges his pesos for dollars (still at 20 pesos per dollar). He gets back $\frac{1}{10}$ the amount he started with. Find how many dollars Andre exchanged for pesos and explain your reasoning. If you get stuck, try writing an equation representing Andre's trip using a variable for the number of dollars he exchanged.

Solution

500 dollars. Sample reasoning: $\frac{20x-9000}{20} = \frac{x}{10}$, where x represents the number of dollars he exchanged. Rewrite the equation as $20x - 9000 = 2x$, and then solve to find $x = 500$.

(From Unit 4, Lesson 5.)

Lesson 10: Applications of the Pythagorean Theorem

Goals

- Describe (orally) situations that use right triangles, and explain how the Pythagorean Theorem could help solve problems in those situations.

- Use the Pythagorean Theorem to solve problems within a context, and explain (orally) how to organize the reasoning.

Learning Targets

- I can use the Pythagorean Theorem to solve problems.

Lesson Narrative

In this lesson students use the Pythagorean Theorem and its converse as a tool to solve application problems. In the first activity, they solve a problem involving the distance and speed of two children walking and riding a bike along different sides of a triangular region. In the second activity they find internal diagonals of rectangular prisms.

Alignments

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.G.B.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

Instructional Routines

- MLR5: Co-Craft Questions

- MLR6: Three Reads

- MLR8: Discussion Supports

- Poll the Class

- Think Pair Share

Student Learning Goals

Let's explore some applications of the Pythagorean Theorem.

10.1 Closest Estimate: Square Roots

Warm Up: 5 minutes
The purpose of this warm-up is for students to reason about square roots by estimating the value of each expression. The values given as choices are close in range to encourage students to use the square roots they know to help them estimate ones they do not. These understandings will be helpful for students in upcoming activities where they will be applying the Pythagorean Theorem.

While four problems are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem.

Addressing
- 8.EE.A.2
- 8.NS.A

Instructional Routines
- MLR8: Discussion Supports

Launch
Display one problem at a time. Give students 30 seconds of quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

> **Student Task Statement**
>
> Which estimate is closest to the actual value of the expression? Explain your reasoning.
>
> 1. $\sqrt{24}$
>
>   ○ 4
>
>   ○ 4.5
>
>   ○ 5
>
> 2. $\sqrt{7}$
>
>   ○ 2
>
>   ○ 2.5
>
>   ○ 3
>
> 3. $\sqrt{42}$
>
>   ○ 6
>
>   ○ 6.5

- ○ 7

4. $\sqrt{10} + \sqrt{97}$

   - ○ 13

   - ○ 13.25

   - ○ 13.5

Student Response

1. 5; Strategies may vary. Sample response: The square root of 25 is 5, and 24 is very close to 25.

2. 2.5; Strategies may vary. Sample response: 7 is just a little over halfway between 2^2 and 3^2

3. 6.5; Strategies may vary. Sample response: 42 is almost halfway between 6^2 and 7^2

4. 13; Strategies may vary. Sample response: The square root of 10 is a little more than 3 and the square root of 97 is just a little bit less than 10.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their responses for all to see. After each student shares, ask if their chosen estimate is more or less than the actual value of each expression.

Access for English Language Learners

Speaking: MLR8 Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimize output (for explanation)

10.2 Cutting Corners

15 minutes
The purpose of this activity is for students to use the Pythagorean Theorem to reason about distances and speeds to figure out who will win a race. Students must translate between the context and the geometric representation of the context and back (MP2). Identify students whose work is clearly labeled and organized to share during the whole-class discussion.

Addressing

- 8.G.B.7

Instructional Routines

- MLR6: Three Reads
- Poll the Class

Launch

Arrange students in groups of 2. Provide students with access to calculators. Ask students to read and answer the first problem and then give a signal when they have done so. Next, poll the class to see who students think will win the race and post the results of the poll for all to see. Select 2–3 students to share why they chose who they chose. The conversation should illuminate a few key points: Mai travels farther than Tyler, but she is also going faster, so it is not immediately clear who will win.

Give 2 minutes of quiet work time for the second problem, followed by partner and then whole-class discussions.

Access for Students with Disabilities

Representation: Access for Perception. Read the situation aloud. Students who both listen to and read the information will benefit from extra processing time.
Supports accessibility for: Language

Access for English Language Learners

Reading: MLR6 Three Reads. Use this routine to support reading comprehension of this problem without solving it for students. In the first read, students read the problem with the goal of comprehending the situation (e.g., Mai and Tyler are racing from one corner of the rectangular field to the opposite corner.). In the second read, ask students to identify important quantities that can be counted or measured (e.g., the length and width of the field; Mai's speed on her bike; Tyler's running speed). After the third read, reveal the question: "Who do you think will win? By how much?" Ask students to brainstorm possible solution strategies to answer the question (e.g., Use the Pythagorean Theorem to calculate the length of the diagonal across the field; Use Mai's and Tyler's speeds to calculate how long it will take them to reach the opposite corner.). This will help students concentrate on making sense of the situation before rushing to a solution or method.
Design Principle(s): Support sense-making

Anticipated Misconceptions

Some students may rush through reading the problem and mix up who is traveling which path at what speed. Encourage these students to label the relevant distances and information on the diagram.

Student Task Statement

Mai and Tyler were standing at one corner of a large rectangular field and decided to race to the opposite corner. Since Mai had a bike and Tyler did not, they thought it would be a fairer race if Mai rode along the sidewalk that surrounds the field while Tyler ran the shorter distance directly across the field. The field is 100 meters long and 80 meters wide. Tyler can run at around 5 meters per second, and Mai can ride her bike at around 7.5 meters per second.

1. Before making any calculations, who do you think will win? By how much? Explain your thinking.

2. Who wins? Show your reasoning.

Student Response

1. Answers vary. Sample response: Mai is going faster, but she also has a longer distance to travel. I think that she might be fast enough to beat Tyler even if she is going a little farther.

2. Mai wins. Mai has 180 meters to travel going 7.5 meters per second, which will take her 24 seconds since $\frac{180}{7.5} = 24$. Using the Pythagorean Theorem, Tyler travels $\sqrt{16,400}$ meters going 5 meter per second, which will take him approximately 25.6 seconds since $\frac{\sqrt{16,400}}{5} \approx 25.6$. Mai beats Tyler to the opposite corner by about 1.6 seconds.

Are You Ready for More?

A calculator may be necessary to answer the following questions. Round answers to the nearest hundredth.

1. If you could give the loser of the race a head start, how much time would they need in order for both people to arrive at the same time?

2. If you could make the winner go slower, how slow would they need to go in order for both people to arrive at the same time?

Student Response

1. Tyler needs roughly 1.6 seconds of head start to beat Mai. He travels approximately 128.06 meters going 5 meter per second, which will take him 25.6 seconds. Mai finishes in 24 seconds.

Unit 8 Lesson 10

2. About 7.03 meters per second. If Mai goes 7.03 meters per second, then she will finish the race in $\frac{180}{7.03} \approx 25.6$ seconds. She will beat Tyler by a fraction of a second.

Activity Synthesis

The purpose of this discussion is for students to share how they organized their work for the problem and see how accurate students' predictions at the start of the activity were. Select 1–2 previously identified students to share their work. Directly draw attention to how careful labeling of provided figures and organization of calculations when problems have multiple steps helps to both solve problems and identify errors. For example, in problems where multiple calculations are completed using a calculator, it is easy to copy an incorrect number, such as writing 14,600 instead of 16,400 for the sum $100^2 + 80^2$.

10.3 Internal Dimensions

15 minutes
The purpose of this task is for students to use the Pythagorean Theorem to calculate which rectangular prism has the longer diagonal length. To complete the activity, students will need to picture or sketch the right triangles necessary to calculate the diagonal length.

Identify groups using well-organized strategies to calculate the diagonals. For example, for Prism K some groups may draw in the diagonal on the bottom face of the cube and label it with an unknown variable, such as s, to set up the calculation for the diagonal as $\sqrt{6^2 + s^2}$. Students would then know that to finish this calculation, they would need to find the value of s, which is can be done using the edge lengths of the face of the prism s is drawn on.

The idea that the Pythagorean Theorem might be applied several times in order to find a missing length is taken even further in the extension, where students can apply it over a dozen times, hopefully noticing some repeated reasoning along the way, in order to find the last length.

Addressing
- 8.G.B.7

Instructional Routines
- MLR5: Co-Craft Questions
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to calculators. Ask students to read and consider the first problem and then give a signal when they have done so. If possible, use an actual rectangular prism, such as a small box, to help students understand what length the diagonal shows in the image. Once students understand what the first problem is asking, poll the class to see which prism they think has the longer diagonal and post the results of the poll for all to see. Select 2–3 students to share why they chose the prism they did. The conversation should illuminate a few

key points: Prism K has one edge with length 6 units, which is longer than any of the edges of Prism L, but it also has a side of length 4 units, which is shorter than any of the edges of Prism L. Prism K also has a smaller volume than Prism L.

Give 1 minute of quiet think time for students to brainstorm how they will calculate the lengths of each diagonal. Ask partners to discuss their strategies before starting their calculations. Follow with a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Provide access to 3-D models of rectangular prisms (for example, a tissue box) for students to view or manipulate. Use color coding to highlight and make connections between corresponding parts.
Supports accessibility for: Visual-spatial processing; Conceptual processing

Access for English Language Learners

Conversing, Writing: MLR5 Co-Craft Questions. Before revealing the questions in this activity, display only the image of the rectangular prisms. Invite students to work with a partner to write mathematical questions that could be asked about the prisms. Select 1–2 groups to share their questions with the class. Listen for and amplify questions about the lengths of the diagonals. For example, "What is the length of the diagonal of Prism K?" and "Which rectangular prism has the longer diagonal length?" If no student asks these questions, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about calculating the diagonal length of a rectangular prism.
Design Principle(s): Maximize meta-awareness

Student Task Statement

Here are two rectangular prisms:
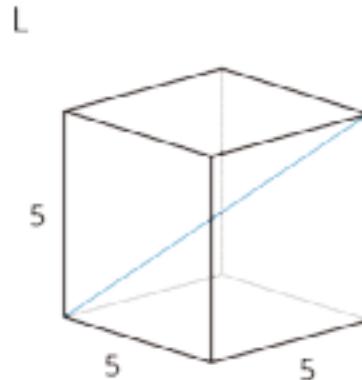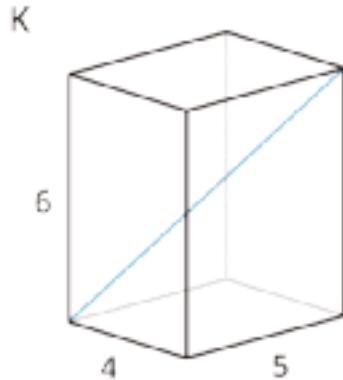
1. Which figure do you think has the longer diagonal? Note that the figures are not drawn to scale.

2. Calculate the lengths of both diagonals. Which one is actually longer?

Student Response

1. Answers vary. Sample Response: I think Prism L has a longer diagonal since it has a larger volume than Prism K and similar dimensions.

2. Prism K has a longer diagonal of $\sqrt{77}$ units. Sample response: for Prism K, the length of the diagonal, d, is $\sqrt{6^2 + s^2}$, where s is the length of the diagonal of the bottom face of the prism, since $6^2 + s^2 = d^2$ by the Pythagorean Theorem. We can calculate s by first recognizing it as the hypotenuse of the triangle with legs of length 4 and 5, which means $s = \sqrt{41}$ since $4^2 + 5^2 = 41$. Putting these together, $d = \sqrt{77}$ since $\sqrt{6^2 + s^2} = \sqrt{6^2 + (\sqrt{41})^2} = \sqrt{36 + 41} = \sqrt{77}$. Noticing that this is the same as $\sqrt{6^2 + 4^2 + 5^2}$, which is the square root of the sum of the squares of the three edge lengths of the prism, we can conclude that the diagonal of Prism L has length $\sqrt{75}$ since $\sqrt{5^2 + 5^2 + 5^2} = \sqrt{75}$.

Activity Synthesis

Ask groups to share how they calculated the diagonal of one of the rectangular prisms. Display the figures in the activity for all to see and label the them while groups share.

If time allows and no groups pointed out how the diagonal length of the rectangular prisms are the square root of the sum of the squares of the three edge lengths, use Prism K and the calculations students shared to do so. For example, to calculate the diagonal of Prism K, students would have to calculate $\sqrt{6^2 + \left(\sqrt{41}\right)^2}$, but $\sqrt{41}$ is from $\sqrt{5^2 + 4^2}$ which means the diagonal length is really $\sqrt{6^2 + 5^2 + 4^2}$ since $\sqrt{6^2 + \left(\sqrt{41}\right)^2} = \sqrt{6^2 + \left(\sqrt{5^2 + 4^2}\right)^2} = \sqrt{6^2 + 5^2 + 4^2}$. So, for a rectangular prism with sides d, e, and f, the length of the diagonal of the prism is just $\sqrt{d^2 + e^2 + f^2}$.

Lesson Synthesis

Tell students that there are many situations in the world where we can use the Pythagorean Theorem to solve problems. Ask, "What situations that you can think of involve right triangles?" Give brief quiet think time, then invite students to share their ideas. For example, thick wires (called guy-wires) are used to keep telephone poles upright and their length depends on how high up the pole they attach and how far away from the pole they hook into the ground.

10.4 Jib Sail

Cool Down: 5 minutes
Addressing
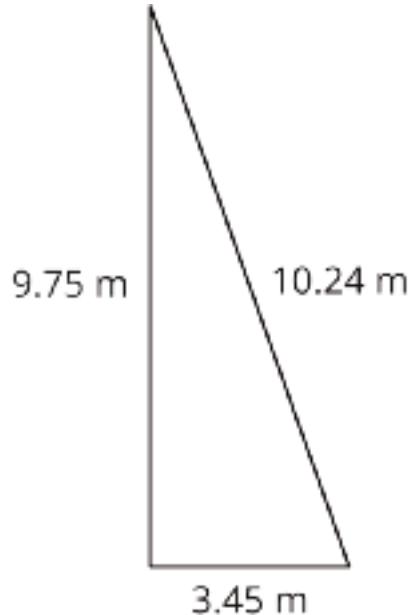
- 8.G.B.7

> **Student Task Statement**
>
>
>
> Sails come in many shapes and sizes. The sail on the right is a triangle. Is it a right triangle? Explain your reasoning.

Student Response

No. The sum of the squares of the two shorter sides is 106.965 square meters, and the square of the longest side is 104.8576 square meters. So by the converse of the Pythagorean Theorem, it is not a right triangle.

> # Student Lesson Summary
>
> The Pythagorean Theorem can be used to solve any problem that can be modeled with a right triangle where the lengths of two sides are known and the length of the other side needs to be found. For example, let's say a cable is being placed on level ground to support a tower. It's a 17-foot cable, and the cable should be connected 15 feet up the tower. How far away from the bottom of the tower should the other end of the cable connect to the ground?
>
> It is often very helpful to draw a diagram of a situation, such as the one shown here:
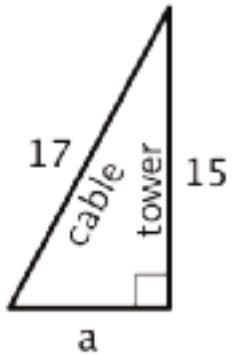
It's assumed that the tower makes a right angle with the ground. Since this is a right triangle, the relationship between its sides is $a^2 + b^2 = c^2$, where c represents the length of the hypotenuse and a and b represent the lengths of the other two sides. The hypotenuse is the side opposite the right angle. Making substitutions gives $a^2 + 15^2 = 17^2$. Solving this for a gives $a = 8$. So, the other end of the cable should connect to the ground 8 feet away from the bottom of the tower.

Lesson 10 Practice Problems
Problem 1

> ### Statement
> A man is trying to zombie-proof his house. He wants to cut a length of wood that will brace a door against a wall. The wall is 4 feet away from the door, and he wants the brace to rest 2 feet up the door. About how long should he cut the brace?
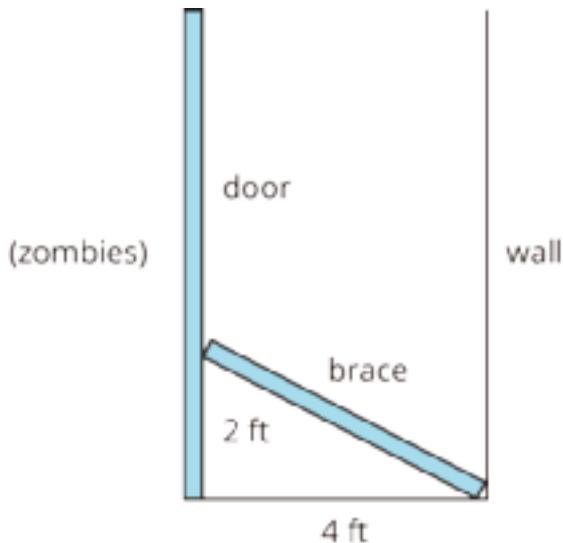
Solution

Around 4.5 feet. Solving $2^2 + 4^2 = c^2$, we get $c = \sqrt{20}$, which is approximately 4.5.

Problem 2

> ### Statement
> At a restaurant, a trash can's opening is rectangular and measures 7 inches by 9 inches. The restaurant serves food on trays that measure 12 inches by 16 inches. Jada says it is impossible for the tray to accidentally fall through the trash can opening because the shortest side of the tray is longer than either edge of the opening. Do you agree or disagree with Jada's explanation? Explain your reasoning.

Solution

I disagree. Explanations vary. Sample explanation: It is impossible for the tray to fall through the opening, but not for the reason Jada gives. The longest dimension of the trash can opening is the diagonal. The diagonal is $\sqrt{130}$ inches long, because $7^2 + 9^2 = 130$. The diagonal is between 11 and 12 inches long, because $11^2 < 130 < 12^2$. The tray cannot fall through the opening because the diagonal is a little shorter than the shortest dimension of the tray.

Problem 3

Statement

Select **all** the sets that are the three side lengths of right triangles.

A. 8, 7, 15

B. 4, 10, $\sqrt{84}$

C. $\sqrt{8}$, 11, $\sqrt{129}$

D. $\sqrt{1}$, 2, $\sqrt{3}$

Solution

["B", "C", "D"]

(From Unit 8, Lesson 9.)

Problem 4

Statement

For each pair of numbers, which of the two numbers is larger? How many times larger?

a. $12 \cdot 10^9$ and $4 \cdot 10^9$

b. $1.5 \cdot 10^{12}$ and $3 \cdot 10^{12}$

c. $20 \cdot 10^4$ and $6 \cdot 10^5$

Solution

a. $12 \cdot 10^9$, 3 times larger

b. $3 \cdot 10^{12}$, 2 times larger

c. $6 \cdot 10^5$, 3 times larger

(From Unit 7, Lesson 10.)

Problem 5

Statement

A line contains the point $(3, 5)$. If the line has negative slope, which of these points could also be on the line?

A. $(2, 0)$

B. $(4, 7)$

C. $(5, 4)$

D. $(6, 5)$

Solution

C

(From Unit 3, Lesson 10.)

Problem 6

Statement

Noah and Han are preparing for a jump rope contest. Noah can jump 40 times in 0.5 minutes. Han can jump y times in x minutes, where $y = 78x$. If they both jump for 2 minutes, who jumps more times? How many more?

Solution

Noah jumps 160 times and Han jumps 156 times, so Han jumps 4 more times.

(From Unit 3, Lesson 4.)

Lesson 11: Finding Distances in the Coordinate Plane

Goals

- Calculate the distance between two points in the coordinate plane by using the Pythagorean Theorem and explain (orally) the solution method.

- Generalize (orally) a method for calculating the length of a line segment in the coordinate plane using the Pythagorean Theorem.

Learning Targets

- I can find the distance between two points in the coordinate plane.

- I can find the length of a diagonal line segment in the coordinate plane.

Lesson Narrative

In this lesson, students continue to apply the Pythagorean Theorem to find distances between points in the coordinate plane.

Students who successfully answer the problems in the second activity use the structure of the coordinate plane to draw a right triangle, an example of looking for and making use of structure in the coordinate plane (MP7).

Alignments
Addressing

- 8.G.B.8: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Building Towards

- 8.G.B.8: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- MLR3: Clarify, Critique, Correct

- MLR7: Compare and Connect

- Poll the Class

- Think Pair Share

> **Student Learning Goals**
>
> Let's find distances in the coordinate plane.

11.1 Closest Distance

Warm Up: 5 minutes
The purpose of this warm-up is for students to find the distance between two points on the same horizontal or vertical line in the coordinate plane. Students are given only the coordinates and no graph to encourage them to notice that to find the distance between two points on the same horizontal or vertical line, they subtract the coordinate that is not the same in both points. (This is an idea students should have encountered as early as grade 6.) This understanding will be important for students in upcoming lessons as they begin using the distance formula between two points in the plane as they apply the Pythagorean Theorem.

Building Towards

- 8.G.B.8

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time followed by 1 minute to compare their responses with a partner. Follow with a whole-class discussion.

> ### Student Task Statement
>
> 1. Order the following pairs of coordinates from closest to farthest apart. Be prepared to explain your reasoning.
>   a. $(2, 4)$ and $(2, 10)$
>
>   b. $(-3, 6)$ and $(5, 6)$
>
>   c. $(-12, -12)$ and $(-12, -1)$
>
>   d. $(7, 0)$ and $(7, -9)$
>
>   e. $(1, -10)$ and $(-4, -10)$
>
> 2. Name another pair of coordinates that would be closer together than the first pair on your list.
>
> 3. Name another pair of coordinates that would be farther apart than the last pair on your list.

Student Response

1. E (5), A (6), B (8), D (9), C (11)

2. Answers vary. Sample response: $(2, 4)$ and $(2, 8)$

3. Answers vary. Sample response: $(12, -10)$ and $(-4, -10)$

Activity Synthesis

Invite students to share their order of the pairs of coordinates from closest to furthest apart. Record and display the list for all to see. After the class agrees on the correct order, ask students to share the distance between a few of the pairs of coordinates and their strategy for finding that distance. Ask 2–3 students to share pairs of coordinates they found that would have a closer or further distance than the ones in the list.

If the following ideas do not arise during the discussion, consider asking the following questions:

- "Why does each pair have one coordinate that is the same?"

- "How did you decide on which coordinate to subtract?"

- "Why didn't you need to subtract the other?"

- "Could we represent this distance with a line segment? How do you know?"

- "Would your strategy work for any pair of coordinates?"

- "Which pairs would it work? Which pairs are we not sure if it would work?"

11.2 How Far Apart?

10 minutes
In this activity, students find distances between points in the coordinate plane. The three points are the vertices of a right triangle, helping students to see that they can find the distance between two points that are not on the same vertical or horizontal line by creating a right triangle.

Addressing

- 8.G.B.8

Instructional Routines

- MLR3: Clarify, Critique, Correct

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time followed by partner and whole-class discussions.

Student Task Statement

Find the distances between the three points shown.

Student Response

12, 30, $\sqrt{1044}$

Activity Synthesis

Invite groups to share their solutions. Then draw two points in the coordinate plane, for example $(-3, -4)$ and $(2, 7)$. Ask students how we can use the problem we just solved to find the distance between these two points. If no students suggest drawing the point $(2, -4)$, draw it and ask how we can use it to find the distance between the first two points.

Reading, Writing, Speaking: MLR3 Clarify, Critique, Correct. Before students share their solutions, display an incorrect solution based on a common error you observe for finding the distance between points with negative coordinates. For example, "The distance between (-14, 9) and (-14, -3) is 6, because $9 - 3 = 6$. The distance between (-14, -3) and (16, -3) is 2, because $16 - 14 = 2$." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who reference the points in the coordinate plane to show that the distances are incorrect. Amplify the language students use to clarify that the horizontal distance between two points is the difference between the x-values and the vertical distance between two points is the difference between the y-values. This routine will engage students in meta-awareness as they critique and correct a common error when finding the distance between points.
Design Principles(s): Cultivate conversation; Maximize meta-awareness

11.3 Perimeters with Pythagoras

Optional: 15 minutes
In this optional activity, students calculate the perimeters of two triangles in the coordinate plane using the Pythagorean Theorem. Use this activity if time allows and your students need practice calculating the distance between two points in a coordinate plane.

Addressing

- 8.G.B.8

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- Poll the Class

- Think Pair Share

Launch

Arrange students in groups of 2. Display the image of the two figures for all to see and then poll the class and ask which figure they think has the longer perimeter. Display the results of the poll for all to see during the activity. Tell the class that each partner will now calculate the perimeter of one of the figures.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Access for English Language Learners

Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time. Use this routine to help students consider audience when preparing to share their work. Ask students to prepare a visual display that shows how they calculated the perimeter of their triangle. Students should consider how to display their calculations so that another student can interpret them. Some students may wish to add notes or details to their drawings to help communicate their thinking. Ask each student to meet with 2–3 other partners for feedback. Display prompts for feedback that will help students strengthen their ideas and clarify their language. For example, "How did you find the distance between these points?", "What is the perimeter of a triangle?", and "How did you find the perimeter of the triangle?" Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students refine both their ideas, and their verbal and written output.
Design Principles(s): Optimize output (for explanation); Maximize meta-awareness

Student Task Statement

1. Which figure do you think has the longer perimeter?

2. Select one figure and calculate its perimeter. Your partner will calculate the perimeter of the other. Were you correct about which figure had the longer perimeter?
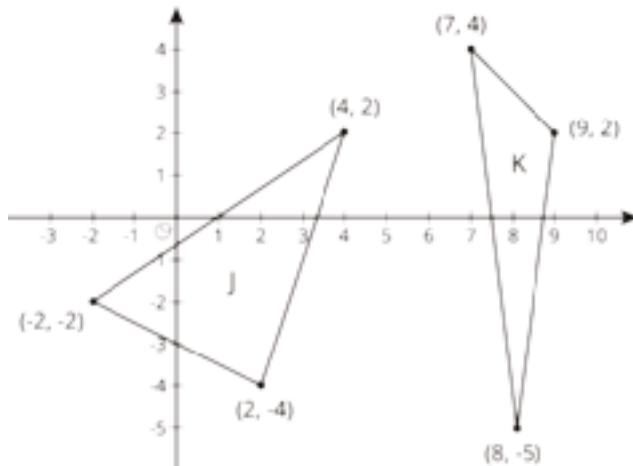
Student Response

1. Answers vary.

2. Figure J has a perimeter of approximately 18 units. The perimeter is the sum
$\sqrt{2^2 + 6^2} + \sqrt{2^2 + 4^2} + \sqrt{4^2 + 6^2} = \sqrt{40} + \sqrt{20} + \sqrt{52} \approx 18$. Figure K has a perimeter of
approximately 19 units. The perimeter is the sum
$\sqrt{9^2 + 1^2} + \sqrt{7^2 + 1^2} + \sqrt{2^2 + 2^2} = \sqrt{82} + \sqrt{50} + \sqrt{8} \approx 19$.

Are You Ready for More?

Quadrilateral $ABCD$ has vertices at $A = $ (-5, 1), $B = $ (-4, 4), $C = $ (2, 2), and $D = $ (1, -1).

1. Use the Pythagorean Theorem to find the lengths of sides AB, BC, CD, and AD.

2. Use the Pythagorean Theorem to find the lengths of the two diagonals, AC and BD.

3. Explain why quadrilateral $ABCD$ is a rectangle.

Student Response

1. Use the Pythagorean Theorem to find the length of each segment. Segment AB has
length $\sqrt{10}$ because $AB^2 = 1^2 + 3^2$. Segment CD also has length $\sqrt{10}$ because the right
triangle used to find AB is congruent to the right triangle used to find CD. The length
of AD is $\sqrt{40}$ because $AD^2 = 6^2 + 2^2$. The triangle used to calculate BC is congruent to the
one used to calculate AD, so the length of BC is also $\sqrt{40}$.

2. The length of AC is $\sqrt{50}$ because $AC^2 = 7^2 + 1^2$. The length of BD is also $\sqrt{50}$ because
$BD^2 = 5^2 + 5^2$.

3. The figure $ABCD$ is a rectangle because it has four right angles. For example, the angle at A is
a right angle by the converse of the Pythagorean Theorem, since we have that
$AD^2 + AB^2 = BD^2$.

Activity Synthesis

Repeat the poll from the start of the activity to see how results have changed. Select 1–2 students to
share their calculations.

11.4 Finding the Right Distance

15 minutes
The purpose of this task is for students to think about a general method for finding the distance
between two points on the coordinate plane. Students do not need to formalize this into a more
traditional representation of the distance formula. In groups of 4, each student will find the distance
between two coordinate pairs and then share how they completed their calculations. The
coordinates are carefully chosen so that the distances are all equal since each pair represents a
possible diameter for a circle centered at (2, -2) with radius 5, though students do not need to know
this in order to complete their calculations.

Identify students who clearly explain their thinking as they work with their group. Notice any groups that discover the points are all on the perimeter of a circle.

Addressing
- 8.G.B.8

Instructional Routines
- MLR7: Compare and Connect

Launch
Arrange students in groups of 4. Tell students that once everyone in their group has calculated the distance between their points they will share how they did their calculations and then answer the problems. Encourage students to listen carefully to the ideas of other members of their group in order to write a clear explanation for the second question.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "I noticed ____ so I . . .", "By comparing triangles I . . .", or "Another strategy could be ____ because . . . ."
Supports accessibility for: Language; Social-emotional skills

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Student Task Statement

Have each person in your group select one of the sets of coordinate pairs shown here. Then calculate the length of the line segment between those two coordinates. Once the values are calculated, have each person in the group briefly share how they did their calculations.

- $(-3, 1)$ and $(5, 7)$
- $(-1, -6)$ and $(5, 2)$
- $(-1, 2)$ and $(5, -6)$
- $(-2, -5)$ and $(6, 1)$

1. How does the value you found compare to the rest of your group?

2. In your own words, write an explanation to another student for how to find the distance between any two coordinate pairs.

Student Response
1. All lengths are 10 units.

2. Answers vary. Sample response: For the coordinate pairs, $(-2, 1)$ and $(6, -5)$, a right triangle can be drawn with the coordinate pairs as vertices. The legs of this triangle are 6 and 8. This means the distance between the coordinate pairs is given by c in the equation $6^2 + 8^2 = c^2$. Then $c = 10$ since $c^2 = 36 + 64 = 100$.

Activity Synthesis

The purpose of this discussion is for students to compare their methods for finding the distance between two points. Select 2–3 previously identified students to share how they found the distance between their points.

If any groups figured out that the points lie on a circle, ask them to share how they did so. Then, ask students to find the distance between one (or both) of their points and the point $(2, -2)$ using the method they described in the second problem. If students calculations are correct, they should get a distance of 5 units, which is the radius of the circle.

Access for English Language Learners

Speaking, Listening: MLR7 Compare and Connect. Ask students to prepare a visual display of their work along with a written explanation for how to find the distance between any two coordinate pairs. As students investigate each other's work, ask them to share what worked well in a particular approach. Listen for and amplify the language students use to explain how they used the Pythagorean Theorem to calculate distances. Then encourage students to explain how the terms a and b in the Pythagorean Theorem are related to the coordinate pairs. For example, a could represent the vertical distance between the coordinate pairs and b could represent the horizontal distance between the coordinate pairs. This will foster students' meta-awareness and support constructive conversations as they compare strategies for finding the distance between coordinate pairs and make connections between the terms a and b in the Pythagorean Theorem and the coordinate pairs.
Design Principles(s): Cultivate conversation; Maximize meta-awareness

---

Lesson Synthesis

The purpose of the discussion is to check student understanding of how to use the Pythagorean Theorem to calculate distances between points in the coordinate plane. Ask:

- "How can you find the distance between points in the coordinate plane?" (If they are on the same horizontal or vertical line, we just subtract the coordinates that are different. If they aren't, we can construct a right triangle and use the Pythagorean Theorem.)

- "What advice would you give someone finding the distance between two points on the coordinate plane using the Pythagorean Theorem?" (Make a sketch!)

11.5 Lengths of Line Segments

Cool Down: 5 minutes
Addressing

- 8.G.B.8

> ### Student Task Statement
>
> Here are two line segments with lengths e and f. Calculate the exact values of e and f. Which is larger?
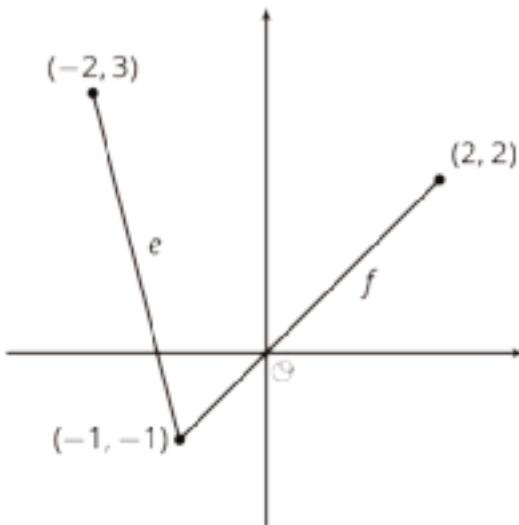
Student Response

The length of e is $\sqrt{17}$ units, and the length of f is $\sqrt{18}$ units. $e = \sqrt{1^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17}$. $f = \sqrt{3^2 + 3^2} = \sqrt{9 + 9} = \sqrt{18}$. Line segment f is longer.

> ## Student Lesson Summary
>
> We can use the Pythagorean Theorem to find the distance between any two points on the coordinate plane. For example, if the coordinates of point A are $(-2, -3)$, and the coordinates of point B are $(-8, 4)$, let's find the distance between them. This distance is also the length of line segment AB. It is a good idea to plot the points first.
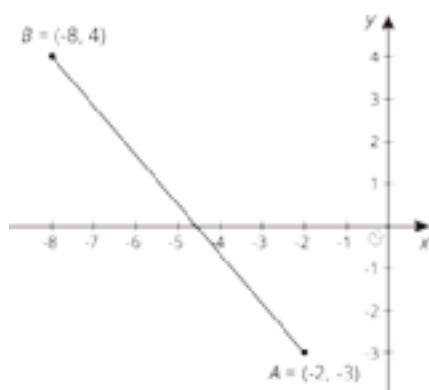
Think of the distance between A and B, or the length of segment AB, as the hypotenuse of a right triangle. The lengths of the legs can be deduced from the coordinates of the points.
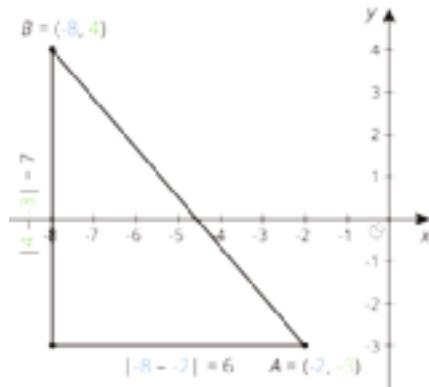
The length of the horizontal leg is 6, which can be seen in the diagram, but it is also the distance between the x-coordinates of A and B since $|\text{-}8 - \text{-}2| = 6$. The length of the vertical leg is 7, which can be seen in the diagram, but this is also the distance between the y-coordinates of A and B since $|4 - \text{-}3| = 7$.

Once the lengths of the legs are known, we use the Pythagorean Theorem to find the length of the hypotenuse, AB, which we can represent with c. Since c is a positive number, there is only one value it can take:

$$6^2 + 7^2 = c^2$$
$$36 + 49 = c^2$$
$$85 = c^2$$
$$\sqrt{85} = c$$

This length is a little longer than 9, since 85 is a little longer than 81. Using a calculator gives a more precise answer, $\sqrt{85} \approx 9.22$.

Lesson 11 Practice Problems

Problem 1

Statement

The right triangles are drawn in the coordinate plane, and the coordinates of their vertices are labeled. For each right triangle, label each leg with its length.
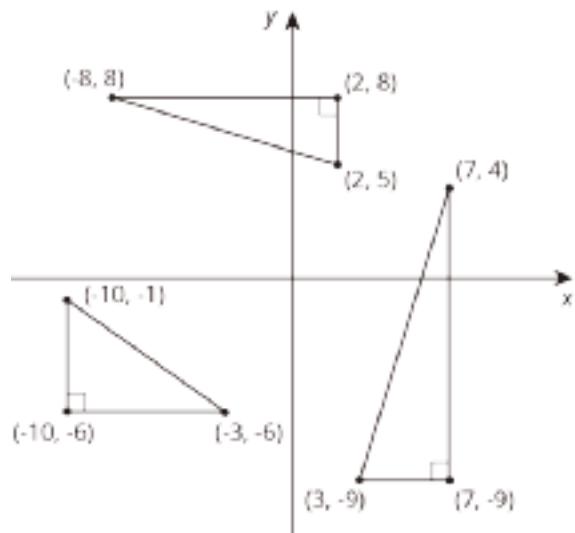
Solution
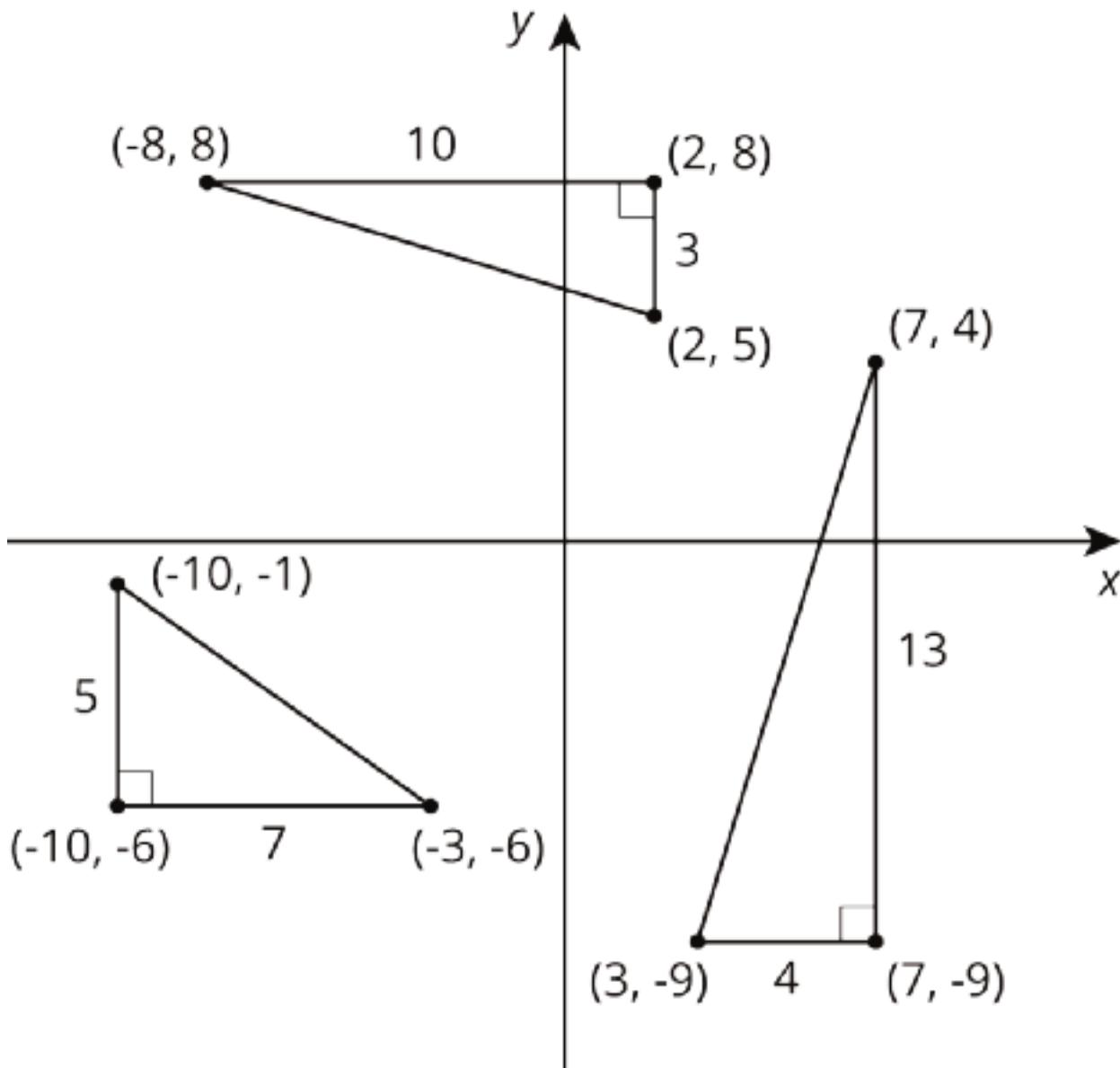
Problem 2

Statement

Find the distance between each pair of points. If you get stuck, try plotting the points on graph paper.

  a. $M = (0, -11)$ and $P = (0, 2)$

  b. $A = (0, 0)$ and $B = (-3, -4)$

  c. $C = (8, 0)$ and $D = (0, -6)$

Solution

    a. 13

    b. 5

    c. 10

Problem 3

Statement

    a. Find an object that contains a right angle. This can be something in nature or something that was made by humans or machines.

    b. Measure the two sides that make the right angle. Then measure the distance from the end of one side to the end of the other.

    c. Draw a diagram of the object, including the measurements.

    d. Use the Pythagorean Theorem to show that your object really does have a right angle.

Solution

Answers vary. A correct reponse will include a labeled diagram and the three measurements inserted into $a^2 + b^2 = c^2$ with enough work to show that the three measurements make this equation true. (Or close-enough, accounting for measurement error.)

(From Unit 8, Lesson 9.)

Problem 4

Statement

Which line has a slope of 0.625, and which line has a slope of 1.6? Explain why the slopes of these lines are 0.625 and 1.6.
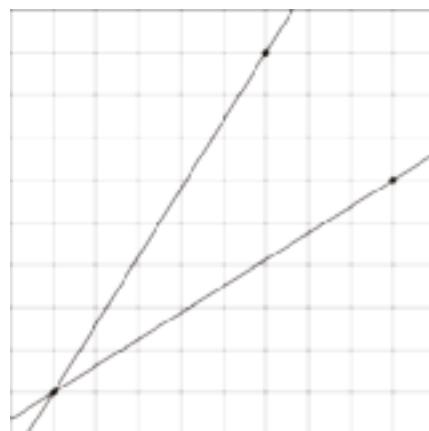
Solution

Slope of 0.625:
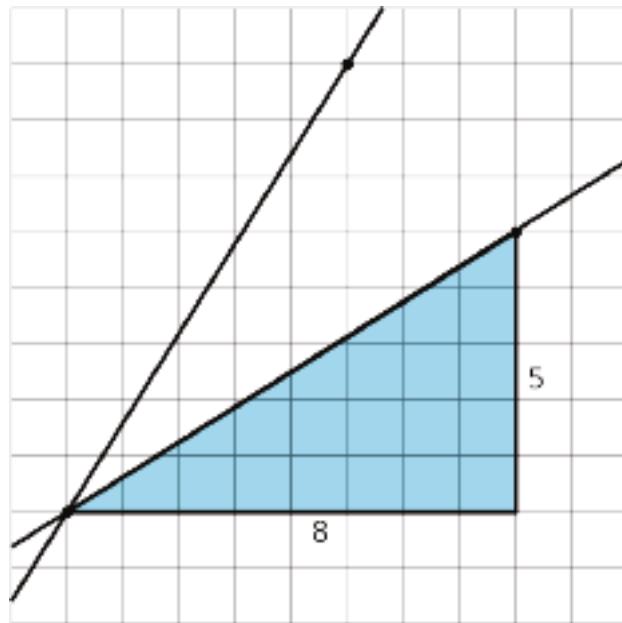
Slope of 1.6:
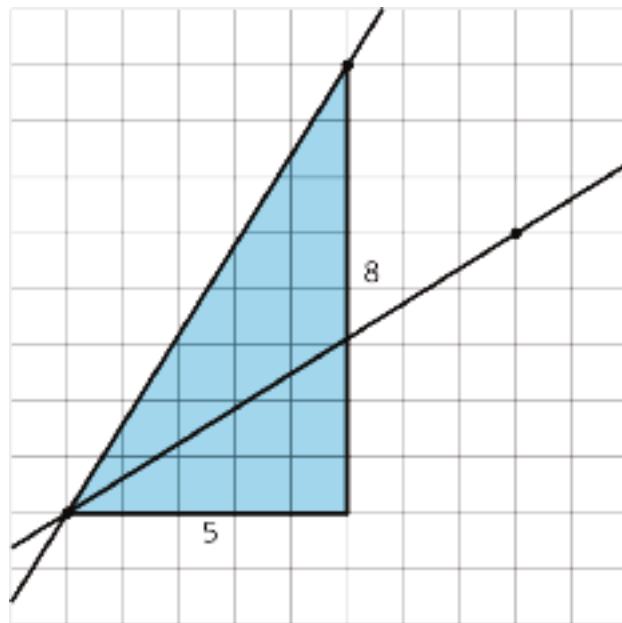
Construct triangles perpendicular to the axes whose hypotenuses lie on their line to find the slopes. The slopes of the lines are then the quotient of the length of the vertical edge by the length of the horizontal edge.

(From Unit 2, Lesson 10.)

Problem 5

Statement

Write an equation for the graph.

Solution

$y = 2x + 1.5$

(From Unit 3, Lesson 7.)

Section: Side Lengths and Volumes of Cubes

Lesson 12: Edge Lengths and Volumes

Goals

- Comprehend the term "cube root of a" (in spoken language) and the notation $\sqrt[3]{a}$ (in written language) to mean the side length of a cube whose volume is a cubic units.

- Coordinate representations of a cube root, including cube root notation, decimal representation, the side length of a cube of given volume, and a point on the number line.

Learning Targets

- I can approximate cube roots.

- I know what a cube root is.

- I understand the meaning of expressions like $\sqrt[3]{5}$.

Lesson Narrative

This is the first of two lessons in which students learn about cube roots. In the first lesson, students learn the notation and meaning of cube roots, e.g., $\sqrt[3]{8}$. In the warm-up, they order solutions to equations of the form $a^2 = 9$ and $b^3 = 8$. They already know about square roots, so in the discussion of the warm-up, they learn about the parallel definition of cube roots. In the following classroom activity, students use cube roots to find the edge length of a cube with given volume. A card sort activity helps them make connections between cube roots as values, as solutions to equations, and as points on the number line.

In the next lesson, students will find out that it is possible to find cube roots of negative numbers.

Alignments

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- MLR8: Discussion Supports

- Take Turns

- Think Pair Share

Required Materials

Pre-printed slips, cut from copies of the Instructional master

Required Preparation

Copies of the Instructional master for this lesson. Prepare 1 copy for every 3 students, and cut them up ahead of time.

> **Student Learning Goals**
>
> Let's explore the relationship between volume and edge lengths of cubes.

12.1 Ordering Squares and Cubes

Warm Up: 10 minutes
The purpose of this warm-up is to introduce students to **cube roots** during the discussion. This activity provides an opportunity to use cube root language and notation during the discussion. Students will explore the possibility of negative cube roots in the next lesson.

At first, students should be able to order the values of a, b, c, and f without a calculator. However, d and e will be easier with a calculator. Encourage students to use estimated values for d and e to order the values before using a calculator. As students work, identify students who use different strategies for ordering.

Addressing

- 8.EE.A.2

Instructional Routines

- Think Pair Share

Launch

Students in groups of 2. Give students 2–3 minutes to order the options to the best of their ability without a calculator, and to share their reasoning with a partner. Pause to discuss which are easy to order (likely f, b, c, and a) and which ones students are not sure about (likely d and e, which are between b and a). Then give students 1 minute with a calculator to finish ordering the options. Follow with a whole-class discussion.

> **Student Task Statement**
>
> Let a, b, c, d, e, and f be positive numbers.
>
> Given these equations, arrange a, b, c, d, e, and f from least to greatest. Explain your reasoning.

- $a^2 = 9$

- $b^3 = 8$

- $c^2 = 10$

- $d^3 = 9$

- $e^2 = 8$

- $f^3 = 7$

Student Response

The order is f, b, d, e, a, c. Sample response: We know that $a = \sqrt{9}$, which is equal to 3. Since $e = \sqrt{8}$, it is slightly less than 3. Since $c = \sqrt{10}$, it is slightly greater than 3. We know that $b = \sqrt[3]{8}$, which is equal to 2 because $2^3 = 8$. Since $d = \sqrt[3]{9}$, it is slightly greater than 2. Since $f = \sqrt[3]{7}$, it is slightly less than 2.

Activity Synthesis

Ask a student to share their order of a, b, c, d, e, and f from least to greatest. Record and display their responses for all to see. Ask the class if they agree or disagree. If the class agrees, select previously identified students to share their strategies for ordering the values. If the class is in disagreement, ask students to share their reasoning until an agreement is reached.

As students share, ask students which were the easiest to find and which were the hardest to find. Introduce students to cube root language and notation. Remind students that they previously learned that the equation $c^2 = 10$ has solution $c = \sqrt{10}$. Similarly, we can say that the equation $d^3 = 9$ has solution $d = \sqrt[3]{9}$. Ask students to write the solution to $f^3 = 7$ ($f = \sqrt[3]{7}$).

Finally, tell students that while square roots are a way to write the exact value of the side length of a square with a known area, cube roots are a way to write the exact value of the edge length of a cube with a known volume, which students will do in a following activity.

12.2 Name That Edge Length!

10 minutes
Now that students have been introduced to the notation for cube roots, the goal of this task is for students to connect their understanding of the relationship between the volume and edge lengths of cubes with that of cube roots.

Addressing

- 8.EE.A.2

Instructional Routines

- MLR8: Discussion Supports

Launch

Students should not use a calculator, and all values should be exact.
If students are unsure what to put for the "volume equation," remind them that the volume of a cube is the cube of the edge length.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, review activities from lesson 2 in which students explored square root by completing a table based on the side length and the area of a square.
Supports accessibility for: Social-emotional skills; Conceptual processing

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Student Task Statement

Fill in the missing values using the information provided:
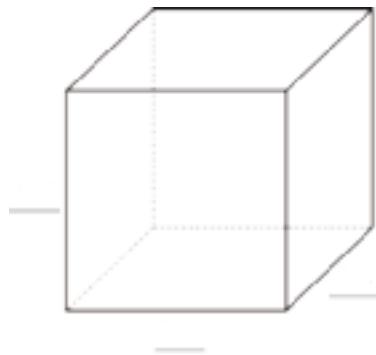
sides	volume	volume equation
	$27\,\text{in}^3$	
$\sqrt[3]{5}$		
		$(\sqrt[3]{16})^3 = 16$

Student Response

1. Each length is 3 in. Equation: $3^3 = 27$.

2. Volume is 5. Equation: $\sqrt[3]{5}^3 = 5$.

3. Each length is $\sqrt[3]{16}$. The volume is 16.

Are You Ready for More?

A cube has a volume of 8 cubic centimeters. A square has the same value for its area as the value for the surface area of the cube. How long is each side of the square?

Student Response

The cube has sides of length 2 and surface area 24, so the square has side lengths $\sqrt{24}$.

Activity Synthesis

Previously, students learned that knowing the area of a square is sufficient information for stating the exact length of the side of the square using square roots. The purpose of this discussion is for students to make that same connection with the volume of cubes and cube roots. The cube root of the volume of a cube represents the *exact* value of the edge length of a cube in ways that measuring or approximating by cubing values do not (except in special cases such as perfect cubes).

Discuss:

- "What integers is $\sqrt[3]{5}$ between?" ($\sqrt[3]{5}$ is between 1 and 2 because $1^3 = 1$ and $2^3 = 8$.)

- "What integers is $\sqrt[3]{16}$ between?" ($\sqrt[3]{16}$ is between 2 and 3 because $2^3 = 8$ and $3^3 = 27$.)

- "Name another volume for a cube with edge lengths between 2 and 3." (A cube with volume 26 has edge lengths between 2 and 3 since since $3^3 = 27$, meaning $\sqrt[3]{26}$ has a value slightly less than 3.)

12.3 Card Sort: Rooted in the Number Line

15 minutes
The purpose of this activity is for students to use rational approximations of irrational numbers and to match irrational numbers to equations they are solutions to. To do this, students sort cards into sets of three consisting of:

- A square or cube root value

- An equation of the form $x^2 = p$ or $x^3 = p$ that the value is a solution to

- A number line showing the location of the value

Identify groups using clear explanations for how they chose to arrange their sets of 3, particularly when matching the cards with a value plotted on a number line.

Addressing

- 8.EE.A.2
- 8.NS.A.2

Instructional Routines

- MLR8: Discussion Supports
- Take Turns

Launch

Arrange students in groups of 2–4. Students should not use a calculator for this activity. Tell students that they will be given cards to sort into sets of three. Distribute 27 pre-cut slips from the Instructional master to each group. Group work time followed by a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity by beginning with fewer cards. For example, give students a subset of the cards (using multiples of 3) to start with and introduce the remaining cards once students have completed their initial set of matches.
Supports accessibility for: Conceptual processing; Organization

---

Anticipated Misconceptions

Some students may mix up cube and square roots. Encourage these students to pay close attention to the notation. They may want to fill in the "2" for the square roots similar to how the "3" is used for cube roots.

> ### Student Task Statement
>
> Your teacher will give your group a set of cards. For each card with a letter and value, find the two other cards that match. One shows the location on a number line where the value exists, and the other shows an equation that the value satisfies. Be prepared to explain your reasoning.

Student Response

The Instructional master shows the solution to the matching.

Activity Synthesis

Select 3–4 previously identified groups to share one of their sets of three cards.

Also discuss:

- "What was a match that was hard to make?" (Two of the number lines have values between 4 and 5, and it took some extra reasoning to figure out that the one with the value closer to 4 was $\sqrt{18}$ while the one with the point closer to 5 was $\sqrt[3]{100}$.

- "If another class was going to sort these cards, what is something you would recommend they have or do that you found helpful?" (I would recommend they have a list of perfect squares and perfect cubes to help think about where the different roots are plotted on the number line.)

Access for English Language Learners

Speaking, Writing: MLR8 Discussion Supports. Display sentence frames to support students as they justify their reasoning for each match. For example, "The solution to _____ ($x^3 = 25$) is _____ ($\sqrt[3]{25}$) and it is between _____ (2 and 3) on the number line, because . . ." The helps students place extra attention on language used to engage in mathematical communication, without reducing the cognitive demand of the task. It also emphasizes the importance of justification.
Design Principle: Maximize meta-awareness

Lesson Synthesis

In this lesson, students learned about cube roots. Similar to square roots, cube roots can be thought of in the context of shapes; a cube with volume 64 cubic units has an edge length of 4 units, which is $\sqrt[3]{64}$, because $4^3 = 64$.

- "If a cube has a volume of 27 cubic inches, what is its side length?" (The cube with volume 27 cubic inches has an edge length of 3 inches, since $\sqrt[3]{27} = 3$.)

- "What is the solution to $x^3 = 150$, and what two integers would it fall between on a number line?" (The solution to $x^3 = 150$ is $\sqrt[3]{150}$ and it is between 5 and 6 on a number line, because $5^3 = 125$ and $6^3 = 216$.)

12.4 Roots of 36

Cool Down: 5 minutes
Addressing

- 8.NS.A.2

> **Student Task Statement**
>
> Plot $\sqrt{36}$ and $\sqrt[3]{36}$ on the number line.
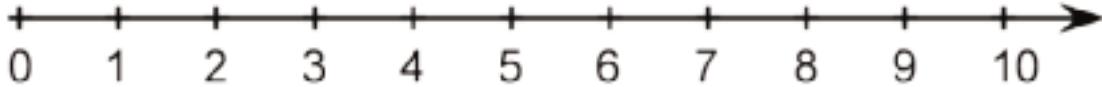
Student Response

$\sqrt{36}$ should be at 6, and $\sqrt[3]{36}$ should be between 3 and 4.

Student Lesson Summary

To review, the side length of the square is the square root of its area. In this diagram, the square has an area of 16 units and a side length of 4 units.

These equations are both true:
$$4^2 = 16$$
$$\sqrt{16} = 4$$
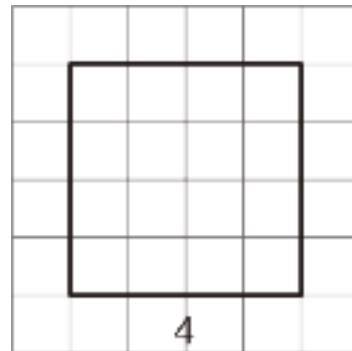
Now think about a solid cube. The cube has a volume, and the edge length of the cube is called the **cube root** of its volume. In this diagram, the cube has a volume of 64 units and an edge length of 4 units:

These equations are both true:

$$4^3 = 64$$

$$\sqrt[3]{64} = 4$$
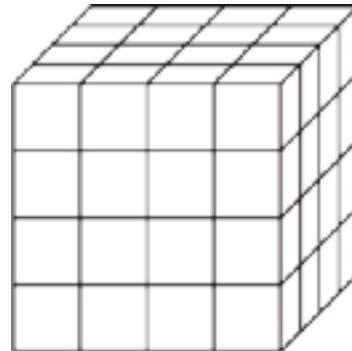
$\sqrt[3]{64}$ is pronounced "The cube root of 64." Here are some other values of cube roots:

$$\sqrt[3]{8} = 2, \text{ because } 2^3 = 8$$

$$\sqrt[3]{27} = 3, \text{ because } 3^3 = 27$$

$$\sqrt[3]{125} = 5, \text{ because } 5^3 = 125$$

Glossary

- cube root

Unit 8 Lesson 12

Lesson 12 Practice Problems
Problem 1
Statement

a. What is the volume of a cube with a side length of

    i. 4 centimeters?

    ii. $\sqrt[3]{11}$ feet?

    iii. s units?

b. What is the side length of a cube with a volume of

    i. 1,000 cubic centimeters?

    ii. 23 cubic inches?

    iii. v cubic units?

Solution

a.

    i. 64 cubic centimeters

    ii. 11 cubic feet

    iii. s^3 cubic units

b.

    i. 10 cm

    ii. $\sqrt[3]{23}$ inches

    iii. $\sqrt[3]{v}$ units

Problem 2
Statement

Write an equivalent expression that doesn't use a cube root symbol.

a. $\sqrt[3]{1}$

b. $\sqrt[3]{216}$

c. $\sqrt[3]{8000}$

d. $\sqrt[3]{\frac{1}{64}}$

e. $\sqrt[3]{\frac{27}{125}}$

f. $\sqrt[3]{0.027}$

g. $\sqrt[3]{0.000125}$

Solution

a. 1

b. 6

c. 20

d. $\frac{1}{4}$

e. $\frac{3}{5}$

f. 0.3

g. 0.05

Problem 3

Statement

Find the distance between each pair of points. If you get stuck, try plotting the points on graph paper.

a. $X = (5, 0)$ and $Y = (\text{-}4, 0)$

b. $K = (\text{-}21, \text{-}29)$ and $L = (0, 0)$

Solution

a. 9

b. $\sqrt{1282}$

(From Unit 8, Lesson 11.)

Problem 4

Statement

Here is a 15-by-8 rectangle divided into triangles. Is the shaded triangle a right triangle? Explain or show your reasoning.
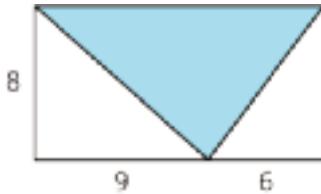
Solution

No, it is not. Use the Pythagorean Theorem to find the length of the interior sides of the triangle: the lengths are $\sqrt{145}$ and 10. The longest side is 15, the length of the rectangle. Now check whether this triangle's side lengths make $a^2 + b^2 = c^2$. Because $145 + 100 = 245$, not 225, the converse of the Pythagorean Theorem states this triangle is not a right triangle.

(From Unit 8, Lesson 9.)

Problem 5

Statement

Here is an equilateral triangle. The length of each side is 2 units. A height is drawn. In an equilateral triangle, the height divides the opposite side into two pieces of equal length.
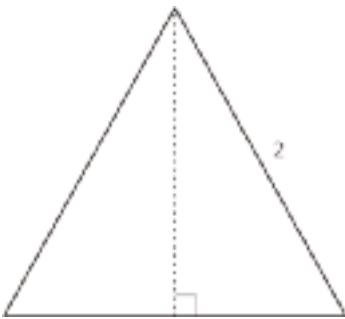
a. Find the exact height.

b. Find the area of the equilateral triangle.

c. (Challenge) Using x for the length of each side in an equilateral triangle, express its area in terms of x.

Solution

a. $\sqrt{3}$ units

b. $\sqrt{3}$ units^2

c. The area is $\frac{x^2\sqrt{3}}{4}$ units^2

(From Unit 8, Lesson 10.)

Unit 8 Lesson 12

Lesson 13: Cube Roots

Goals

- Determine the whole numbers that a cube root lies between, and explain (orally) the reasoning.

- Generalize a process for approximating the value of a cube root, and justify (orally and in writing) that if $x^3 = a$, then $x = \sqrt[3]{a}$.

Learning Targets

- When I have a cube root, I can reason about which two whole numbers it is between.

Lesson Narrative

In this lesson, students continue to work with cube roots, moving away from the geometric interpretation in favor of the algebraic definition. They approximate cube roots and locate them on the number line. They see their first negative cube root, and locate it on the number line.

Alignments

Addressing

- 8.EE.A.2: Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

- 8.NS.A.2: Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- MLR7: Compare and Connect

- MLR8: Discussion Supports

- Think Pair Share

- True or False

> **Student Learning Goals**
>
> Let's compare cube roots.

13.1 True or False: Cubed

Warm Up: 5 minutes

The purpose of this warm-up is for students to analyze symbolic statements about cube roots and decide if they are true or not based on the meaning of the cube root symbol.

Addressing

- 8.EE.A.2

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

> **Student Task Statement**
>
> Decide if each statement is true or false.
>
> $$\left(\sqrt[3]{5}\right)^3 = 5$$
>
> $$\left(\sqrt[3]{27}\right)^3 = 3$$
>
> $$7 = \left(\sqrt[3]{7}\right)^3$$
>
> $$\left(\sqrt[3]{10}\right)^3 = 1{,}000$$
>
> $$\left(\sqrt[3]{64}\right) = 2^3$$

Student Response

True, false, true, false, false

Activity Synthesis

Poll students on their response for each problem. Record and display their responses for all to see. If all students agree, ask 1 or 2 students to share their reasoning. If there is disagreement, ask students to share their reasoning until an agreement is reached.

13.2 Cube Root Values

10 minutes

The purpose of this activity is for students to think about cube roots in relation to the two whole number values they are closest to. Students are encouraged to use the fact that $\sqrt[3]{a}$ is a solution to the equation $x^3 = a$. Students can draw a number line if that helps them reason about the

magnitude of the given cube roots, but this is not required. However students reason, they need to explain their thinking (MP3).

Monitor students multiplying non-integers by hand to try and approximate. While this isn't what the problem is asking for, their work could be used to think about which integer the square root is closest to and should be brought up during the whole-class discussion.

Addressing

- 8.EE.A.2

- 8.NS.A.2

Instructional Routines

- MLR8: Discussion Supports

- Think Pair Share

Launch

Do not give students access to calculators. Students in groups of 2. Two minutes of quiet work time, followed by partner then whole-class discussion.

Access for Students with Disabilities

Representation: *Internalize Comprehension.* Activate or supply background knowledge. Provide students with access to a number line that includes rational numbers to support information processing.
Supports accessibility for: Visual-spatial processing; Organization

---

> **Student Task Statement**
>
> What two whole numbers does each cube root lie between? Be prepared to explain your reasoning.
>
> 1. $\sqrt[3]{5}$
>
> 2. $\sqrt[3]{23}$
>
> 3. $\sqrt[3]{81}$
>
> 4. $\sqrt[3]{999}$

Student Response

1. 1 and 2. $1^3 = 1$ and $2^3 = 8$, so $\sqrt[3]{5}$ is between 1 and 2.

2. 2 and 3. $2^3 = 8$ and $3^3 = 27$, so $\sqrt[3]{23}$ is between 2 and 3.

3. 4 and 5. $4^3 = 64$ and $5^3 = 125$, so $\sqrt[3]{81}$ is between 4 and 5.

4. 9 and 10. $10^3 = 1,000$ so $\sqrt[3]{999}$ is a little bit less than 10.

Activity Synthesis

Discuss:

- "What strategy did you use to figure out the two whole numbers?"
  (I made a list of perfect cubes and then found which two the number was between.)

- "How can we write one of these answers using inequality symbols?" (For example, $2 < \sqrt[3]{23} < 3$.)

Access for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. Call on students to use mathematical language to restate and/or revoice the strategy (or strategies) presented. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language that describes strategies to figure out which two whole numbers each cube root lies between.
Design Principle(s): Support sense-making; Maximize meta-awareness

13.3 Solutions on a Number Line

10 minutes
The purpose of this activity is for students to use rational approximations of irrational numbers to place both rational and irrational numbers on a number line, and to reinforce the definition of a cube root as a solution to the equation of the form $x^3 = a$. This is also the first time that students have thought about negative cube roots.

Addressing

- 8.EE.A.2

- 8.NS.A.2

Instructional Routines

- MLR7: Compare and Connect

- Think Pair Share

Launch

No access to calculators. Students in groups of 2. Two minutes of quiet work time, followed by partner then whole-class discussion.

Student Task Statement

The numbers x, y, and z are positive, and:

$$x^3 = 5 \qquad\qquad y^3 = 27 \qquad\qquad z^3 = 700$$
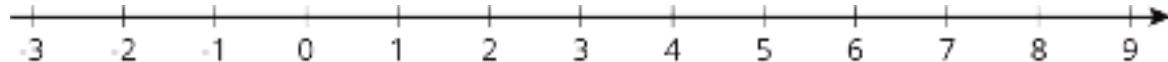
1. Plot x, y, and z on the number line. Be prepared to share your reasoning with the class.

2. Plot $-\sqrt[3]{2}$ on the number line.

Student Response

The point $\sqrt[3]{5}$ should be between 1 and 2, the point 3 should be labeled, the point $\sqrt[3]{700}$ should be between 8 and 9, and the point $-\sqrt[3]{2}$ should be between -1 and -2.

Are You Ready for More?

Diego knows that $8^2 = 64$ and that $4^3 = 64$. He says that this means the following are all true:

- $\sqrt{64} = 8$
- $\sqrt[3]{64} = 4$
- $\sqrt{-64} = -8$
- $\sqrt[3]{-64} = -4$

Is he correct? Explain how you know.

Student Response

Three of Diego's statements are correct, but $\sqrt{-64}$ does not equal -8 because $(-8)^2$ is 64, not -64.

Activity Synthesis

Display the number line from the activity for all to see. Select groups to share how they chose to place values onto the number line. Place the values on the displayed number line as groups share, and after each placement poll the class to ask if students used the same reasoning or different reasoning. If any students used different reasoning, invite them to share with the class.

Conclude the discussion by asking students to share how they placed $-\sqrt[3]{2}$ on the number line.

Access for English Language Learners

Representing, Conversing, Listening: MLR7 Compare and Connect. As students prepare their work before the discussion, look for approaches that favor the algebraic definition, and approaches that use the geometric interpretation. Call students' attention to the different ways students describe how they operated on numbers cubed to determine the approximate cube roots, and to the different ways these operations are made visible in each representation (e.g., using the fact that $\sqrt[3]{a}$ is a solution to the equation $x^3 = a$, and finding rational approximations of the irrational numbers to plot the cube roots on the number line). Emphasize language used to make sense of strategies reinforcing the definition of a cube root as a solution to the equation of the form $x^3 = a$.
Design Principle(s): Maximize meta-awareness; Support sense-making

Lesson Synthesis

The purpose of this discussion is to reinforce the definition of a cube root.

- "What is the solution to the equation $a^3 = 47$?" ($\sqrt[3]{47}$)

- "What is the solution to the equation $a^3 = 64$?" ($\sqrt[3]{64} = 4$)

- "What is the solution to the equation $a^3 = -64$?" ($\sqrt[3]{-64} = -4$)

- "How can we plot cube roots on the number line?" (Find the two whole numbers they lie between, and determine the approximate location between them.)

13.4 Different Types of Roots

Cool Down: 5 minutes
Addressing

- 8.EE.A.2

Lin is asked to place a point on a number line to represent the value of $\sqrt[3]{49}$ and she writes:
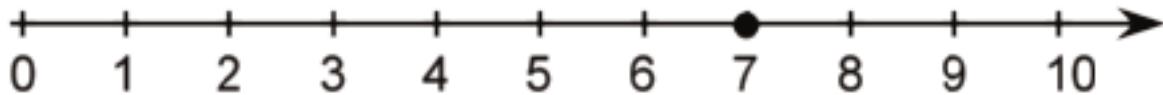
Where should $\sqrt[3]{49}$ actually be on the number line? How do you think Lin got the answer she did?

Student Response

Answers vary. Sample response: $\sqrt[3]{49}$ should be between 3 and 4 on the number line. I think Lin placed the point at $\sqrt{49}$ because she forgot it was a cube root instead of a square root.

Student Lesson Summary

Remember that square roots of whole numbers are defined as side lengths of squares. For example, $\sqrt{17}$ is the side length of a square whose area is 17. We define cube roots similarly, but using cubes instead of squares. The number $\sqrt[3]{17}$, pronounced "the cube root of 17," is the edge length of a cube which has a volume of 17.

We can approximate the values of cube roots by observing the whole numbers around it and remembering the relationship between cube roots and cubes. For example, $\sqrt[3]{20}$ is between 2 and 3 since $2^3 = 8$ and $3^3 = 27$, and 20 is between 8 and 27. Similarly, since 100 is between 4^3 and 5^3, we know $\sqrt[3]{100}$ is between 4 and 5. Many calculators have a cube root function which can be used to approximate the value of a cube root more precisely. Using our numbers from before, a calculator will show that $\sqrt[3]{20} \approx 2.7144$ and that $\sqrt[3]{100} \approx 4.6416$.

Also like square roots, most cube roots of whole numbers are irrational. The only time the cube root of a number is a whole number is when the original number is a perfect cube.

Lesson 13 Practice Problems

Problem 1

Statement

Find the positive solution to each equation. If the solution is irrational, write the solution using square root or cube root notation.

   a. $t^3 = 216$

   b. $a^2 = 15$

   c. $m^3 = 8$

   d. $c^3 = 343$

   e. $f^3 = 181$

Solution

   a. $t = 6$

   b. $a = \sqrt{15}$

   c. $m = 2$

   d. $c = 7$

   e. $f = \sqrt[3]{181}$

Problem 2

Statement

For each cube root, find the two whole numbers that it lies between.

   a. $\sqrt[3]{11}$

   b. $\sqrt[3]{80}$

   c. $\sqrt[3]{120}$

   d. $\sqrt[3]{250}$

Solution

   a. 2 and 3

   b. 4 and 5

c. 4 and 5

d. 6 and 7

Problem 3

Statement

Order the following values from least to greatest:

$$\sqrt[3]{530}, \ \sqrt{48}, \ \pi, \ \sqrt{121}, \ \sqrt[3]{27}, \ \frac{19}{2}$$

Solution

$\sqrt[3]{27}, \ \pi, \ \sqrt{48}, \ \sqrt[3]{530}, \ \frac{19}{2}, \ \sqrt{121}$

Problem 4

Statement

Select **all** the equations that have a solution of $\frac{2}{7}$:

A. $x^2 = \frac{2}{7}$

B. $x^2 = \frac{4}{14}$

C. $x^2 = \frac{4}{49}$

D. $x^3 = \frac{6}{21}$

E. $x^3 = \frac{8}{343}$

F. $x^3 = \frac{6}{7}$

Solution

["C", "E"]

Problem 5

Statement

The equation $x^2 = 25$ has *two* solutions. This is because both $5 \cdot 5 = 25$, and also
$-5 \cdot -5 = 25$. So, 5 is a solution, and also -5 is a solution. But! The equation $x^3 = 125$ only has
one solution, which is 5. This is because $5 \cdot 5 \cdot 5 = 125$, and there are no other numbers you
can cube to make 125. (Think about why -5 is not a solution!)

Find all the solutions to each equation.

    a. $x^3 = 8$

    b. $\sqrt[3]{x} = 3$

    c. $x^2 = 49$

    d. $x^3 = \frac{64}{125}$

Solution

    a. 2

    b. 27

    c. 7 and -7

    d. $\frac{4}{5}$

Problem 6

Statement

Find the value of each variable, to the nearest tenth.

a.
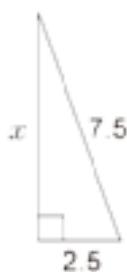
b.

c.

Solution

a. $x \approx 7.1$

b. $f \approx 5.4$

c. $d \approx 15.1$

(From Unit 8, Lesson 8.)

Problem 7

Statement

A standard city block in Manhattan is a rectangle measuring 80 m by 270 m. A resident wants to get from one corner of a block to the opposite corner of a block that contains a park. She wonders about the difference between cutting across the diagonal through the park compared to going around the park, along the streets. How much shorter would her walk be going through the park? Round your answer to the nearest meter.

Solution

The walk through the park is about 68 m shorter than the walk around the park. Along the streets: $80 + 270 = 350$. Along the diagonal: solve $80^2 + 270^2 = x^2$ and get approximately 282. $350 - 282 = 68$.

(From Unit 8, Lesson 10.)

Unit 8 Lesson 13

Section: Decimal Representation of Rational and Irrational Numbers

Lesson 14: Decimal Representations of Rational Numbers

Goals

- Comprehend that a rational number is a fraction or its opposite, and that a rational number can be represented with a decimal expansion that "repeats" or "terminates".

- Represent rational numbers as equivalent decimals and fractions, and explain (orally) the solution method.

Learning Targets

- I can write a fraction as a repeating decimal.

- I understand that every number has a decimal expansion.

Lesson Narrative

In the last two lessons in this unit, students explore decimal representations of rational and irrational numbers. The zooming number line representation used in these lessons supports students' understanding of place value and helps them form mental images of the two different ways a decimal expansion may go on forever (depending on whether the number is rational or irrational).

This first lesson explores the different forms of rational numbers. The warm-up reviews the idea of rational numbers as fractions of the form $\frac{a}{b}$ using tape diagrams. The first classroom activity, which is optional, continues with the same fractions by writing them as decimals.

In the second classroom activity students work with a variety of rational numbers written in different forms, including fractions, decimals and square roots. They see that it is not the symbols used to write a number that makes it rational but rather the fact that it can be rewritten in the form $\frac{a}{b}$ where a and b are integers, e.g. $\sqrt{\frac{1}{9}} = \frac{1}{3}$.

In the last activity students explore the decimal expansion of $\frac{2}{11}$. They use long division with repeated reasoning (MP8) to find that $\frac{2}{11} = 0.1818\ldots$. Students realize that they could easily keep zooming in on $\frac{2}{11}$ because of the pattern of alternating between the intervals $\frac{1}{10} - \frac{2}{10}$ and $\frac{8}{10} - \frac{9}{10}$ of the previous line. The goal is for students to notice and appreciate the predictability of repeating decimals and see how that connects with the $\frac{a}{b}$ structure.

By the end of this lesson students have seen that rational numbers can have decimal representations that terminate or that eventually repeat. This begs the question if there are

numbers with non-terminating decimal representations that do not repeat. This leads into the next lesson.

Alignments

Building On

- 7.NS.A.2.d: Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Addressing

- 8.EE.A: Work with radicals and integer exponents.

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

- 8.NS.A.1: Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Building Towards

- 8.NS.A: Know that there are numbers that are not rational, and approximate them by rational numbers.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR1: Stronger and Clearer Each Time

- MLR3: Clarify, Critique, Correct

- Notice and Wonder

- Think Pair Share

> **Student Learning Goals**
>
> Let's learn more about how rational numbers can be represented.

14.1 Notice and Wonder: Shaded Bars

Warm Up: 5 minutes
The purpose of this warm-up is for students to review what rational numbers are—fractions and their opposites. In earlier lessons, students explored square and cube roots and now return to strictly rational numbers. While students do not need to be able to recite a definition for rational numbers, this type of diagram is one students are likely familiar with and gives students an opportunity to use related language such as part, whole, halves, 1 out of 8 parts, one eighth, etc. After students work with different representations of rational numbers in upcoming lessons, they will expand their definition and understanding of irrational numbers.

**Building Towards**

- 8.NS.A

Instructional Routines

- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

> **Student Task Statement**
>
> What do you notice? What do you wonder?
>
>
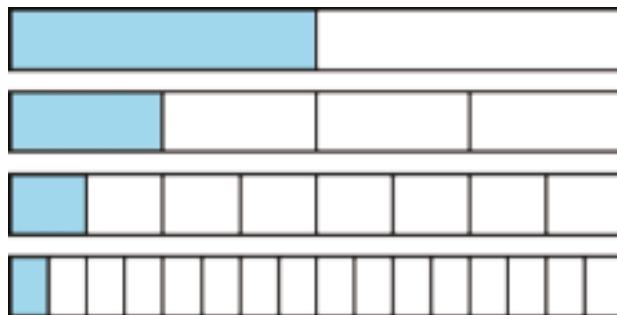
Student Response

Answers vary. Possible responses:

Students may notice:

- The highlighted areas can be written as fractions, parts of wholes.
- Each bar is broken into equal-sized parts.
- They look like tape diagrams broken into halves, fourths, eighths, sixteenths, etc.
- The whole could be any value, not necessarily 1.
- If we knew the whole, we could find the value of any of the highlighted areas by finding half of it, a fourth of it, etc.

Students may wonder:

- What is the value of the whole?
- What is the value of each highlighted area?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time.

14.2 Halving the Length

Optional: 10 minutes

The purpose of this optional activity is for students to see what the figure in the warm up didn't explore: as 1 is successively divided in half, the number of digits needed to accurately describe the resulting value increases. Students likely have memorized that a half is 0.5 and a fourth is 0.25, but what is half of 0.25? How can you calculate it? Use this activity if you think your students need a reminder about place value or long division with decimals.

Addressing

- 8.NS.A

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Do not provide access to calculators. Give students 2 minutes quiet work time followed by partner then whole-class discussion.

Access for English Language Learners

Writing, Speaking: MLR1 Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their response to: "How do the decimal representations change?" Ask each student to meet with 2–3 other partners for feedback. Display prompts for feedback that will help students strengthen their ideas and clarify their language. For example, "Can you explain how. . . ?", "You should expand on. . . .", etc. Give students 1–2 minutes to revise their writing based on the feedback they received.
Design Principle(s): Optimize output (for generalization)

Student Task Statement

Here is a number line from 0 to 1.

1. Mark the midpoint between 0 and 1. What is the decimal representation of that number?

2. Mark the midpoint between 0 and the newest point. What is the decimal representation of that number?

3. Repeat step two. How did you find the value of this number?

4. Describe how the value of the midpoints you have added to the number line keep changing as you find more. How do the decimal representations change?

Student Response

1. The decimal representation of the midpoint between 0 and 1 is 0.5.

2. The decimal representation of the midpoint between 0 and 0.5 is 0.25.

3. The decimal representation of the midpoint between 0 and .025 is 0.125. Answers vary. Sample response: I divided 0.25 by 2 using long division.

4. Answers vary. Sample response: the value of the midpoint keeps halving while the number of digits needed to write the value of the midpoint increases by 1 each time.

Activity Synthesis

Discuss:

- "How can you use long division to answer the third problem?"

- "What are these values when written in fraction notation? How do you know?"

Access for Students with Disabilities

Representation: *Develop Language and Symbols*. Create a display of important terms and vocabulary. Include the following terms and maintain the display for reference throughout the unit: decimal form and fraction form. Provide step by step instructions for how to convert fractions to decimals and vice versa. Be sure to provide an example using long division.
Supports accessibility for: *Memory; Language*

14.3 Recalculating Rational Numbers

20 minutes
The purpose of this task is for students to rewrite rational numbers with terminating decimal expansions in fraction form and fractions with terminating decimal expansions as decimals. This activity is the first of a series of three in which students rewrite numbers in different ways, supporting their understanding of what rational and irrational numbers are and how they can be represented.

Monitor for students who write $\frac{1}{5}$ and $\frac{2}{10}$ for 0.2.

Addressing

- 8.EE.A

- 8.NS.A

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR3: Clarify, Critique, Correct

Launch

Arrange students in groups of 2. Do not provide access to calculators.

Remind students that all rational numbers are fractions and their opposites, and they can all be written in the form $\frac{a}{b}$ where a and b are integers (with $b \neq 0$). In fact, there are many equivalent fractions that represent a single rational number. For example, 5 is equivalent to $\frac{10}{2}$ and $\frac{15}{3}$.

Students should complete the problems individually, then compare with their partners and come to resolutions over any differences. Repeat this process for the second problem.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills, check in with students within the first 2-3 minutes of work time. Look for students who find equivalent ways to represent fractions and decimals. Check for precision in calculations and encourage partners to seek clarification in their discussions.
Supports accessibility for: Memory; Organization

Access for English Language Learners

Writing: MLR3 Clarify, Critique, Correct. Present an incorrect statement that reflects a possible misunderstanding from the class. For example, finding the decimal representations of rational numbers, an incorrect statement is: "For finding the decimal representation of $\frac{3}{8}$, divide 8 by 3 using long division ". Prompt students to identify the error, and then write a correct version. This helps students evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Maximize meta-awareness

Anticipated Misconceptions

Some students may write that $0.333 = \frac{1}{3}$. If so, ask them to check their work by calculating the decimal representation of $\frac{1}{3}$.

Student Task Statement

1. Rational numbers are fractions and their opposites. All of these numbers are rational numbers. Show that they are rational by writing them in the form $\frac{a}{b}$ or $-\frac{a}{b}$.

    a. 0.2

    b. $-\sqrt{4}$

    c. 0.333

    d. $\sqrt[3]{1000}$

    e. -1.000001

    f. $\sqrt{\frac{1}{9}}$

2. All rational numbers have decimal representations, too. Find the decimal representation of each of these rational numbers.

    a. $\frac{3}{8}$

    b. $\frac{7}{5}$

    c. $\frac{999}{1000}$

    d. $\frac{111}{2}$

    e. $\sqrt[3]{\frac{1}{8}}$

Student Response

1. Answers vary. Sample responses: $\frac{2}{10}, -\frac{2}{1}, \frac{333}{1000}, \frac{10}{1}, -\frac{1000001}{1000000}, \frac{1}{3}$

2. 0.375, 1.4, 0.999, 55.5, 0.5

Activity Synthesis

The purpose of this discussion is to highlight different strategies for rewriting rational numbers in different forms. Select previously identified students to share their solutions, including one student who wrote $\frac{1}{5}$ for 0.2 and another who wrote $\frac{2}{10}$. For the problems with roots, the values were

purposefully chosen to emphasize to students that just because a number is written with a square or cube root does not mean it is not rational.

For 0.2, draw a number line with the numbers 0, 1, and 2 with plenty of space between the integers. Subdivide the segment from 0 to 1 into 5 equal pieces, and then ask where to plot $\frac{1}{5}$. Then label $\frac{1}{5}$ and ask how we can see that this is $\frac{2}{10}$. (Subdivide each fifth into two equal pieces—now each one is $\frac{1}{10}$.) Then label the point $\frac{2}{10}$ and 0.2.
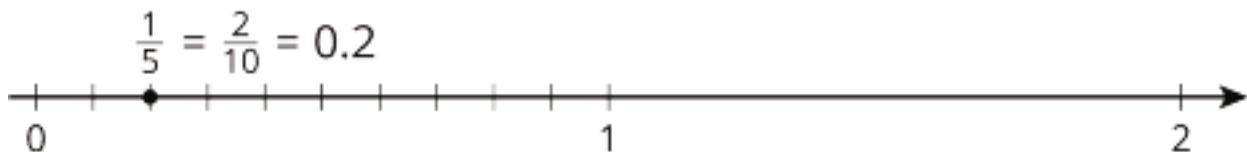
Remind students that the point corresponds to a rational number, and we have a lot of different ways we can represent that number.

14.4 Zooming In On $\frac{2}{11}$

10 minutes
This activity continues with the work from the previous activity by examining a decimal representation of a rational number that, when written as a decimal, repeats forever. The purpose of this activity is for students to use repeated reasoning with division to justify to themselves that $0.1818\ldots$ repeats the digits 1 and 8 forever (MP8).

Building On

- 7.NS.A.2.d

Addressing

- 8.NS.A.1

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Do not provide access to calculators. Tell students that now we are going to think about the decimal representation of $\frac{2}{11}$. Remind students that they should be prepared to explain their reasoning for each step in the activity.

If you think students need a reminder of how the zooming number lines work, which were used in an earlier unit, demonstrate how to show where $\frac{1}{8}$ is using 3 number lines, starting with one from 0 to 1.

Students in groups of 2. Give 2 minutes for students to begin individually and then ask students to discuss their work with their partner and resolve any differences. Follow with a whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Begin with a physical demonstration of how to use a zooming number line using a simpler fraction to support connections between new situations and prior understandings. Consider using these prompts: "What does the second number line represent?", or "How can you relate this example to the fraction mentioned in the task statement?"
Supports accessibility for: Conceptual processing; Visual-spatial processing

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Student Task Statement
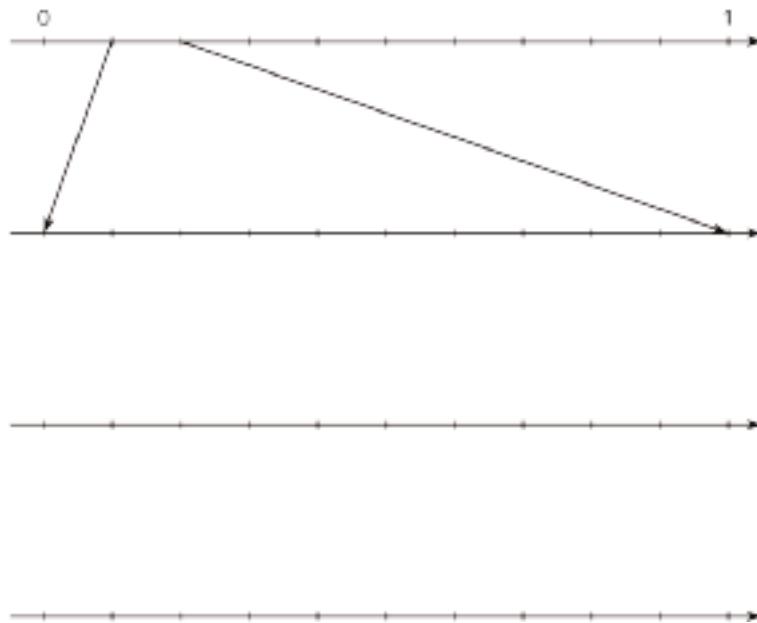
1. On the topmost number line, label the tick marks. Next, find the first decimal place of $\frac{2}{11}$ using long division and estimate where $\frac{2}{11}$ should be placed on the top number line.

2. Label the tick marks of the second number line. Find the next decimal place of $\frac{2}{11}$ by continuing the long division and estimate where $\frac{2}{11}$ should be placed on the second number line. Add arrows from the second to the third number line to zoom in on the location of $\frac{2}{11}$.

3. Repeat the earlier step for the remaining number lines.

4. What do you think the decimal expansion of $\frac{2}{11}$ is?

Student Response

1. A number line from 0 to 1, each tick mark increasing by 0.1. A point for $\frac{2}{11}$ is located at 0.1.

2. A number line from 0.1 to 0.2, each tick mark increasing by 0.01. A point for $\frac{2}{11}$ is located at 0.18.

3. A number line from 0.18 to 0.19, each tick mark increasing by 0.001. A point for $\frac{2}{11}$ is located at 0.181.

4. Answers vary. Sample response: I think the decimal expansion of $\frac{2}{11}$ is $0.1818\ldots$

> **Are You Ready for More?**
>
> Let $x = \frac{25}{11} = 2.272727\ldots$ and $y = \frac{58}{33} = 1.75757575\ldots$
>
> For each of the following questions, first decide whether the fraction or decimal representations of the numbers are more helpful to answer the question, and then find the answer.
>
> - Which of x or y is closer to 2?
>
> - Find x^2.

Student Response

- y is closer to 2. From the decimal expansion, we can see that y is less than .25 units away from 2, but x is more than .25 units away. We could also this from the fraction representation, though it seems slightly more time-consuming.

- $x^2 = \frac{625}{121}$. This is an easy calculation from the fraction representation, but would be very challenging from the decimal representation. Indeed, the decimal expansion for $\frac{625}{121}$ is $5.\overline{165289256198347107438 0}$.

Activity Synthesis

The purpose of this discussion is to explicitly state the repeated reasoning and successive approximation used to calculate each digit after the decimal point for $\frac{2}{11}$. Select one or two students to share their reasoning about the decimal representation of $\frac{2}{11}$.

Tell students that we often find that the decimal representation of a rational number repeats like this, and we have a special notation to represent it. Then write

$$0.18181818181818181818\ldots = 0.\overline{18}$$

Lesson Synthesis

This lesson was about rational numbers and their decimal representations.

- "What is a rational number?" (A fraction (or its equivalent) or its opposite.)

- "What do we know about the decimal expansion of rational numbers?" (The decimal expansion always eventually *repeats*. Sometime the repeating part is zeros, like 0.250000 . . . in which case we can also say it *terminates*.)

14.5 An Unknown Rational Number

Cool Down: 5 minutes
Addressing

- 8.NS.A

> **Student Task Statement**
>
> Explain how you know that -3.4 is a rational number.

Student Response

Answer vary. Sample response: $-3.4 = -\frac{34}{10}$, so it is the opposite of a fraction and therefore rational.

> **Student Lesson Summary**
>
> We learned earlier that rational numbers are a fraction or the opposite of a fraction. For example, $\frac{3}{4}$ and $-\frac{5}{2}$ are both rational numbers. A complicated-looking numerical expression can also be a rational number as long as the value of the expression is a positive or negative fraction. For example, $\sqrt{64}$ and $-\sqrt[3]{\frac{1}{8}}$ are rational numbers because $\sqrt{64} = 8$ and
> $-\sqrt[3]{\frac{1}{8}} = -\frac{1}{2}$.
>
> Rational numbers can also be written using decimal notation. Some have finite decimal expansions, like 0.75, -2.5, or -0.5. Other rational numbers have infinite decimal expansions, like 0.7434343 . . . where the 43s repeat forever. To avoid writing the **repeating** part over and over, we use the notation $0.7\overline{43}$ for this number. The bar over part of the expansion tells us the part which is to repeat forever.
>
> A decimal expansion of a number helps us plot it accurately on a number line divided into tenths. For example, $0.7\overline{43}$ should be between 0.7 and 0.8. Each further decimal digit increases the accuracy of our plotting. For example, the number $0.7\overline{43}$ is between 0.743 and 0.744.

> **Glossary**
>
> - repeating decimal

Lesson 14 Practice Problems
Problem 1
Statement

Andre and Jada are discussing how to write $\frac{17}{20}$ as a decimal.

Andre says he can use long division to divide 17 by 20 to get the decimal.

Jada says she can write an equivalent fraction with a denominator of 100 by multiplying by $\frac{5}{5}$, then writing the number of hundredths as a decimal.

    a. Do both of these strategies work?

    b. Which strategy do you prefer? Explain your reasoning.

    c. Write $\frac{17}{20}$ as a decimal. Explain or show your reasoning.

Solution

    a. Yes, both strategies are effective.

    b. Answers vary. Sample responses:
- I prefer Jada's method because I can calculate it mentally.

- I prefer Andre's method because it always works, even if the denominator is not a factor of 100.

    c. 0.85. Explanations vary. Sample explanation: $\frac{17}{20} \cdot \frac{5}{5} = \frac{85}{100}$, so $\frac{17}{20}$ equals 0.85.

Problem 2
Statement

Write each fraction as a decimal.

    a. $\sqrt{\frac{9}{100}}$

    b. $\frac{99}{100}$

    c. $\sqrt{\frac{9}{16}}$

    d. $\frac{23}{10}$

Solution

    a. 0.3

b. 0.99

c. 0.75

d. 2.3

Problem 3

Statement

Write each decimal as a fraction.

    a. $\sqrt{0.81}$

    b. 0.0276

    c. $\sqrt{0.04}$

    d. 10.01

Solution

    a. $\frac{9}{10}$

    b. $\frac{276}{10000}$ (or equivalent)

    c. $\frac{1}{5}$ (or equivalent)

    d. $\frac{1001}{100}$ (or equivalent)

Problem 4

Statement

Find the positive solution to each equation. If the solution is irrational, write the solution using square root or cube root notation.

    a. $x^2 = 90$

    b. $p^3 = 90$

    c. $z^2 = 1$

    d. $y^3 = 1$

    e. $w^2 = 36$

    f. $h^3 = 64$

Solution

a. $x = \sqrt{90}$

b. $p = \sqrt[3]{90}$

c. $z = 1$

d. $y = 1$

e. $w = 6$

f. $h = 4$

(From Unit 8, Lesson 13.)

Problem 5

Statement

Here is a right square pyramid.
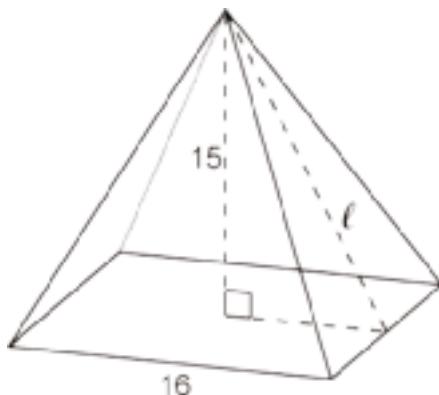
a. What is the measurement of the slant height ℓ of the triangular face of the pyramid? If you get stuck, use a cross section of the pyramid.

b. What is the surface area of the pyramid?

Solution

a. 17 units ($15^2 + 8^2 = 289$ and $\sqrt{289} = 17$)

b. 800 square units (The pyramid is made from a square and four triangles. The square's area, in square units, is $16^2 = 256$. Each triangle's area, in square units, is $\frac{1}{2} \cdot 16 \cdot 17 = 136$. The surface area, in square units, is $256 + 4 \cdot 136 = 800$.)

(From Unit 8, Lesson 10.)

Unit 8 Lesson 14

Lesson 15: Infinite Decimal Expansions

Goals

- Compare and contrast (orally) decimal expansions for rational and irrational numbers.

- Coordinate (orally and in writing) repeating decimal expansions and rational numbers that represent the same number.

Learning Targets

- I can write a repeating decimal as a fraction.

- I understand that every number has a decimal expansion.

Lesson Narrative

In this lesson, students further explore finding decimal expansions of rational numbers as well as irrational numbers. In the warm-up, students find the decimal expansion of $\frac{3}{7}$, which starts to repeat as late as the seventh decimal place. However, once the first repeating digit shows up, repeated reasoning allows the students to stop the long-division process (MP8). The discussion of the warm-up is a good place to introduce students to the overline notation for repeating decimal expansions.

In the first classroom activity, students learn how to take a repeating decimal expansion and rewrite it in fraction form. The activity uses cards with the steps and explanations of the process and asks students to put these cards in order. Once they have the correct order, they use the same steps on different decimal expansions. While the numbers are different, the structure of the method is the same (MP7).

In the last activity of this lesson, and of this unit, students investigate how to approximate decimal expansions of irrational numbers. In an earlier lesson, students learned that $\sqrt{2}$ cannot be written as a fraction and they estimated its location on the number line. Now they use "successive approximation," a process of zooming in on the number line to find more and more digits of the decimal expansion of $\sqrt{2}$. They also use given circumference and diameter values to find more precise approximations of π, another irrational number students know. In contrast to the previous lesson, students see that there is no easy was to keep zooming in on these irrational numbers. They are not predictable like a repeating decimal. Because it is not possible to write out the complete decimal expansion of an irrational number we use symbols to name them. However, in practice we use approximations that are good enough for a given purpose.

Alignments

Building On

- 7.NS.A.2.d: Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Addressing

- 8.NS.A.1: Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Instructional Routines

- MLR1: Stronger and Clearer Each Time

- MLR8: Discussion Supports

- Think Pair Share

Required Materials

Pre-printed slips, cut from copies of the Instructional master

Required Preparation

Prepare enough copies of the Some Numbers are Rational Instructional master for each group of 2 to have a set of 6 cards.

> **Student Learning Goals**
>
> Let's think about infinite decimals.

15.1 Searching for Digits

Warm Up: 5 minutes
The purpose of this warm-up is to give students practice rewriting numbers in different forms. Students have re-written rational numbers with terminating decimals in fraction form and the reverse, as well as calculated the decimal form of of $\frac{2}{11}$, which has an infinitely repeating two-digit pattern. Students are only asked to get to the first digit that repeats, which for $\frac{3}{7}$ is the 7th digit after the decimal point.

Since students are expected to notice a repeating pattern, they should not use a calculator for this activity.

Building On

- 7.NS.A.2.d

Addressing

- 8.NS.A.1

Instructional Routines

- Think Pair Share

Launch

Remind students of the previous activity where the decimal expansion of $\frac{2}{11}$ was shown to be $0.1818\ldots$ using long division and repeated reasoning.

Students in groups of 2. 1 minute of quiet work time followed by partner and whole-class discussion.

> ### Student Task Statement
>
> The first 3 digits after the decimal for the decimal expansion of $\frac{3}{7}$ have been calculated. Find the next 4 digits.
>
> ```
>        0 . 4  2  8
>  7  / 3
>      - 2  8
>        2  0
>      -  1  4
>           6  0
>        -  5  6
>              4
> ```

Student Response

5714

Activity Synthesis

Ask students what the next 4 digits are, and record them on the calculation for all to see. Ask students, "Without calculating, what number, do you think will be next? Why?" (I think 2 will be the next digit because I can see the starting pattern has begun again.)

Continue the calculation and verify that 2 comes next. Repeat this process until reaching 4 again. Point out that this cycle will continue indefinitely, because we can completely predict what will happen at each step because it is exactly like what happened 7 steps ago.

Define a *repeating decimal* and show students overline notation again. State that all rational numbers have a decimal expansion that eventually repeats. Sometimes it eventually repeats 0's like $\frac{1}{2} = 0.5000000\ldots$, and we call this *terminating decimal*. Be careful in the use of the word "pattern" as it can be ambiguous. For example there is a pattern to the digits of the number 0.12112111211112..., but the number is not rational.

15.2 Some Numbers Are Rational

15 minutes

The purpose of this activity is for students to learn and practice a strategy for rewriting rational numbers with decimal representations that repeat eventually into their fraction representations. Students begin by arranging cards in order that show how the strategy was used to show $0.4\overline{85} = \frac{481}{990}$. Next, students use the strategy to calculate the fraction representations of two other values.

The extension problem asks students to repeat the task for decimal expansions with a single repeating digit, such as $0.\overline{3}$ and $0.\overline{9}$. While simpler to process algebraically, it may be counter-intuitive for students to conclude that $0.\overline{9} = 1$. The fact that one number can have two different decimal expansions is often surprising, and hints at the fact that the number line has some quite subtle aspects to it. Students might insist that $0.\overline{9}$ must be strictly less than 1, which can invite an interesting discussion as to the nature of infinite decimal expansions. If such a discussion arises, invite students to try to make their argument precise, and to try to explain where they feel there is a flaw in their argument that $0.\overline{9} = 1$. Discussion points to help resolve lingering dissonance include:

- Asking whether any numbers on the number line could be *between* $0.\overline{9}$ and 1.

- Whether expressions like $0.\overline{95}$ make any sense. (They do not, since the notation $0.\overline{9}$ notation means the pattern continues *forever*.)

- A second argument for those who accept that $\frac{1}{3} = 0.\overline{3}$ is to multiply both sides by 3.

Addressing

- 8.NS.A.1

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Demonstrate the algorithm with an example such as converting $0.\overline{12} = \frac{12}{99}$.

Arrange students in groups of 2. Do not provide access to calculators. Distribute a set of the slips cut out from the Instructional master to each group. Tell students that once they have the cards arranged, they should work on the second problem individually and then compare their work with their partner. Finish with a whole class discussion.

Student Task Statement

Your teacher will give your group a set of cards. Each card will have a calculations side and an explanation side.

1. The cards show Noah's work calculating the fraction representation of $0.4\overline{85}$. Arrange these in order to see how he figured out that $0.4\overline{85} = \frac{481}{990}$ without needing a calculator.

2. Use Noah's method to calculate the fraction representation of:

   a. $0.1\overline{86}$

   b. $0.7\overline{88}$

Student Response

1. See Instructional master for the correct

order. 2.

   a. $\frac{185}{990}$ or equivalent. Sample response:

$$x = 0.1\overline{86}$$
$$100x = 18.6\overline{86}$$

219

$$100x - x = 18.6\overline{86} - x$$
$$99x = 18.5$$
$$990x = 185$$
$$x = \frac{185}{990}$$

b. $\frac{71}{90}$ or equivalent. Sample response:

$$x = 0.7\overline{88}$$
$$10x = 7.8\overline{8}$$
$$10x - x = 7.8\overline{8} - x$$
$$9x = 7.1$$
$$90x = 71$$
$$x = \frac{71}{90}$$

Are You Ready for More?

Use this technique to find fractional representations for $0.\overline{3}$ and $0.\overline{9}$.

Student Response

We have $0.\overline{3} = \frac{1}{3}$ and $0.\overline{9} = \frac{9}{9} = 1$.

Activity Synthesis

The goal of this discussion is to help students build a more robust understanding of how the strategy works. Begin the discussion by selecting 2–3 students to share their work for the second problem, displaying each step for all to see. Ask if anyone completed the problem in a different way and, if so, have those students also share.

If no students notice it, point out that when rewriting $0.7\overline{88}$, we can multiply by 100, but multiplying by 10 also works since the part that repeats is only 1 digit long.

End the discussion by asking students to rewrite $0.3\overline{0}$ using this strategy. This is like using a sledgehammer for a nail, but it works and is reflective of the work they did in an earlier activity.

15.3 Some Numbers Are Not Rational

15 minutes
Students first encountered irrational numbers at the start of this unit as a way to denote the side lengths of squares. They also spent time attempting to find a number of the form $\frac{a}{b}$, where a and b are integers, that is equal to $\sqrt{2}$ only to have it revealed that $\sqrt{2}$ is irrational. Now that students have recently done a lot of thinking about decimal representations of rational numbers and how to convert the infinitely repeating decimal representation of a number into a fraction representation, we revisit irrationals to further the point that no such fraction representation exists for these numbers. For many irrationals, long division doesn't work as a tool to calculate a decimal

approximation because there are no two integers to divide. It can be said that a square of area 2 has a side length of $\sqrt{2}$ or that $\sqrt{2}$ is a number that, when squared, has a value of 2 (with $-\sqrt{2}$ being the other number). All this means is that different methods to approximate the value need to be discussed.

In this activity, students will approximate the value of $\sqrt{2}$ using successive approximation and discuss how they might figure out a value for π using measurements of circles. Students will also plot these values on number lines accurately to the thousandth place to reinforce the idea that irrational numbers are numbers. Therefore, irrational numbers have a place on the number line even if they cannot be written as a fraction of integers.

Addressing

- 8.NS.A.1

Instructional Routines

- MLR8: Discussion Supports

Launch

Calculators are okay, but students should not use the root button or the π button. Before beginning, remind students that we have previously found decimal representations for fractions, and that knowing these representations made it easier to plot numbers on a number line. Today we are going to do the same thing with irrational numbers. Ask students "Earlier, we used long division to find decimal representations of numbers. Why can't we do that today with irrational numbers?" (Irrational numbers are ones that cannot be written as fractions.)

Follow work time with a whole-class discussion.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organization; Attention


Student Task Statement

1. a. Why is $\sqrt{2}$ between 1 and 2 on the number line?

   b. Why is $\sqrt{2}$ between 1.4 and 1.5 on the number line?

   c. How can you figure out an approximation for $\sqrt{2}$ accurate to 3 decimal places?

d. Label all of the tick marks. Plot $\sqrt{2}$ on all three number lines. Make sure to add arrows from the second to the third number lines.
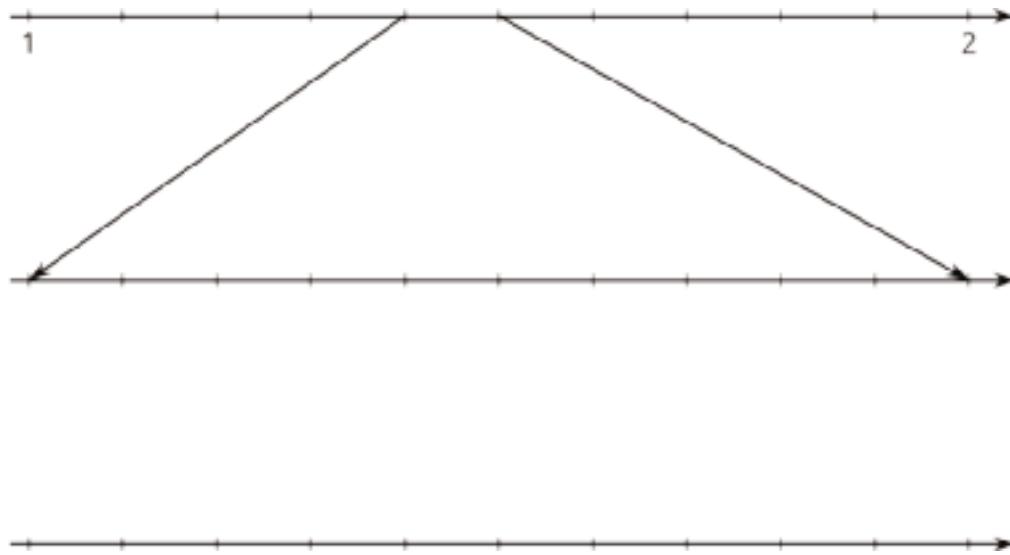
2. a. Elena notices a beaker in science class says it has a diameter of 9 cm and measures its circumference to be 28.3 cm. What value do you get for π using these values and the equation for circumference, $C = 2\pi r$?

b. Diego learned that one of the space shuttle fuel tanks had a diameter of 840 cm and a circumference of 2,639 cm. What value do you get for π using these values and the equation for circumference, $C = 2\pi r$?

c. Label all of the tick marks on the number lines. Use a calculator to get a very accurate approximation of π and plot that number on all three number lines.
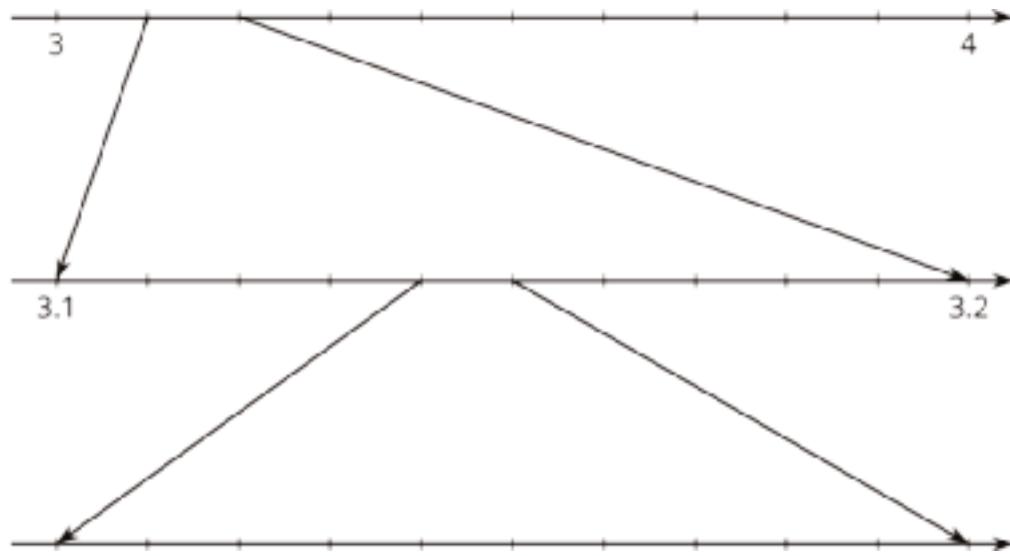
d. How can you explain the differences between these calculations of π?

Student Response

1.

a. Answers vary. Sample response: $\sqrt{2}^2$ is between 1^2 and 2^2, so $\sqrt{2}$ must be between 1 and 2.

b. Answers vary. Sample response: $\sqrt{2}^2$ is between 1.4^2 and 1.5^2, so $\sqrt{2}$ must be between 1.4 and 1.5.
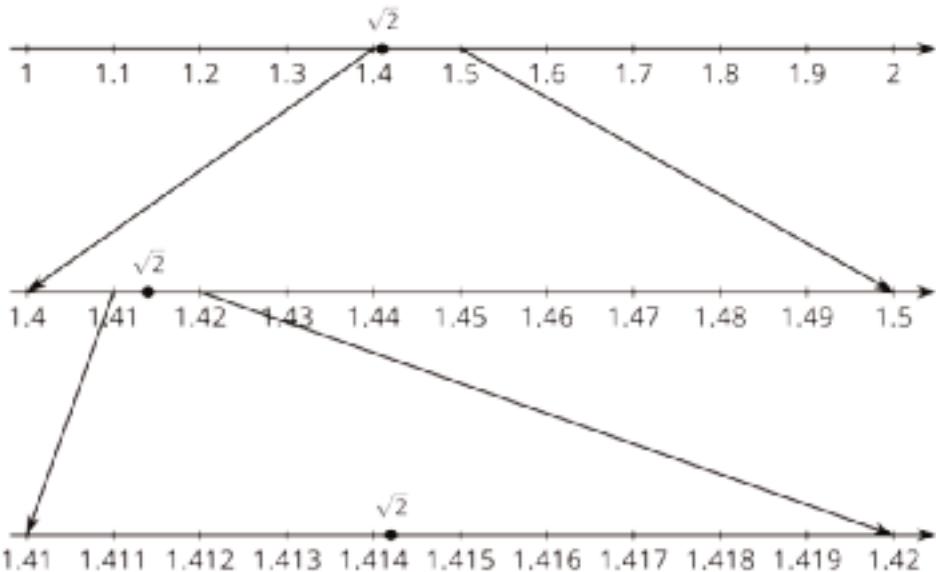
c.

d. Answers vary. Sample response: Checking the squares of values from 1.40 to 1.50, 1.41^2 is the closest to $\sqrt{2}^2$. Next, checking the squares of values from 1.410 to 1.419, 1.414^2 is the closest to $\sqrt{2}^2$. So, 1.414 is approximately equal to $\sqrt{2}$.

2.

a. $28.3 \div 9 = 3.1\overline{4}$

b. $2639 \div 840 = 3.141\overline{6}$

c. A calculator will give something similar to 3.141592654 as a value for π.

π

3   4

3.1 3.11 3.12 3.13 3.14 3.15 3.16 3.17 3.18 3.19 3.2

π

π

3.14 3.141 3.142 3.143 3.144 3.145 3.146 3.147 3.148 3.149 3.15

π
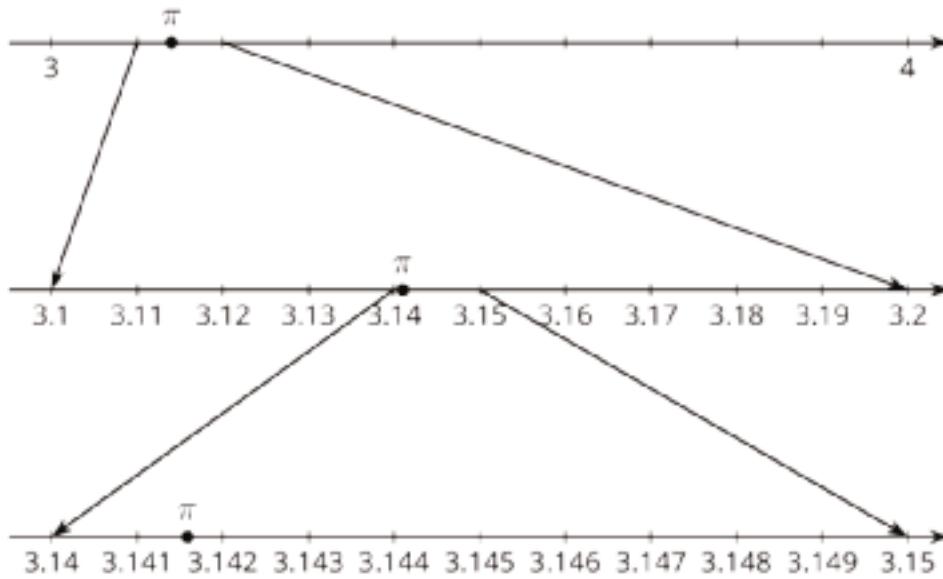
    d. Answers vary. Sample response: Diego and Elena's calculations are using measurements made from actual objects, which means they were approximated with things like tape measures. The objects are probably also not perfectly round. This makes them less accurate than my calculator, which has many digits of π programmed into it.

Activity Synthesis

The purpose of this discussion is to deepen students understanding that irrational numbers are not fractions. As such, their values are approximated using different methods than for rational numbers. Begin the discussion by asking:

- "How long do you think you could keep using the method in the first problem to find more digits of the decimal representation of $\sqrt{2}$?" (My calculator only shows 9 digits to the left of the decimal, so that's as far as I could go.)

- "Would this method work for any root?" (Yes, I could find the two perfect squares the square of the root is between and then start approximating the value of the root from there.)

Tell students that what they did in the first problem is a strategy called "successive approximation." It takes time, but successive approximation works for finding more and more precise approximations of irrational numbers so long as you have a clear value to check against. In the case of $\sqrt{2}$, since $\sqrt{2}^2 = 2$, there was a clear value to test approximations against whether they are too high or too low.

Lastly, select 2–3 students to share their response to the last part of the second problem. Make sure students understand that since measuring has limitations of accuracy, any calculation of π using measurement, such as with the beaker and the fuel tank, will have its own accuracy limited.

Lesson Synthesis

To wrap up the lesson, emphasize that $\sqrt{2}$ and π are irrational numbers. Their decimal expansions never end and never repeat like rational numbers do. But they are still numbers! The number $\sqrt{2}$ is the length of the diagonal of a square with a side length of 1 unit. It has to be a number because you can see it when you draw a square. Similarly, π has to be a number because you can see it when you draw a circle. The big take away is that we have now learned about numbers that are real, but that are not fractions or their opposites. Here are some questions to discuss:

- "What are some decimals for which our method of rewriting decimals as fractions will work?" (Any repeating decimal.)

- "We've seen that the decimal expansion of $\sqrt{2}$ does not repeat. What would happen if we tried to use Noah's method on $\sqrt{2}$? On π?" (It would not work, because those numbers are irrational.)

15.4 Repeating in Different Ways

Cool Down: 5 minutes
Addressing

- 8.NS.A.1

> **Student Task Statement**
>
> Let $x = 0.147$ and let $y = 0.\overline{147}$.
>
> - Is x a rational number?
>
> - Is y a rational number?
>
> - Which is larger, x or y?

Student Response

- Yes. $0.147 = \frac{147}{1000}$ is rational.

- Yes. $0.\overline{147}$ is an infinite repeating decimal, so is rational. (In fact, $0.\overline{147} = \frac{49}{333}$).

- y is larger than x, since y would be to the right of x on the number line.

Student Lesson Summary

Not every number is rational. Earlier we tried to find a fraction whose square is equal to 2. That turns out to be impossible, although we can get pretty close (try squaring $\frac{7}{5}$). Since there is no fraction equal to $\sqrt{2}$ it is not a rational number, which is why we call it an irrational number. Another well-known irrational number is π.

Any number, rational or irrational, has a decimal expansion. Sometimes it goes on forever. For example, the rational number $\frac{2}{11}$ has the decimal expansion $0.181818\ldots$ with the 18s repeating forever. Every rational number has a decimal expansion that either stops at some point or ends up in a repeating pattern like $\frac{2}{11}$. Irrational numbers also have infinite decimal expansions, but they don't end up in a repeating pattern. From the decimal point of view we can see that rational numbers are pretty special. Most numbers are irrational, even though the numbers we use on a daily basis are more frequently rational.

Lesson 15 Practice Problems
Problem 1
Statement

Elena and Han are discussing how to write the repeating decimal $x = 0.13\overline{7}$ as a fraction. Han says that $0.13\overline{7}$ equals $\frac{13764}{99900}$. "I calculated $1000x = 137.7\overline{7}$ because the decimal begins repeating after 3 digits. Then I subtracted to get $999x = 137.64$. Then I multiplied by 100 to get rid of the decimal: $99900x = 13764$. And finally I divided to get $x = \frac{13764}{99900}$." Elena says that $0.13\overline{7}$ equals $\frac{124}{900}$. "I calculated $10x = 1.3\overline{7}$ because one digit repeats. Then I subtracted to get $9x = 1.24$. Then I did what Han did to get $900x = 124$ and $x = \frac{124}{900}$."

Do you agree with either of them? Explain your reasoning.

Solution

Both strategies are valid. Han and Elena both get fractions that are equal to $0.13\overline{7}$. These are equivalent fractions, but Elena's fraction has fewer common factors in the numerator and denominator. The equivalent fraction with the lowest possible denominator is $\frac{31}{225}$.

Problem 2
Statement

How are the numbers 0.444 and $0.\overline{4}$ the same? How are they different?

Solution

Answers vary. Sample response: They are the same in that they are both rational numbers between 0.4 and 0.5, and the first three digits in their decimal expansions are the same. They are different in that $0.\overline{4}$ is greater than 0.444 because it has a greater digit in the ten-thousandths place. 0.444 is a terminating decimal, while $0.\overline{4}$ is an infinitely repeating decimal.

Problem 3
Statement

a. Write each fraction as a decimal.

   i. $\frac{2}{3}$

   ii. $\frac{126}{37}$

b. Write each decimal as a fraction.

   i. $0.\overline{75}$

ii. $0.\overline{3}$

Solution

   a.

      i. $0.\overline{6}$

      ii. $3.\overline{405}$

   b.

      i. $\frac{75}{99}$ (or equivalent)

      ii. $\frac{1}{3}$ (or equivalent)

Problem 4

Statement

Write each fraction as a decimal.

   a. $\frac{5}{9}$

   b. $\frac{5}{4}$

   c. $\frac{48}{99}$

   d. $\frac{5}{99}$

   e. $\frac{7}{100}$

   f. $\frac{53}{90}$

Solution

   a. $0.\overline{5}$

   b. 1.25

   c. $0.\overline{48}$

   d. $0.\overline{05}$

   e. 0.07

   f. $0.5\overline{8}$

Problem 5

Statement

Write each decimal as a fraction.

    a. $0.\overline{7}$

    b. $0.\overline{2}$

    c. $0.1\overline{3}$

    d. $0.\overline{14}$

    e. $0.\overline{03}$

    f. $0.6\overline{38}$

    g. $0.52\overline{4}$

    h. $0.1\overline{5}$

Solution

a. $\frac{7}{9}$ (or equivalent)

b. $\frac{2}{9}$ (or equivalent)

c. $\frac{2}{15}$ (or equivalent)

d. $\frac{14}{99}$ (or equivalent)

e. $\frac{3}{99}$ (or equivalent)

f. $\frac{632}{990}$ (or equivalent)

g. $\frac{472}{900}$ (or equivalent)

h. $\frac{14}{90}$ (or equivalent)

Problem 6

Statement

$2.2^2 = 4.84$ and $2.3^2 = 5.29$. This gives some information about $\sqrt{5}$.

Without directly calculating the square root, plot $\sqrt{5}$ on all three number lines using successive approximation.
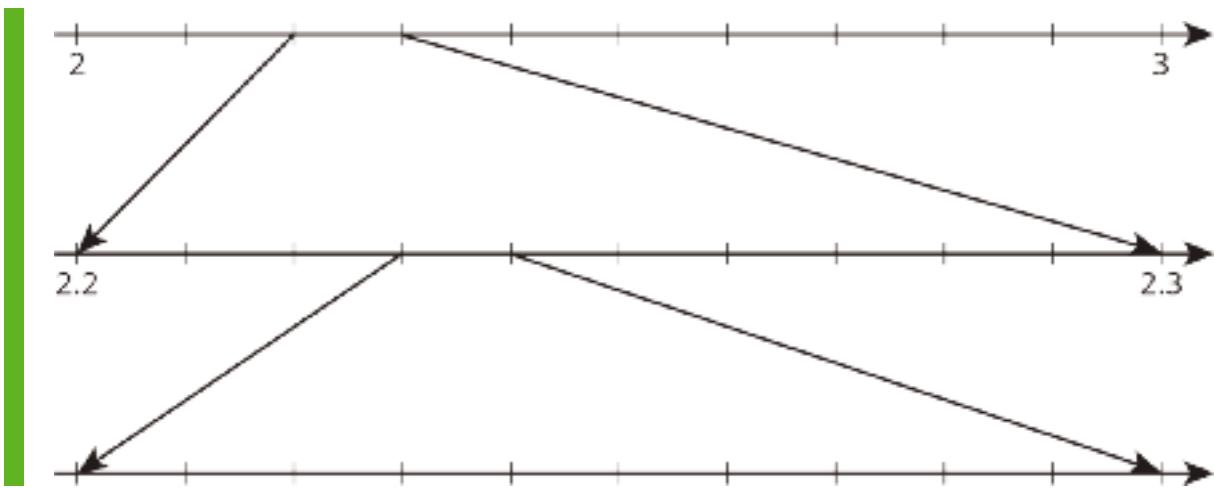
Solution
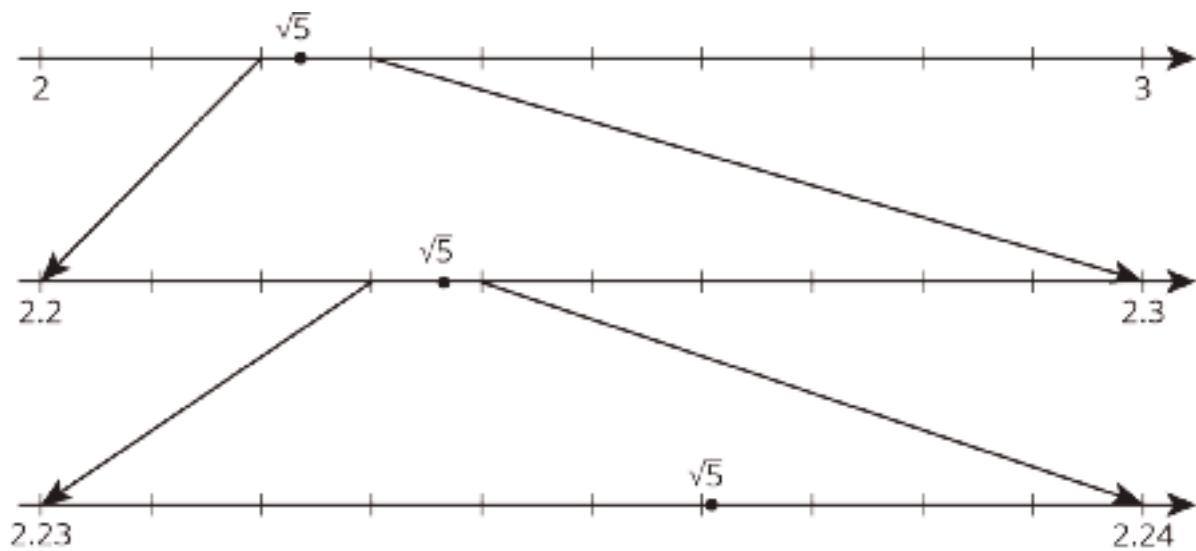
Section: Let's Put it to Work

Lesson 16: When Is the Same Size Not the Same Size?

Goals

- Apply ratios and the Pythagorean Theorem to solve a problem involving the aspect ratio of screens or photos, and explain (orally) the reasoning.

- Describe (in writing and using other representations) characteristics of rectangles with the same aspect ratio or with different aspect ratios.

Learning Targets

- I can apply what I have learned about the Pythagorean Theorem to solve a more complicated problem.

- I can decide what information I need to know to be able to solve a real-world problem using the Pythagorean Theorem.

Lesson Narrative

Before 2017, smartphones by major manufacturers all had screens with a $16 : 9$ aspect ratio. In 2017, two major brands released phones with screens in an $18.5 : 9$ aspect ratio. However they still reported their screen size using the same diagonal length of the earlier phones, 5.8 inches. How did the change in aspect ratio affect the screen size, if at all?

There is an element of mathematical modeling (MP4) in the last activity, because in order to quantify the screens' sizes to compare them, students need to refine the question that is asked.

Alignments

Building On

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

- 6.RP.A.1: Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, "The ratio of wings to beaks in the bird house at the zoo was $2 : 1$, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."

Addressing

- 8.G.B.7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Instructional Routines

- MLR6: Three Reads

Required Materials

Scientific calculators

> ### Student Learning Goals
>
> - Let's figure out how aspect ratio affects screen area.

16.1 Three Figures

Warm Up: 5 minutes
The purpose of this activity is to notice that rectangles can have the same diagonal length but different areas. The concept of aspect ratio is also introduced in the activity synthesis.

Monitor for students checking whether the diagonals are all the same length by using a ruler, compass, or edge of a piece of paper to compare their lengths. Also monitor for students mentioning that the rectangles have different areas.

Building On

- 6.G.A.1

- 6.RP.A.1

Launch

Provide access to rulers, or suggest students use the edge of a blank piece of paper if they would like to compare lengths.

> ### Student Task Statement
>
> How are these shapes the same? How are they different?
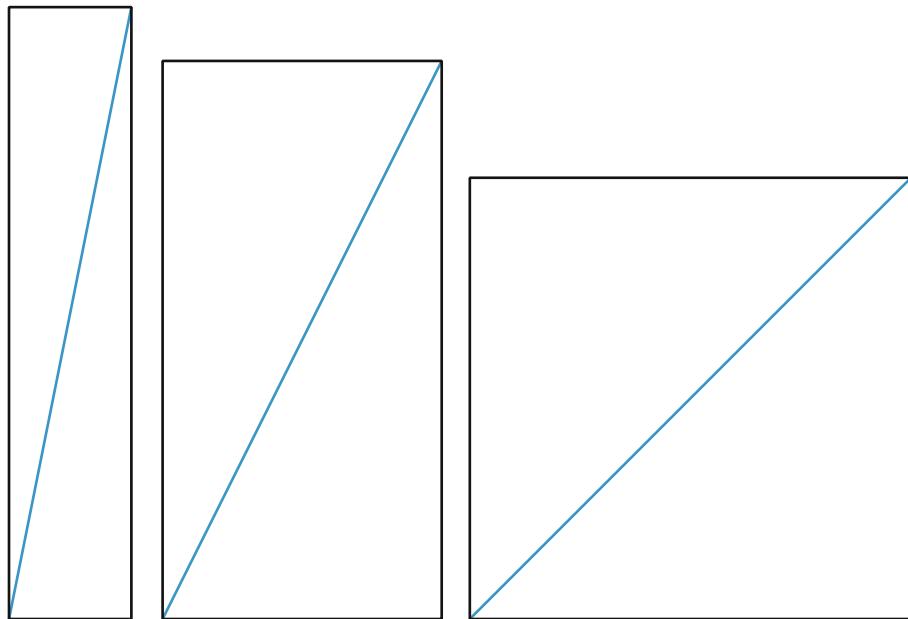
Student Response

They are the same because they are all rectangles. Also, their diagonals are all the same length. They are different because they have different areas and aspect ratios. (If students mention the aspect ratio, they are likely to use informal language.)

Activity Synthesis

Ask selected students to share their observations. Ensure that these ideas are mentioned:

- The diagonals of the rectangles are all the same length.

- The rectangles have different areas. The leftmost rectangle has a relatively small area compared to the others, and the rightmost rectangle has a relatively large area.

- Each rectangle has a different ratio of height to base. Students might mention the slope of the diagonals, which is related to this idea.

Tell students that in photography, film, and some consumer electronics with a screen, the ratio of the two sides of a rectangle is often called its *aspect ratio*. In the rectangles in this activity, the aspect ratios are $5 : 1, 2 : 1$, and $1 : 1$.

Demonstrate how the length of one side is a multiple of the other, on each rectangle. Students may be familiar with selecting an aspect ratio when taking or editing photos. Some common aspect ratios for photos are $1 : 1, 4 : 3$, and $16 : 9$. Also, from ordering school pictures, 5 by 7 and 8 by 10 may be common sizes they've heard of.

16.2 A $4 : 3$ Rectangle

20 minutes
The purpose of this activity is to really understand what an aspect ratio means when one side is not a multiple of the other, and to think about how you can figure out the side lengths if you know the rectangle's aspect ratio and some other information. This problem is a simpler version of the type of work needed for the more complicated activity that follows.

Building On
- 6.RP.A.1

Addressing
- 8.G.B.7

Launch
Provide access to calculators that can take the square root of a number.

Make sure that students understand what it means for the rectangle to have a $4 : 3$ aspect ratio before they set to work on figuring out the side lengths. Give them time to productively struggle before showing any strategies. It may be necessary to clarify that the rectangle's diagonal refers to the segment that connects opposite corners (which is not drawn).

Access for Students with Disabilities

Representation: Internalize Comprehension. Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems and other text-based content.
Supports accessibility for: Language; Conceptual processing

> **Student Task Statement**
> A typical aspect ratio for photos is $4 : 3$. Here's a rectangle with a $4 : 3$ aspect ratio.
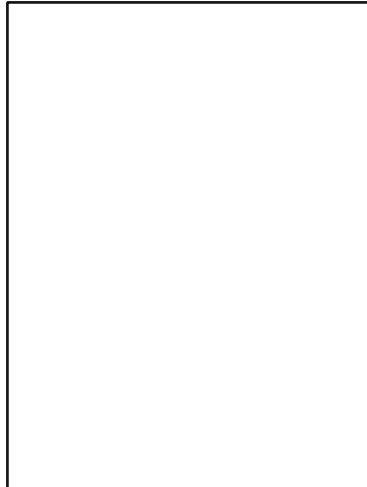
1. What does it mean that the aspect ratio is $4 : 3$? Mark up the diagram to show what that means.

2. If the shorter side of the rectangle measures 15 inches:
    a. What is the length of the longer side?

    b. What is the length of the rectangle's diagonal?

3. If the diagonal of the $4 : 3$ rectangle measures 10 inches, how long are its sides?

4. If the diagonal of the $4 : 3$ rectangle measures 6 inches, how long are its sides?

Student Response
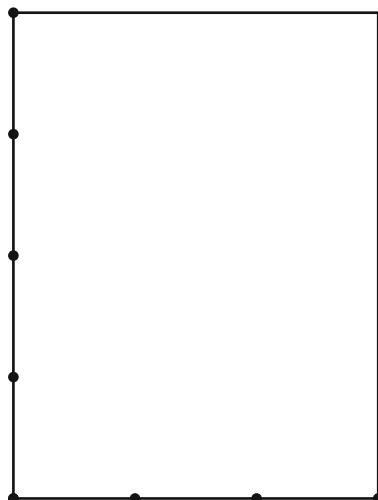
1. See diagram.

2. With a shorter side of 15 inches, each equal "piece" along the two subdivided sides is 5 inches.
    a. 20 inches

b. 25 inches, by solving $15^2 + 20^2 = c^2$, or scaling up a 3-4-5 right triangle.

3. 6 inches and 8 inches. Students may be able to figure this out if they know that 6-8-10 is a Pythagorean triple, recognizing that 6 and 8 are in the desired ratio.

4. 3.6 inches and 4.8 inches. Possible strategies:
   a. Solve the equation $(3x)^2 + (4x)^2 = 6^2$, where x represents the length of one of the little subdivisions of sides in the diagram. x is $\frac{6}{5}$, which can be multiplied by 3 and 4 to find the side lengths.

   b. Scale the 6 and 8 found previously by a factor of $\frac{6}{10}$, since this triangle would necessarily be similar.

Activity Synthesis

Invite students to share their solutions. If any students solved an equation such as $(3x)^2 + (4x)^2 = 6^2$ for the last question, ensure they have an opportunity to demonstrate their approach.

16.3 The Screen Is the Same Size . . . Or Is It?

20 minutes
The purpose of this activity is to give students an opportunity to solve a relatively complicated application problem that requires an understanding of aspect ratio, the Pythagorean Theorem, and realizing that a good way to compare the sizes of two screens is to compare their areas. The previous activities in this lesson are meant to prepare students to understand the situation and suggest some strategies for tackling the problem.

Monitor for students using different approaches and strategies. Students may benefit from more time to think about this problem than is available during a typical class meeting.

Building On
- 6.RP.A.1

Addressing
- 8.G.B.7

Instructional Routines
- MLR6: Three Reads

Launch

Provide access to calculators that can take the square root of a number. The task statement is wordy, so consider using the Three Reads protocol to ensure students understand what the problem is saying and what it is asking.

We chose not to provide diagrams drawn to scale in the student materials, since it makes it somewhat obvious that the new phone design has a smaller screen area. However if desired, here is an image to show or provide to students:
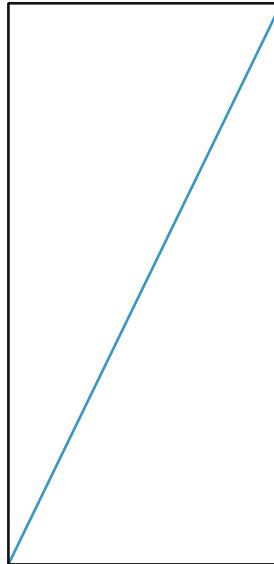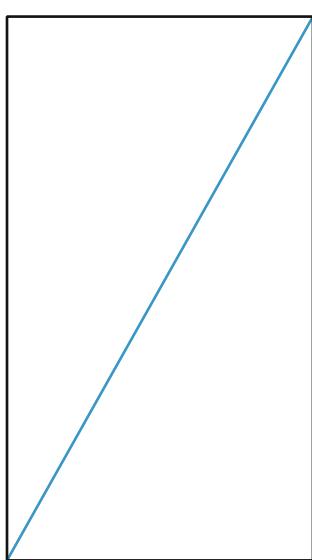
Access for English Language Learners

Reading, Listening, Conversing: MLR6 Three Reads. Use this routine to support reading comprehension of this word problem. Use the first read to orient students to the situation. Ask students to describe what the situation is about without using numbers (customers think that newly released smartphones have smaller screens). Use the second read to identify quantities and relationships. Ask students what can be counted or measured without focusing on the values (aspect ratios for phones manufactured before and after 2017). After the third read, invite students to brainstorm possible strategies they can use determine whether or not the new phones have a smaller screen, and by how much.
Design Principle(s): Support sense-making

Student Task Statement

Before 2017, a smart phone manufacturer's phones had a diagonal length of 5.8 inches and an aspect ratio of $16 : 9$. In 2017, they released a new phone that also had a 5.8-inch diagonal length, but an aspect ratio of $18.5 : 9$. Some customers complained that the new phones had a smaller screen. Were they correct? If so, how much smaller was the new screen compared to the old screen?

Student Response

Sample response: For the $16 : 9$ screen, it's approximately 5.06 inches by 2.84 inches, for an area of approximately 14.4 in^2. This can be found by solving $(16x)^2 + (9x)^2 = 5.8^2$ which gives $x \approx 0.316$.

Multiply this value by 16 and 9 to find the lengths of the two sides. Then multiply the lengths of the sides to find the area in square inches.

For the $18.5 : 9$ screen, it's approximately 5.22 inches by 2.54 inches, for an area of approximately 13.3 in^2. This can be found by solving $(18.5x)^2 + (9x)^2 = 5.8^2$ which gives $x \approx 0.282$. Multiply this value by 18.5 and 9 to find the lengths of the two sides. Then multiply the lengths of the sides to find the area in square inches.

Since $13.3 < 14.4$, the newer phones did, in fact, have a smaller screen when measured in terms of area. The difference was about 1.1 square inches.

This isn't the only possible solution method. For example, you could find that the diagonal of a 16 by 9 rectangle is approximately 18.36, and then scale each of the three measures down by a factor of $\frac{5.8}{18.36}$.

Activity Synthesis

Invite students to share their ideas and progress with the class. If appropriate, students may benefit from an opportunity to clearly present their solution in writing.

Lesson Synthesis

The debrief and presentation of student work provides opportunities to summarize takeaways from this lesson. Aside from opportunities to point out how the Pythagorean Theorem can help us tackle difficult problems, this lesson makes explicit some aspects of mathematical modeling. Highlight instances where students had to figure out what additional information they would need to make progress, or restate a question in mathematical terms.

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Family Support
Materials

Family Support Materials

Pythagorean Theorem and Irrational Numbers

Here are the video lesson summaries for Grade 8, Unit 8: Pythagorean Theorem and Irrational Numbers. Each video highlights key concepts and vocabulary that students learn across one or more lessons in the unit. The content of these video lesson summaries is based on the written Lesson Summaries found at the end of lessons in the curriculum. The goal of these videos is to support students in reviewing and checking their understanding of important concepts and vocabulary. Here are some possible ways families can use these videos:

- Keep informed on concepts and vocabulary students are learning about in class.
- Watch with their student and pause at key points to predict what comes next or think up other examples of vocabulary terms (the bolded words).
- Consider following the Connecting to Other Units links to review the math concepts that led up to this unit or to preview where the concepts in this unit lead to in future units.

Grade 8, Unit 8: Pythagorean Theorem and Irrational Numbers	Vimeo	YouTube
Video 1: Side Lengths and Areas of Squares (Lessons 1–2)	Link	Link
Video 2: Square Roots on the Number Line (Lesson 3–5)	Link	Link
Video 3: The Pythagorean Theorem (Lessons 6–8)	Link	Link
Video 4: Using the Pythagorean Theorem (Lessons 9–11)	Link	Link
Video 5: Cube Roots and Decimal Representations (Lessons 12–15)	Link	Link

Video 1

Video 'VLS G8U8V1 Side Lengths and Areas of Squares (Lessons 1–2)' available here: https://player.vimeo.com/video/521945003.

Video 2

Video 'VLS G8U8V2 Square Roots on the Number Line (Lesson 3–5)' available here: https://player.vimeo.com/video/523872469.

Video 3

Video 'VLS G8U8V3 The Pythagorean Theorem (Lessons 6–8)' available here: https://player.vimeo.com/video/526965535.

Video 4

Video 'VLS G8U8V4 Using the Pythagorean Theorem (Lessons 9–11)' available here: https://player.vimeo.com/video/526969582.

Video 5

Video 'VLS G8U8V5 Cube Roots and Decimal Representations (Lessons 12–15)' available here: https://player.vimeo.com/video/526956953.

Connecting to Other Units

- *Coming soon*

Side Lengths and Areas of Squares

Family Support Materials 1

This week your student will be working with the relationship between the side length and area of squares. We know two main ways to find the area of a square:

- Multiply the square's side length by itself.

- Decompose and rearrange the square so that we can see how many square units are inside. For example, if we decompose and rearrange the tilted square in the diagram, we can see that its area is 10 square units.
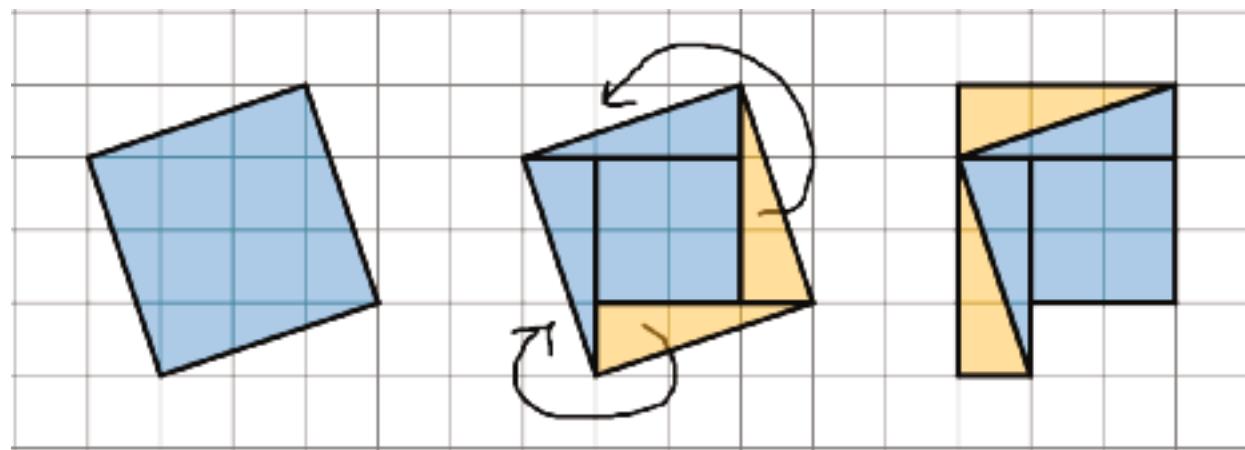
But what is the side length of this tilted square? It cannot be 3 units since $3^2 = 9$ and it cannot be 4 units since $4^2 = 16$. In order to write "the side length of a square whose area is 10 square units," we use notation called a **square root**. We write "the square root of 10" as $\sqrt{10}$ and it means "the length of a side of a square whose area is 10 square units." All of these statements are true:

- $\sqrt{9} = 3$ because $3^2 = 9$

- $\sqrt{16} = 4$ because $4^2 = 16$

- $\sqrt{10}$ is the side length of a square whose area is 10 square units, and $\left(\sqrt{10}\right)^2 = 10$
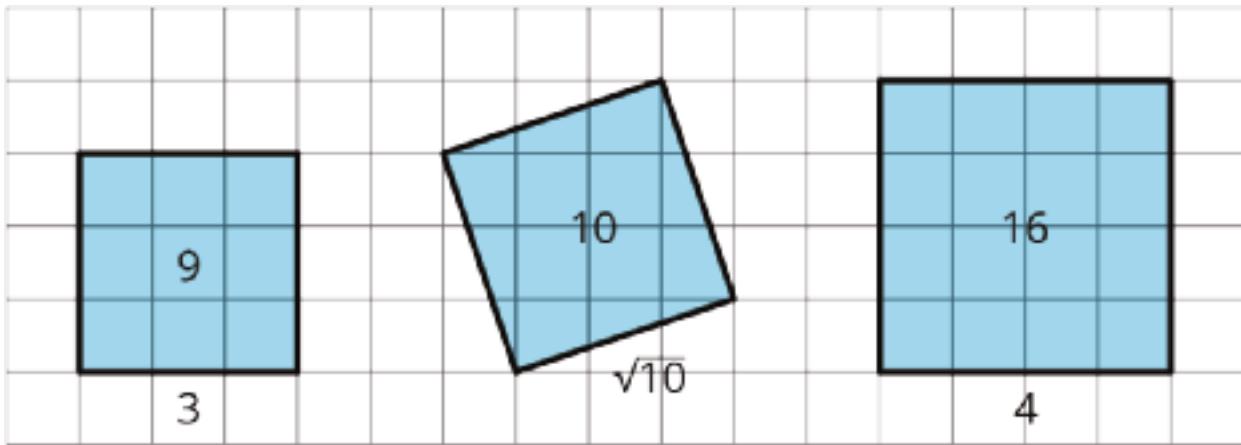
Here is a task to try with your student:

If each grid square represents 1 square unit, what is the side length of this titled square? Explain your reasoning.
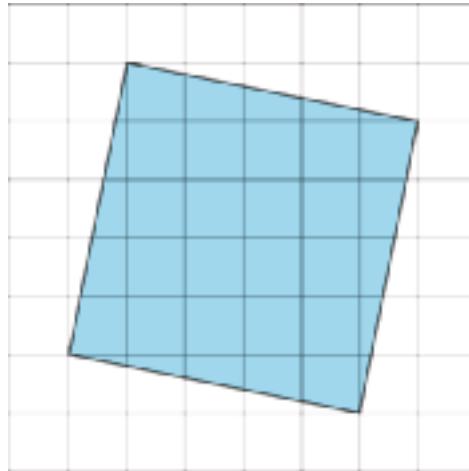
Solution:

The side length is $\sqrt{26}$ because the area of the square is 26 square units and the square root of the area of a square is the side length.

The Pythagorean Theorem

Family Support Materials 2

This week your student will work with the Pythagorean Theorem, which describes the relationship between the sides of any right triangle. A right triangle is any triangle with a right angle. The side opposite the right angle is called the hypotenuse, and the two other sides are called the legs.

Here we have a triangle with hypotenuse c and legs a and b. The Pythagorean Theorem states that for any right triangle, the sum of the squares of the legs are equal to the square of the hypotenuse. In other words, $a^2 + b^2 = c^2$.
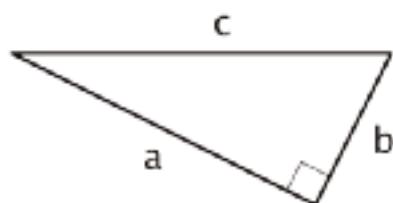
We can use the Pythagorean Theorem to tell if a triangle is a right triangle or not, to find the value of one side length of a right triangle if we know the other two, and to answer questions about situations that can be modeled with right triangles. For example, let's say we wanted to find the length of this line segment:
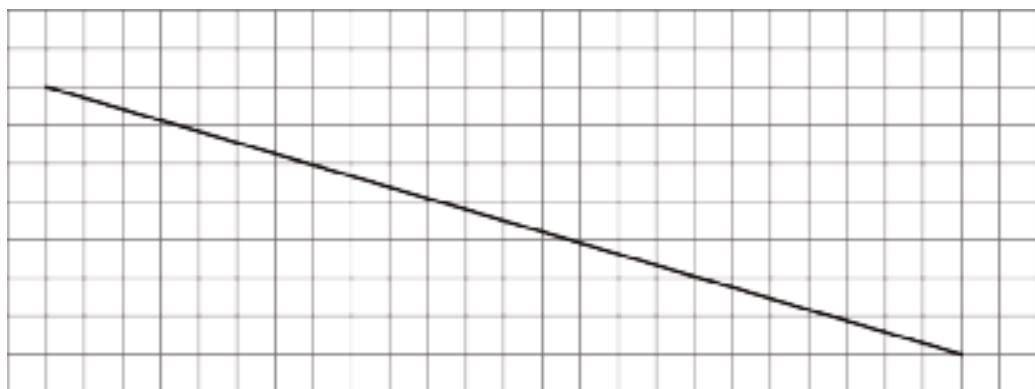
We can first draw a right triangle and determine the lengths of the two legs:
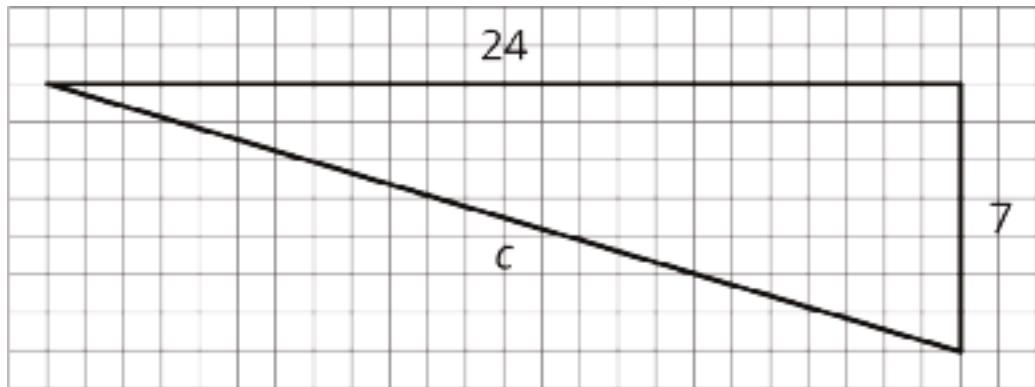
Next, since this is a right triangle, we know that $24^2 + 7^2 = c^2$, which means the length of the line segment is 25 units.

Here is a task to try with your student:

1. Find the length of the hypotenuse as an exact answer using a square root.
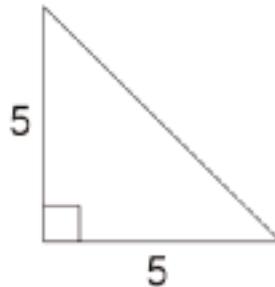
2. What is the length of line segment p? Explain or show your reasoning. (Each grid square represents 1 square unit.)
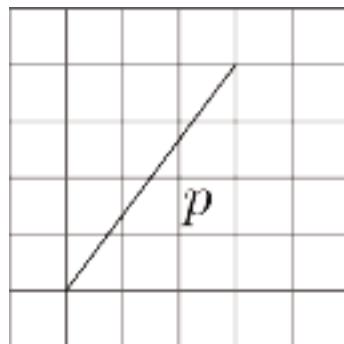
Solution:

1. The length of the hypotenuse is $\sqrt{50}$ units. With legs a and b both equal to 5 and an unknown value for the hypotenuse, c, we know the relationship $5^2 + 5^2 = c^2$ is true. That means $50 = c^2$, so c must be $\sqrt{50}$ units.

2. The length of p is $\sqrt{25}$ or 5 units. If we draw in the right triangle, we have legs of length 3 and 4 and hypotenuse p, so the relationship $3^2 + 4^2 = p^2$ is true. Since $3^2 + 4^2 = 25 = p^2$, p must equal $\sqrt{25}$ or 5 units.

Side Lengths and Volumes of Cubes

Family Support Materials 3

This week your student will learn about cube roots. We previously learned that a square root is the side length of a square with a certain area. For example, if a square has an area of 16 square units then its edge length is 4 units because $\sqrt{16} = 4$. Now, think about a solid cube. The cube has a volume, and the edge length of the cube is called the cube root of its volume. In this diagram, the cube has a volume of 64 cubic units:
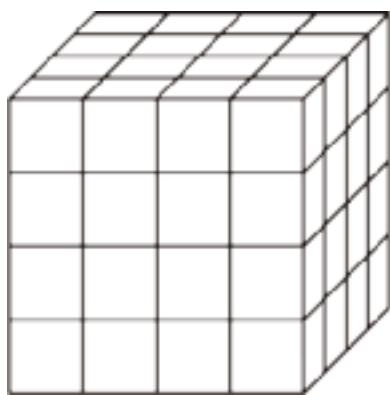
Even without the useful grid, we can calculate that the edge length is 4 from the volume since $\sqrt[3]{64} = 4$.

Cube roots that are not integers are still numbers that we can plot on a number line. If we have the three numbers $\sqrt{40}$, $\sqrt[3]{30}$, and $\sqrt[3]{64}$, we can plot them on the number line by estimating what integers they are near.

For example, $\sqrt{40}$ is between 6 and 7, since $\sqrt{36} < \sqrt{40} < \sqrt{49}$ and $\sqrt{36} = 6$ while $\sqrt{49} = 7$. Similarly, $\sqrt[3]{30}$ is between 3 and 4 because 30 is between 27 and 64. Our number line will look like this:
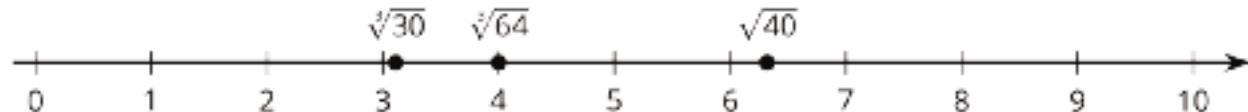
Here is a task to try with your student:

Plot the given numbers on the number line: $\sqrt{28}$, $\sqrt[3]{27}$, $\sqrt[3]{50}$
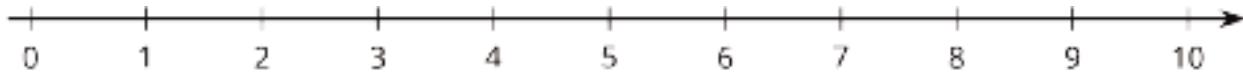
Solution:

Since $3^3 = 27$ means $\sqrt[3]{27} = 3$, we can plot $\sqrt[3]{27}$ at 3. $\sqrt[3]{50}$ is between 3 and 4 because 50 is between $3^3 = 27$ and $4^3 = 64$. $\sqrt{28}$ is between 5 and 6 because 28 is between $5^2 = 25$ and $6^2 = 36$.

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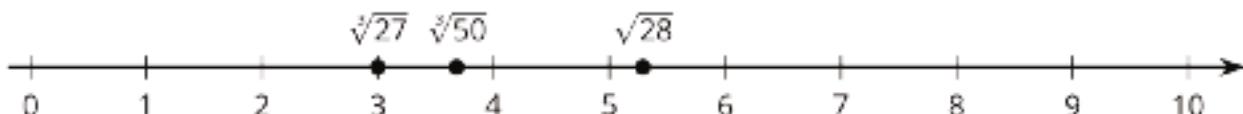
Unit Assessments

Check Your Readiness A and B
End-of-Unit Assessment A and B

Pythagorean Theorem and Irrational Numbers: Check Your Readiness (A)

Do not use a calculator.

1. How long is the segment from (-5, 2) to (-5, -8)?

2. Evaluate each expression for the given value.

   a. a^2 when $a = \frac{3}{4}$

   b. b^3 when $b = 1.1$

3. Locate these numbers on the number line.

   $\frac{3}{4}$, -1.5, 3^2, 0.5^3
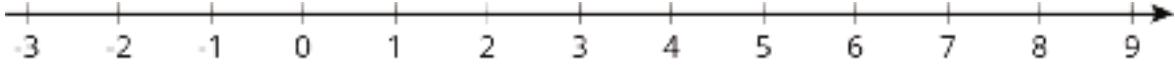
4. Find a fraction that is equal to each decimal.

   a. 0.4

   b. 1.15

   c. 0.125

5. Find a decimal that is equal to each fraction.

   a. $\frac{3}{5}$

   b. $\frac{271}{100}$

   c. $\frac{1}{9}$

6. What is the area of this triangle, in square units? Explain or show your reasoning.
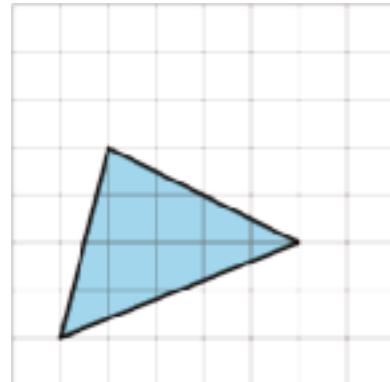
7. Find a solution for each equation.

   a. $a^2 = 25$

   b. $b^3 = 8$

   c. $10^c = 1,000$

Pythagorean Theorem and Irrational Numbers: Check Your Readiness (B)

Do not use a calculator.

1. Which segment has a length of 10 units?

   A. $(3, 4)$ to $(30, 40)$

   B. $(0, 0)$ to $(10, 10)$

   C. $(-1, -2)$ to $(9, -2)$

   D. $(-2, 8)$ to $(-3, 7)$

2. Evaluate each expression for the given value.

   a. a^2 when $a = \frac{5}{2}$

   b. b^3 when $b = 0.2$

3. Select **all** the numbers that are greater than 1.

   A. 2^3

   B. 0.6^3

   C. $\frac{7}{5}$

   D. $\frac{5}{7}$

   E. $\left(\frac{1}{3}\right)^2$

   F. $\left(\frac{3}{2}\right)^2$

4. Find a fraction that is equal to each decimal.

    a. 0.22

    b. 5.2

    c. 0.225

5. Find a decimal that is equal to each fraction.

    a. $\frac{721}{100}$

    b. $\frac{7}{5}$

    c. $\frac{1}{6}$

6. What is the area of this square, in square units? Explain or show your reasoning.
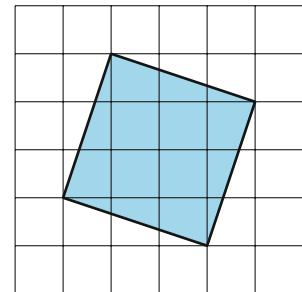
7. Find a solution for each equation.

   a. $a^2 = 100$

   b. $4^b = 64$

   c. $c^3 = -125$

Pythagorean Theorem and Irrational Numbers: End-of-Unit Assessment (A)

Do not use a calculator.

1. Select **all** the numbers that are solutions to the equation $x^3 = 27$.

    A. $\sqrt{27}$

    B. 3

    C. $\sqrt[3]{27}$

    D. 27^3

    E. 9

2. Select **all** the right triangles, given the lengths of the sides.
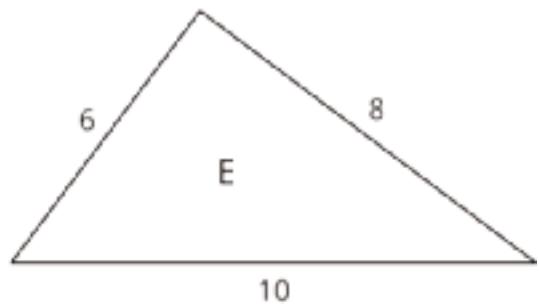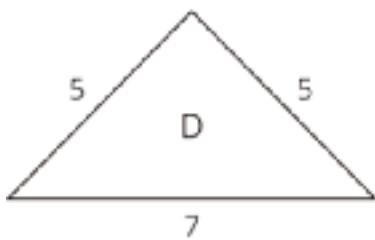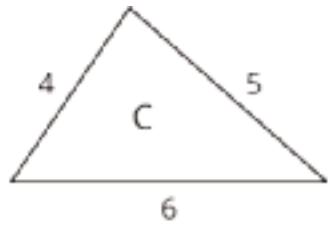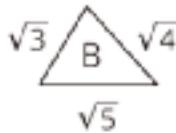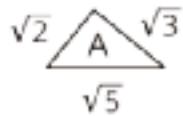
    A. A

    B. B

    C. C

    D. D

    E. E

3. Which of these is equal to $0.1\overline{3}$?

    A. $\dfrac{13}{99}$

    B. $\dfrac{12}{90}$

    C. $\dfrac{1}{3}$

    D. $1\dfrac{1}{3}$

4. Plot these numbers on the number line:

$$\sqrt{2}, \sqrt{5}, \sqrt[3]{8}, \sqrt{9}, \sqrt{15}, \sqrt[3]{25}$$

5. Find the length of the segment that joins the points $(-5, 4)$ and $(6, -3)$.
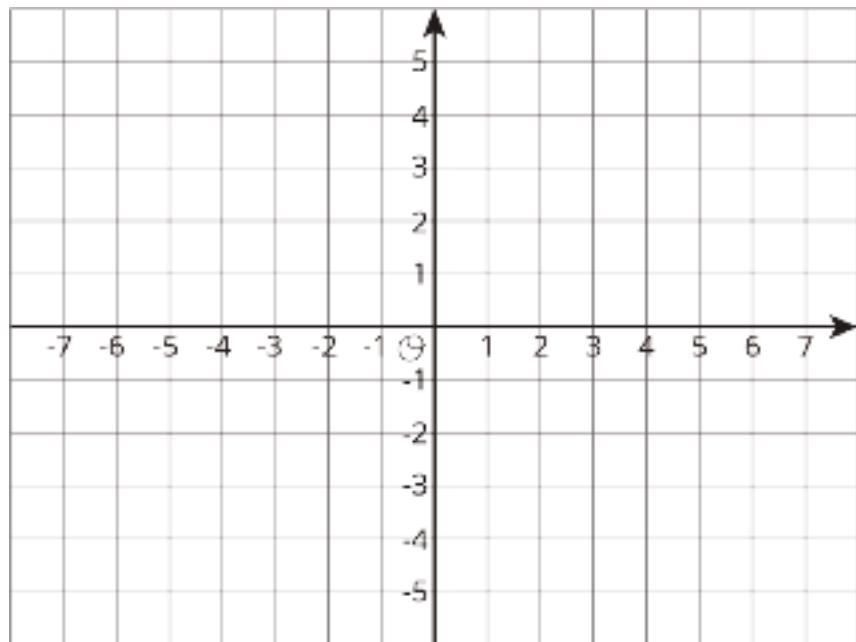
6. Mai's younger brother tells her that $\frac{10}{7}$ is equal to $\sqrt{2}$. Mai knows this can't be right, because $\frac{10}{7}$ is rational and $\sqrt{2}$ is irrational. Write an explanation that Mai could use to convince her brother that $\frac{10}{7}$ cannot be the square root of 2.

7. Elena wonders how much water it would take to fill her cup. She drops her pencil in her cup and notices that it just fits diagonally. (See the diagram.) The pencil is 17 cm long and the cup is 15 cm tall. How much water can the cup hold? Explain or show your reasoning.

   (The surface area of a cylinder is $2\pi r^2 + 2\pi rh$. The volume of a cylinder is $\pi r^2 h$.)
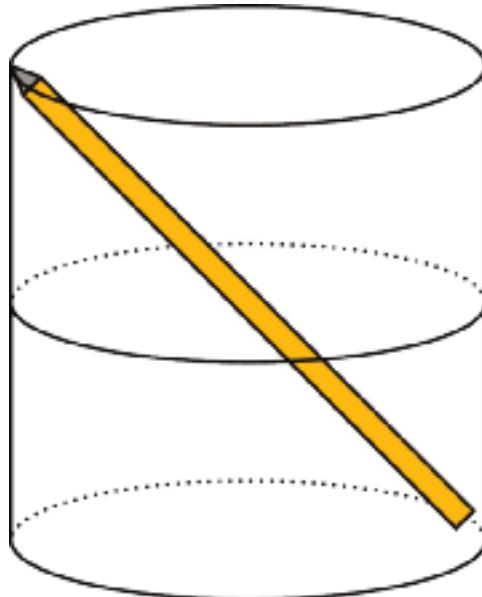
Pythagorean Theorem and Irrational Numbers: End-of-Unit Assessment (B)

Do not use a calculator.

1. Select **all** the numbers that are solutions to the equation $x^2 = 15$.

    A. 225

    B. $\sqrt{225}$

    C. 7.5

    D. $\sqrt{15}$

    E. $-\sqrt{15}$

2. What is the length of the line segment?
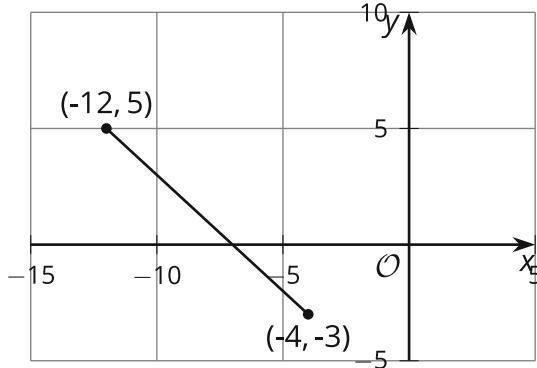
    A. 8

    B. $\sqrt{128}$

    C. 16

    D. $\sqrt{260}$

3. Each of the following gives the lengths, in inches, of the sides of a triangle. Which one is a right triangle?

   A. $3, 4, 6$

   B. $4, 6, 8$

   C. $\sqrt{5}, \sqrt{12}, \sqrt{13}$

   D. $\sqrt{5}, \sqrt{5}, \sqrt{10}$

4. What is the decimal expansion of $\frac{16}{9}$?

5. Put these numbers in order from least to greatest: $\sqrt{36}, \sqrt[3]{27}, \sqrt[3]{16}, \sqrt{10}, \sqrt{8}, \sqrt[3]{64}$

6. Jada plotted $\sqrt{3}$ on the number line as shown. How could you convince Jada that this location could not possibly represent $\sqrt{3}$?

7. Clare has a $\frac{1}{2}$-liter bottle full of water. A cone-shaped paper cup has diameter 10 cm and slant height 13 cm as shown. Can she pour all the water into one paper cup, or will it overflow? Explain your reasoning.
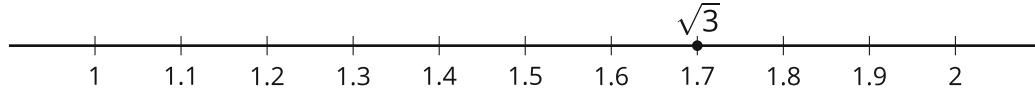
(The volume of a cone is $\frac{1}{3}\pi r^2 h$ and $\frac{1}{2}$ liter = 500 cubic centimeters.)

10cm

13 cm

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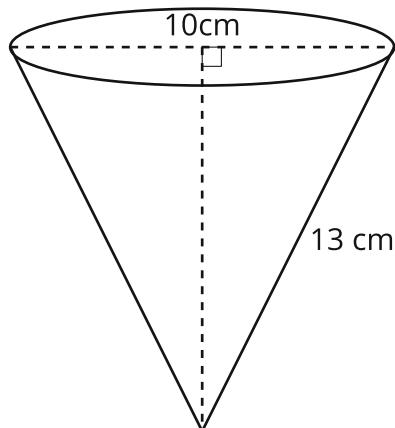
Assessment
Answer Keys

Check Your Readiness A and B
End-of-Unit Assessment A and B

Assessment Answer Keys

Assessment : Check Your Readiness (A)

Teacher Instructions

Calculators should not be used.

Student Instructions

Do not use a calculator.

Problem 1

The content assessed in this problem is first encountered in Lesson 11: Finding Distances in the Coordinate Plane.

Students find the distance between two points that share the same x- or y-coordinate in preparation for their work on the Pythagorean Theorem.

If most students struggle with this item, plan to spend more than the suggested time on the Warm-Up, and include grid paper or a coordinate plane for support. Ensure that students make the generalization that when the points share either the same x-coordinate or the same y-coordinate, we can subtract the coordinate that is not the same in both points to find the distance.

> ### Statement
> How long is the segment from (-5, 2) to (-5, -8)?

Solution

10 units

Aligned Standards

6.G.A.3

Problem 2

The content assessed in this problem is first encountered in Lesson 12: Edge Lengths and Volumes.

The content assessed in this problem is first encountered in Lesson 3: Rational and Irrational Numbers.

In this unit, students investigate whether $\sqrt{2}$ is a rational number. To do this, they square various fractions in an attempt to get an answer of 2.

If most students struggle with this item, plan to revisit what it means to evaluate an algebraic expression given a value for the variable. Students may need to see an example prior to the Lesson 3 Warm-up of how we can substitute a given value into the expression. If students still struggle, show them how to successfully square fractions and decimals using a calculator, and provide four-function calculators for Lessons 3 and 12.

Statement

Evaluate each expression for the given value.

1. a^2 when $a = \frac{3}{4}$

2. b^3 when $b = 1.1$

Solution

1. $a^2 = \frac{9}{16}$ (or equivalent)

2. $b^3 = 1.331$ (or equivalent)

Aligned Standards

6.EE.A.1

Problem 3

The content assessed in this problem is first encountered in Lesson 3: Rational and Irrational Numbers.

This problem also involves practice squaring rational numbers in preparation for studying the square root of 2. An important idea here is that positive numbers less than 1 become smaller after being squared, while positive numbers greater than 1 become larger.

If most students struggle with this item, plan to spend time in Activity 3 on the placement of rational numbers on the number line in the Activity Synthesis. Consider using this item as part of the Cool-Down, replacing the 0.5^3 with 0.5^2.

Statement

Locate these numbers on the number line.

$\frac{3}{4}$, -1.5, 3^2, 0.5^3
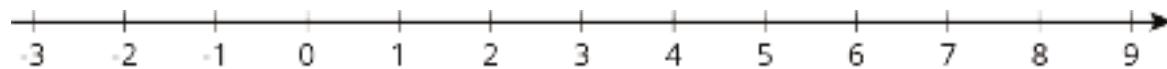
Solution
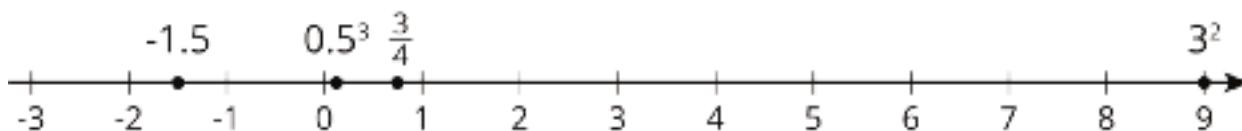
The numbers $\frac{3}{4}$, -1.5, 9, 0.125 are plotted in their approximate location on the number line.

Aligned Standards

6.EE.A.1, 6.NS.C.6.c

Problem 4

The content assessed in this problem is first encountered in Lesson 2: Side Lengths and Areas.

In this unit, students convert decimal expansions that repeat into rational numbers. They may not have much experience doing the same with terminating decimals. If students struggle here, give them a little more practice in ramping up to the relevant lessons in this unit. Emphasize that fractions like $\frac{4}{10}$ are correct and are generally much simpler to find than an equivalent fraction in lowest terms.

If most students struggle with this item, plan to take opportunities starting in Lesson 2 to practice writing numbers in different forms. There are several opportunities to discuss which form of numbers is easier to deal with in given situations. This skill will be particularly important in Lessons 14 and 15.

> ## Statement
>
> Find a fraction that is equal to each decimal.
>
>   1. 0.4
>
>   2. 1.15
>
>   3. 0.125

Solution

1. $\frac{4}{10}$ or $\frac{2}{5}$ (or equivalent)

2. $\frac{115}{100}$ or $\frac{23}{20}$ (or equivalent)

3. $\frac{125}{1000}$ or $\frac{1}{8}$ (or equivalent)

Aligned Standards

4.NF.C, 5.NBT.A.3

Problem 5

The content assessed in this problem is first encountered in Lesson 2: Side Lengths and Areas.

This unit includes work on decimals, including infinitely repeating decimals. Students may have difficulty with $\frac{1}{9}$ because it is a repeating decimal. Students using long division to tackle this third problem should be successful, while students who look for benchmark comparisons or look for a power of 10 in the denominator need to try a different method.

If most students struggle with this item, plan to take opportunities starting in Lesson 2 to practice writing numbers in different forms. There are several opportunities to discuss which form of numbers is easier to deal with in given situations. This skill will be particularly important in Lessons 14 and 15.

Statement

Find a decimal that is equal to each fraction.

1. $\frac{3}{5}$

2. $\frac{271}{100}$

3. $\frac{1}{9}$

Solution

1. 0.6

2. 2.71

3. $0.\overline{1}$

Aligned Standards

7.NS.A.2.d

Problem 6

The content assessed in this problem is first encountered in Lesson 1: The Areas of Squares and Their Side Lengths.

Students are asked to find the area of triangles and quadrilaterals on a grid in this unit using any strategy. Examples of strategies are drawing a surrounding rectangle and decomposing and rearranging. Students will decompose, rearrange, and calculate areas of squares and triangles when proving the Pythagorean Theorem.

If most students do well with this item, it may be possible to skip or move faster through the Warm-Up and Activity 2.

Statement

What is the area of this triangle, in square units? Explain or show your reasoning.

Solution

9 square units. Explanations vary. Sample explanation: Draw a rectangle around the triangle. This rectangle has area 20 square units. Then subtract away the area of three right triangles. These triangles have area 5, 4, and 2 square units, so the original triangle has area 9 square units.

Aligned Standards

6.G.A.1

Problem 7

The content assessed in this problem is first encountered in Lesson 3: Rational and Irrational Numbers.

This problem anticipates students' work with square and cube roots. The square root of a positive number a is defined as the positive number whose square is a. Likewise, the cube root of a number a is a solution to the equation $x^3 = a$.

If most students struggle with this item, plan to begin the Warm-Up with an equation such as

$$x \cdot x = 36$$

and connect it to the equation

$$x^2 = 36$$

. Note that students are not expected to "take the square root of each side" to solve equations like this. If needed, provide a list of perfect squares and cubes, or other opportunities to practice recognizing perfect squares and cubes, throughout the unit.

Statement

Find a solution for each equation.

1. $a^2 = 25$

2. $b^3 = 8$

3. $10^c = 1{,}000$

Solution

1. $a = 5$ or $a = $ -5

2. $b = 2$

3. $c = 3$

Aligned Standards

6.EE.A.1, 8.EE.A

Assessment: Check Your Readiness (A)

Assessment : Check Your Readiness (B)

Teacher Instructions

Calculators should not be used.

Student Instructions

Do not use a calculator.

Problem 1

The content assessed in this problem is first encountered in Lesson 11: Finding Distances in the Coordinate Plane.

Students find the distance between two points that share the same x- or y-coordinate in preparation for their work on the Pythagorean Theorem.

If most students struggle with this item, plan to spend more than the suggested time on the Warm-Up, and include grid paper or a coordinate plane for support. Ensure that students make the generalization that when the points share either the same x-coordinate or the same y-coordinate, we can subtract the coordinate that is not the same in both points to find the distance.

Statement

Which segment has a length of 10 units?

   A. $(3, 4)$ to $(30, 40)$

   B. $(0, 0)$ to $(10, 10)$

   C. (-1, -2) to (9, -2)

   D. (-2, 8) to (-3, 7)

Solution

C

Aligned Standards

6.G.A.3

Problem 2

The content assessed in this problem is first encountered in Lesson 12: Edge Lengths and Volumes.

The content assessed in this problem is first encountered in Lesson 3: Rational and Irrational Numbers.

In this unit, students investigate whether $\sqrt{2}$ is a rational number. To do this, they square various fractions in an attempt to get an answer of 2.

If most students struggle with this item, plan to revisit what it means to evaluate an algebraic expression given a value for the variable. Students may need to see an example prior to the Lesson 3 Warm-up of how we can substitute a given value into the expression. If students still struggle, show them how to successfully square fractions and decimals using a calculator, and provide four-function calculators for Lessons 3 and 12.

Statement

Evaluate each expression for the given value.

1. a^2 when $a = \frac{5}{2}$

2. b^3 when $b = 0.2$

Solution

1. $a^2 = \frac{25}{4}$ (or equivalent)

2. $b^3 = 0.008$ (or equivalent)

Aligned Standards

6.EE.A.1

Problem 3

The content assessed in this problem is first encountered in Lesson 3: Rational and Irrational Numbers.

This problem also involves practice squaring rational numbers in preparation for studying the square root of 2. An important idea here is that positive numbers less than 1 become smaller after being squared or cubed, while positive numbers greater than 1 become larger.

Students selecting C may have ignored the negative sign. Students selecting D or F may need additional practice squaring and cubing fractions and decimals. Students who fail to select E may have ignored the negative sign.

If most students struggle with this item, plan to spend time in Activity 3 on the placement of rational numbers on the number line in the Activity Synthesis. Consider using this item as part of the Cool-Down, replacing the 0.5^3 with 0.5^2.

Statement

Select **all** the numbers that are greater than 1.

Assessment: Check Your Readiness (B)

A. 2^3

B. 0.6^3

C. $\frac{7}{5}$

D. $\frac{5}{7}$

E. $\left(\frac{1}{3}\right)^2$

F. $\left(\frac{3}{2}\right)^2$

Solution

["A", "C", "F"]

Aligned Standards

6.EE.A.1, 6.NS.C.6.c

Problem 4

The content assessed in this problem is first encountered in Lesson 2: Side Lengths and Areas.

In this unit, students convert decimal expansions that repeat into rational numbers. They may not have much experience doing the same with terminating decimals. If students struggle here, give them a little more practice in ramping up to the relevant lessons in this unit. Emphasize that fractions like $\frac{22}{100}$ are correct and are generally much simpler to find than an equivalent fraction in lowest terms.

If most students struggle with this item, plan to take opportunities starting in Lesson 2 to practice writing numbers in different forms. There are several opportunities to discuss which form of numbers is easier to deal with in given situations. This skill will be particularly important in Lessons 14 and 15.

Statement

Find a fraction that is equal to each decimal.

1. 0.22

2. 5.2

3. 0.225

Solution

1. $\frac{22}{100}$ or $\frac{11}{50}$ (or equivalent)

2. $\frac{52}{10}$ or $\frac{26}{5}$ (or equivalent)

3. $\frac{225}{1000}$ or $\frac{9}{40}$ (or equivalent)

Aligned Standards

4.NF.C, 5.NBT.A.3

Problem 5

The content assessed in this problem is first encountered in Lesson 2: Side Lengths and Areas.

This unit includes work on decimals, including infinitely repeating decimals. Students may have difficulty with $\frac{1}{6}$ because it is a repeating decimal. Students using long division to tackle this third problem should be successful, while students who look for benchmark comparisons or look for a power of 10 in the denominator need to try a different method.

If most students struggle with this item, plan to take opportunities starting in Lesson 2 to practice writing numbers in different forms. There are several opportunities to discuss which form of numbers is easier to deal with in given situations. This skill will be particularly important in Lessons 14 and 15.

Statement

Find a decimal that is equal to each fraction.

1. $\frac{721}{100}$

2. $\frac{7}{5}$

3. $\frac{1}{6}$

Solution

1. 7.21

2. 1.4

3. $0.1\overline{6}$ or $0.1666\ldots$

Aligned Standards

7.NS.A.2.d

Problem 6

The content assessed in this problem is first encountered in Lesson 1: The Areas of Squares and Their Side Lengths.

Students are asked to find the area of triangles and quadrilaterals on a grid in this unit using any strategy. Examples of strategies are drawing a surrounding rectangle and decomposing and

rearranging. Students will decompose, rearrange, and calculate areas of squares and triangles when proving the Pythagorean Theorem.

If most students do well with this item, it may be possible to skip or move faster through the Warm-Up and Activity 2.

Statement

What is the area of this square, in square units? Explain or show your reasoning.

Solution

10 square units. Explanations vary. Sample explanation: Draw a larger square around the square. This new square has area 16 square units. Then subtract away the area of four right triangles. These triangles have each have area of 1.5 square units, and $16 - 4(1.5) = 10$.

Aligned Standards

6.G.A.1

Problem 7

The content assessed in this problem is first encountered in Lesson 3: Rational and Irrational Numbers.

This problem anticipates students' work with square and cube roots. The square root of a positive number a is defined as the positive number whose square is a. Likewise, the cube root of a number a is a solution to the equation $x^3 = a$.

If most students struggle with this item, plan to begin the Warm-Up with an equation such as

$$x \cdot x = 36$$

and connect it to the equation

$$x^2 = 36$$

. Note that students are not expected to "take the square root of each side" to solve equations like this. If needed, provide a list of perfect squares and cubes, or other opportunities to practice recognizing perfect squares and cubes, throughout the unit.

Statement

Find a solution for each equation.

1. $a^2 = 100$

2. $4^b = 64$

3. $c^3 = -125$

Solution

1. $a = 10$ or $a = -10$

2. $b = 3$

3. $c = -5$

Aligned Standards

6.EE.A.1, 8.EE.A

Assessment : End-of-Unit Assessment (A)

Teacher Instructions

Calculators should not be used, especially since some of the problems involve estimation of square roots and calculation of decimals and fractions.

Student Instructions

Do not use a calculator.

Problem 1

Students selecting A are confused between the square root and cube root symbols. Students selecting D may need more work understanding what it means for a number to be a solution to an equation. Students selecting E may think x^3 and $3x$ have the same meaning.

Statement

Select **all** the numbers that are solutions to the equation $x^3 = 27$.

   A. $\sqrt{27}$

   B. 3

   C. $\sqrt[3]{27}$

   D. 27^3

   E. 9

Solution

["B", "C"]

Aligned Standards

8.EE.A.2

Problem 2

Students selecting B instead of A may have ignored the square-root symbols, or may have seen "3-4-5" and skipped calculation. Students selecting C or D may be eyeballing the triangles instead of checking whether or not $a^2 + b^2 = c^2$ for the side lengths. Students failing to select E may have made an error in arithmetic.

Statement

Select **all** the right triangles, given the lengths of the sides.
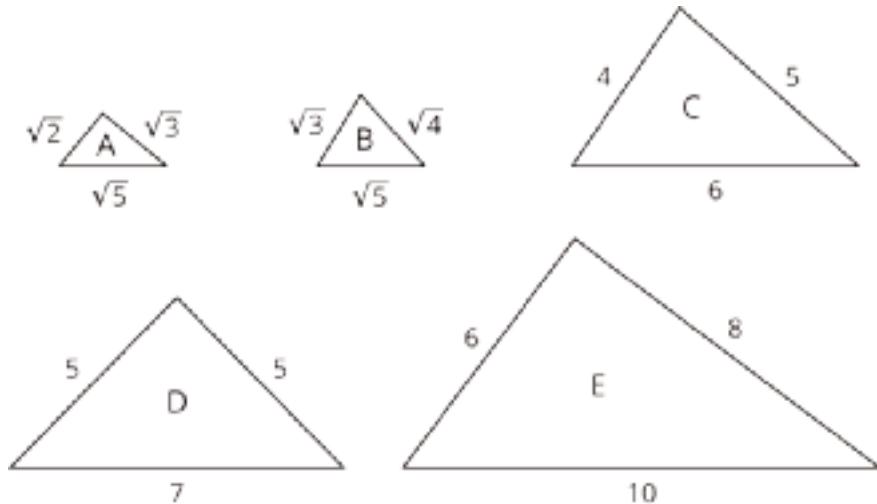
A. A

B. B

C. C

D. D

E. E

Solution

["A", "E"]

Aligned Standards

8.G.B

Problem 3

Students can use the method of expressing repeating decimals as fractions they studied in the unit. Savvier students might eliminate choices C and D, since they already know or can easily find their decimal expansions.

Students selecting C may be incorrectly remembering the decimal expansion of $\frac{1}{3}$, or misreading the given decimal as $0.\overline{3}$. Students selecting D are giving the fraction form of $1.\overline{3}$, not $0.1\overline{3}$. Choice A is the fraction form of $0.\overline{13}$; students making this choice are using the "repeating decimal" bar incorrectly.

Statement

Which of these is equal to $0.1\overline{3}$?

A. $\dfrac{13}{99}$

B. $\dfrac{12}{90}$

C. $\dfrac{1}{3}$

D. $1\dfrac{1}{3}$

Solution

B

Aligned Standards

8.NS.A.1

Problem 4

Watch for students confusing square roots and cube roots here. Students' placement of each number should be accurate to within the correct half of each unit interval. For example, they should be able to place $\sqrt{15}$ closer to 4 than to 3.

Reasoning students may use to determine the approximate location of each number: $\sqrt{2}$ is about 1.4. $\sqrt[3]{8}$ is equal to 2, since $2^3 = 8$. $\sqrt{5}$ is a little more than 2, since $2^2 = 4$. $\sqrt[3]{25}$ is a little less than 3, since $3^3 = 27$. $\sqrt{9}$ is equal to 3, since $3^2 = 9$. $\sqrt{15}$ is a little less than 4, since $4^2 = 16$.

Statement

Plot these numbers on the number line:
$$\sqrt{2}, \sqrt{5}, \sqrt[3]{8}, \sqrt{9}, \sqrt{15}, \sqrt[3]{25}$$

Solution
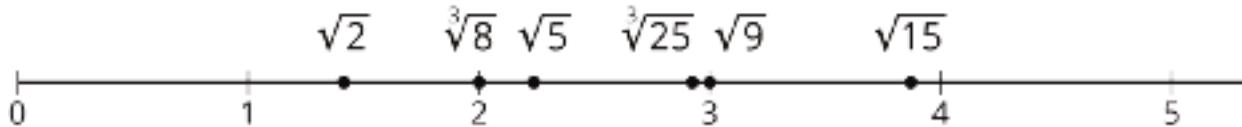
Minimal Tier 1 response:

- Work is complete and correct.

- Sample: See number line.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.

- Sample errors: some of the irrational numbers are placed in the correct unit interval but not within the correct half of the interval; one or two points are plotted completely incorrectly; all points are correct but at least two are not labeled (so it is not possible to tell which point represents which number).

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.

- Sample errors: three or more points are placed completely incorrectly.

Aligned Standards

8.EE.A.2, 8.NS.A.2

Problem 5

Students will probably sketch the segment on the coordinate axes provided, though this is not required. Look out for students making sign errors. Students might use either a horizontal or vertical length of 1, thinking that $4 - 3$ or $6 - 5 = 1$. Drawing the diagram will make this error much less likely. Students who use signed distances like -11 or -7 and square them incorrectly might wind up with an incorrect answer like $\sqrt{72}$, which is $\sqrt{11^2 - 7^2}$.

Statement

Find the length of the segment that joins the points $(-5, 4)$ and $(6, -3)$.
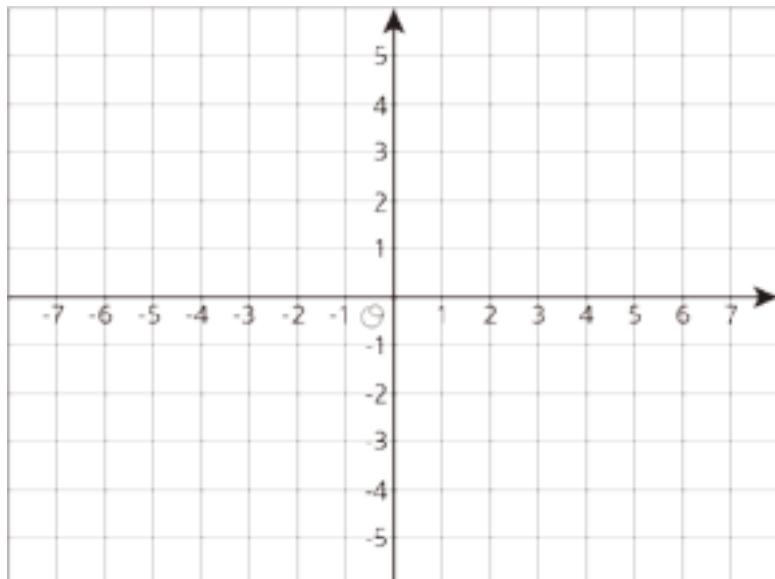
Solution

$\sqrt{170}$ units

Aligned Standards

8.G.B.8, 8.NS.A.2

Problem 6

The explanation that most closely follows the development in this unit is the first sample explanation: $(\frac{10}{7})^2 \neq 2$. Students may also use their knowledge that $\sqrt{2} \approx 1.41$ to argue that the decimal expansion of $\frac{10}{7}$ is not the same.

> ### Statement
>
> Mai's younger brother tells her that $\frac{10}{7}$ is equal to $\sqrt{2}$. Mai knows this can't be right, because $\frac{10}{7}$ is rational and $\sqrt{2}$ is irrational. Write an explanation that Mai could use to convince her brother that $\frac{10}{7}$ cannot be the square root of 2.

Solution

Explanations vary. Sample responses:

- $\frac{10}{7}$ squared is $\frac{100}{49}$, which does not equal 2. Therefore, $\frac{10}{7}$ does not equal $\sqrt{2}$.

- (With accompanying long division) $\frac{10}{7}$ is about 1.43, while $\sqrt{2}$ is about 1.41, so they cannot be the same number.

Minimal Tier 1 response:

- Work is complete and correct.

- Sample: $(\frac{10}{7})^2 = \frac{100}{49}$, which is not 2.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.

- Sample errors: response does not explicitly compare $\frac{100}{49}$ to 2; response compares $\frac{10}{7}$ to $\sqrt{2} \approx 1.4$, which is not an accurate enough approximation of $\sqrt{2}$ to show that the two numbers are different; arithmetic error in squaring $\frac{10}{7}$; response simply states that $\frac{10}{7}$ is rational but $\sqrt{2}$ is irrational.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.

- Sample errors: explanation does not appeal to the work of the unit; no explanation or unreadable explanation.

Aligned Standards

8.EE.A.2, 8.NS.A.2

Problem 7

In order to get started, students will need to realize that the pencil, diameter of the cup, and height of the cup form a right triangle. This may be tricky for some students because of the round base of the cup. Students will also need to recognize that this question is asking for a volume.

Statement

Elena wonders how much water it would take to fill her cup. She drops her pencil in her cup and notices that it just fits diagonally. (See the diagram.) The pencil is 17 cm long and the cup is 15 cm tall. How much water can the cup hold? Explain or show your reasoning.

(The surface area of a cylinder is $2\pi r^2 + 2\pi rh$. The volume of a cylinder is $\pi r^2 h$.)
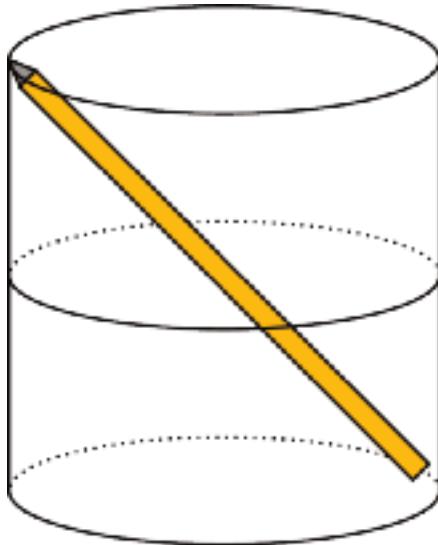
Solution

240π cm^3 or approximately 754 cm^3. The diameter of the cylindrical cup is 8 cm, because $\sqrt{17^2 - 15^2} = 8$. That means the radius is 4 cm. The amount of water the cup can hold is the volume of the cylinder: $V = \pi \cdot 4^2 \cdot 15 \approx 754$.

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.

- Sample: $\sqrt{17^2 - 15^2} = 8$. $V = \pi \cdot 4^2 \cdot 15 \approx 754$. The cup can hold about 754 cubic centimeters of water.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.

- Sample errors: omission of units in the final answer; correct mathematical calculations but failure to state how much water the cup can hold; use of 8 cm as the radius of the cup.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.

- Sample errors: correct use of the Pythagorean Theorem to calculate the diameter of the cup, but no significant further progress; calculation of the surface area of the cup rather than the volume; work involves visual estimation of the radius of the cylinder rather than calculation; calculation for the diameter of the cup treats this length as the hypotenuse of the triangle.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.

- Sample errors: work uses neither the Pythagorean Theorem nor the volume formula for a cylinder; two or more error types under Tier 3 response.

Aligned Standards

8.G.B.7, 8.G.C.9

Assessment : End-of-Unit Assessment (B)

Teacher Instructions

Calculators should not be used, especially since some of the problems involve estimation of square roots and calculation of decimals and fractions.

Student Instructions

Do not use a calculator.

Problem 1

Students selecting C may think that x^2 and $2x$ have the same meaning. Students that fail to select E may need to be reminded that the number 15 has both a positive and a negative square root. Students selecting A may be confusing square and square roots since $225 = 15^2$.

> ### Statement
>
> Select **all** the numbers that are solutions to the equation $x^2 = 15$.
>
> A. 225
>
> B. $\sqrt{225}$
>
> C. 7.5
>
> D. $\sqrt{15}$
>
> E. $-\sqrt{15}$

Solution

["D", "E"]

Aligned Standards

8.EE.A.2

Problem 2

Students who selected A or C may be focusing on vertical and horizontal lengths and need help applying the Pythagorean Theorem correctly. Students who selected D may have ignored the negative signs on -4 and -3 when subtracting the coordinates to find the lengths of the legs. Students who fail to select B, may have made miscounted the lengths or made a computation error.

> ### Statement
>
> What is the length of the line segment?
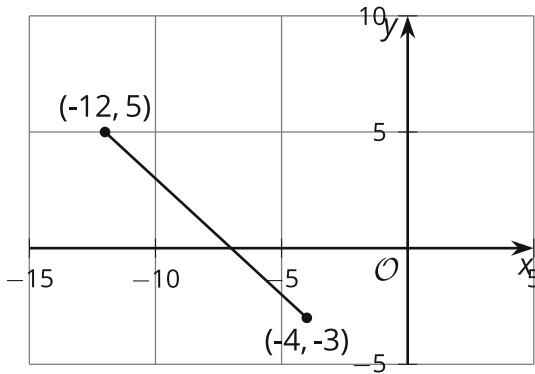
A. 8

B. $\sqrt{128}$

C. 16

D. $\sqrt{260}$

Solution

B

Aligned Standards

8.G.B.8, 8.NS.A.2

Problem 3

Students selecting A may have confused "3-4-6" with "3-4-5". Students selecting B may have made an arithmetic error. Students selecting C may have ignored the square roots and focused on "5-12-13". Students failing to select D may have made an error in arithmetic.

Statement

Each of the following gives the lengths, in inches, of the sides of a triangle. Which one is a right triangle?

A. $3, 4, 6$

B. $4, 6, 8$

C. $\sqrt{5}, \sqrt{12}, \sqrt{13}$

D. $\sqrt{5}, \sqrt{5}, \sqrt{10}$

Solution

D

Aligned Standards

8.G.B

Problem 4

Students can use long division and repeated reasoning to write the decimal expansion of the given fraction. Students who give the answer 1.7 or 1.8 may need additional help working with decimal expansions.

> ## Statement
>
> What is the decimal expansion of $\frac{16}{9}$?

Solution

$1.\overline{7}$ or $1.777\ldots$

Aligned Standards

8.NS.A.1

Problem 5

Watch for students confusing square roots and cube roots here. Students may struggle to place $\sqrt{8}$ and $\sqrt[3]{16}$ in the proper order.

Reasoning students may use to determine the approximate location of each number: $\sqrt[3]{27}$, $\sqrt[3]{64}$, and $\sqrt{36}$ are equal to 3, 4, and 6. The $\sqrt{10}$ is a little more than 3 since $3^2 = 9$ and $4^2 = 16$. $\sqrt{8}$ is just a little less than 3 because $3^2 = 9$. The $\sqrt[3]{16}$ should be closer to 2 since $2^3 = 8$ and $3^3 = 27$ and 16 is closer to 8 than it is to 27.

> ## Statement
>
> Put these numbers in order from least to greatest: $\sqrt{36}, \sqrt[3]{27}, \sqrt[3]{16}, \sqrt{10}, \sqrt{8}, \sqrt[3]{64}$

Solution

$\sqrt[3]{16}, \sqrt{8}, \sqrt[3]{27}, \sqrt{10}, \sqrt[3]{64}, \sqrt{36}$

Aligned Standards

8.EE.A.2, 8.NS.A.2

Problem 6

The explanation that most closely follows the development in this unit is the first sample explanation: $1.7^2 \neq 3$. Students may also use their knowledge that $\sqrt{3} \approx 1.73$ which is $\frac{3}{100}$ larger than 1.7.

Statement

Jada plotted $\sqrt{3}$ on the number line as shown. How could you convince Jada that this location could not possibly represent $\sqrt{3}$?
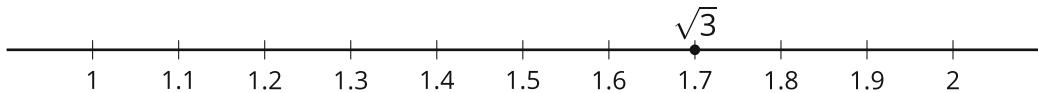
Solution

Explanations vary. Sample response:

- Jada thinks that $\sqrt{3}$ is equal to 1.7. If the $\sqrt{3}$ is equal to 1.7, then 1.7^2 would be equal to 3, but $(1.7) \cdot (1.7) = 2.89$.

- Jada needs to put the point between 1.7 and 1.8. The actual location of $\sqrt{3}$ is between 1.7 and 1.8 because $(1.7) \cdot (1.7) = 2.89$ and $(1.8) \cdot (1.8) = 3.24$.

Minimal Tier 1 response:

- Work is complete and correct.

- Sample: Jada thinks $\sqrt{3} = 1.7$, but $1.7^2 = 2.89$ which is not 3.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.

- Sample errors: response does not explicitly compare 1.7^2 to 3; arithmetic error in squaring 1.7; response simply states that 1.7 is rational but $\sqrt{3}$ is irrational without talking about placement on the number line.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.

- Sample errors: explanation does not appeal to the work of the unit; no explanation or unreadable explanation.

Aligned Standards

8.EE.A.2, 8.NS.A.2

Problem 7

In order to get started, students will need to use the slant height and diameter to determine the height of the cone. Also, students will need to use the fact that one-half liter is the same as 500

cubic centimeters. Students will also need to recognize that this question is asking for a volume and use the given formula to compute the volume of the cone-shaped cup.

Statement

Clare has a $\frac{1}{2}$-liter bottle full of water. A cone-shaped paper cup has diameter 10 cm and slant height 13 cm as shown. Can she pour all the water into one paper cup, or will it overflow? Explain your reasoning.

(The volume of a cone is $\frac{1}{3}\pi r^2 h$ and $\frac{1}{2}$ liter = 500 cubic centimeters.)
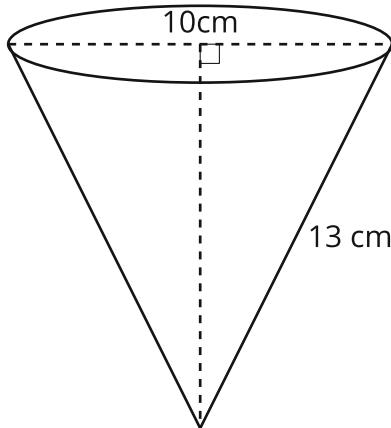
Solution

No. The cup will overflow. $500 > 100\pi$. The volume of the cup is 100π cubic centimeters and Clare has 500 cubic centimeters in her water bottle. The radius of the cup is 5 cm because the diameter is 10 cm. The height of cone is 12 cm because $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$. The amount of water the cup can hold is the volume of the cone: $V = \frac{\pi}{3}(5)^2(12) = 100\pi$.

Since $\pi \approx 3.14$, 100π is approximately 314.

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.

- Sample: $\sqrt{13^2 - 5^2} = 12$. $V = \frac{\pi}{3}(5)^2(12) \approx 314$. The cup can hold about 314 cubic centimeters of water. She cannot pour 500 cubic centimeters into this size cup without it overflowing.

Tier 2 response:

Assessment: End-of-Unit Assessment (B)

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.

- Sample errors: omission of units in the final answer; correct mathematical calculations but failure to state how much water the cup can hold; use of 10 cm as the radius of the cup.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.

- Sample errors: incorrect conclusion about water overflowing but volume of cup is correct; correct use of the Pythagorean Theorem to calculate the height of the cup, but no significant further progress; calculation of the surface area of the cup rather than the volume; work involves visual estimation of the radius of the cylinder rather than calculation; calculation for the volume uses the slant height for the height; does not compare volume to the amount of water in a half-liter bottle.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.

- Sample errors: work uses neither the Pythagorean Theorem nor the volume formula for a cone; two or more error types under Tier 3 response.

Aligned Standards

8.G.B.7, 8.G.C.9

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Lesson
Cool Downs

Lesson 1: The Areas of Squares and Their Side Lengths

Cool Down: It's a Square

Find the area and side length of square $ACEG$.
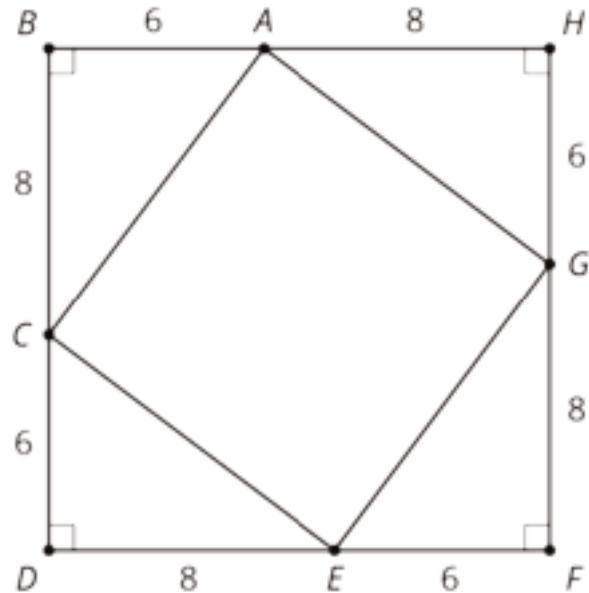
Lesson 2: Side Lengths and Areas

Cool Down: What Is the Side Length?

Write the exact value of the side length of a square with an area of

1. 100 square units.

2. 95 square units.

3. 36 square units.

4. 30 square units.

If the exact value is not a whole number, estimate the length.

Lesson 3: Rational and Irrational Numbers

Cool Down: Types of Solutions

1. In your own words, say what a rational number is. Give at least three different examples of rational numbers.

2. In your own words, say what an irrational number is. Give at least two examples.

Lesson 4: Square Roots on the Number Line

Cool Down: Approximating $\sqrt{18}$

Plot $\sqrt{18}$ on the x-axis. Consider using the grid to help.
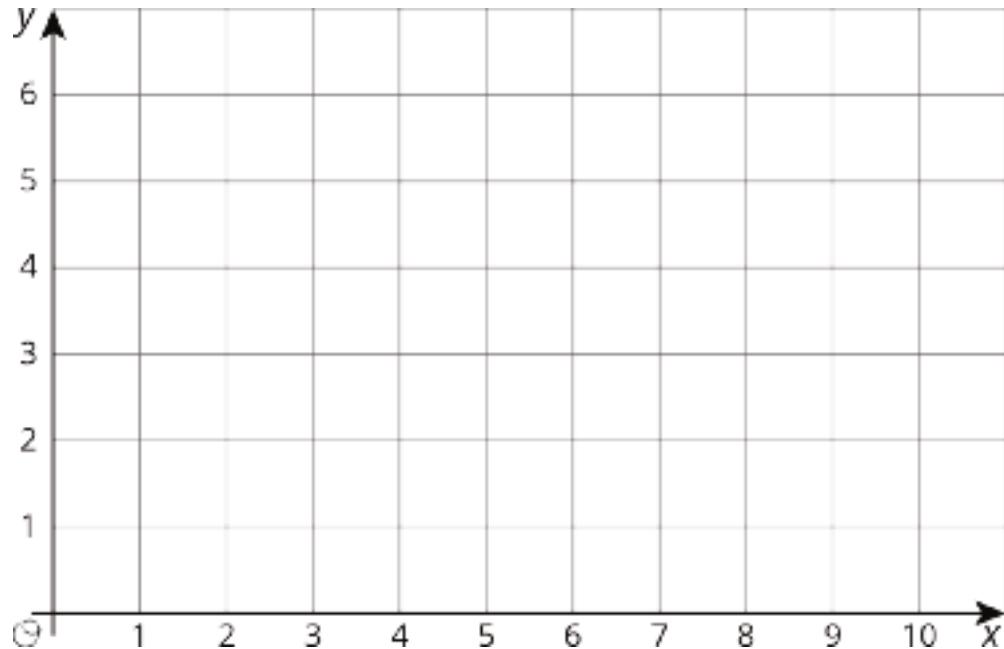
Lesson 5: Reasoning About Square Roots

Cool Down: Betweens

Which of the following numbers are greater than 6 and less than 8? Explain how you know.

- $\sqrt{7}$

- $\sqrt{60}$

- $\sqrt{80}$

Lesson 6: Finding Side Lengths of Triangles

Cool Down: Does a Squared Plus b Squared Equal c Squared?

For each of the following triangles, determine if $a^2 + b^2 = c^2$, where a, b, and c are side lengths of the triangle. Explain how you know.
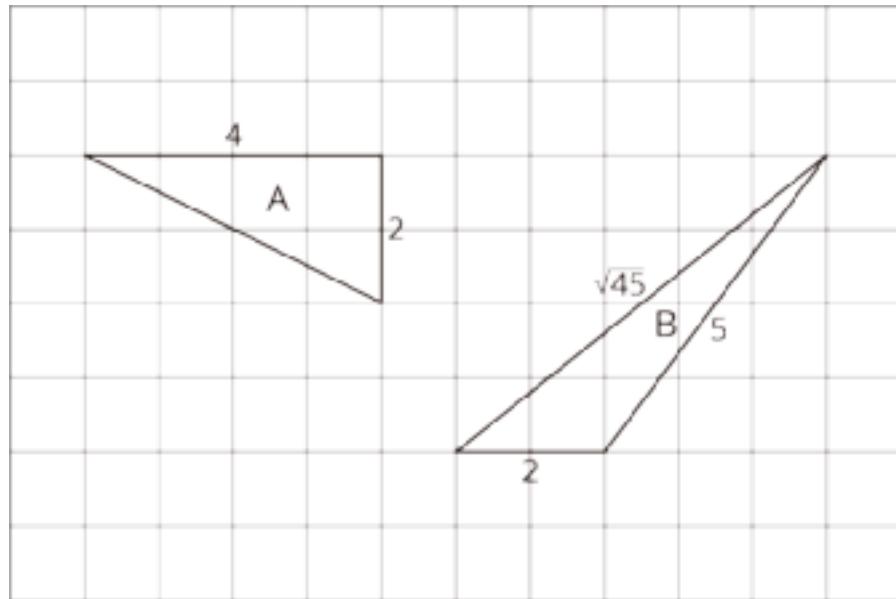
Lesson 7: A Proof of the Pythagorean Theorem

Cool Down: When is it True?

The Pythagorean Theorem is

1. True for all triangles

2. True for all right triangles

3. True for some right triangles

4. Never true

Lesson 8: Finding Unknown Side Lengths

Cool Down: Could be the Hypotenuse, Could be a Leg

A right triangle has sides of length 3, 4, and x.

    1. Find x if it is the hypotenuse.

    2. Find x if it is one of the legs.

Lesson 9: The Converse

Cool Down: Is It a Right Triangle?

The triangle has side lengths 7, 10, and 12. Is it a right triangle? Explain your reasoning.

Lesson 10: Applications of the Pythagorean Theorem

Cool Down: Jib Sail
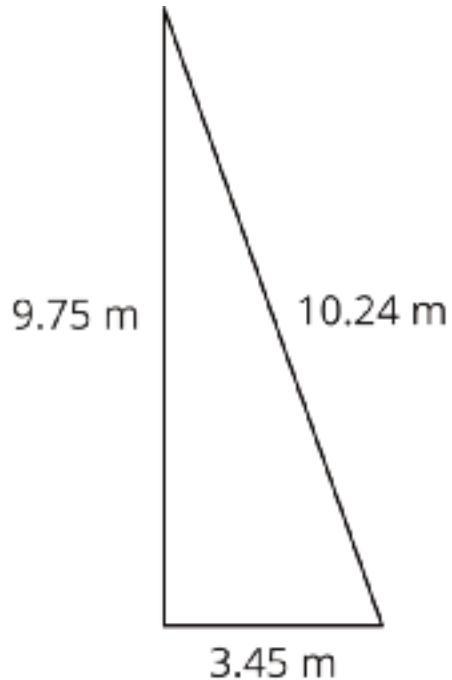
Sails come in many shapes and sizes. The sail on the right is a triangle. Is it a right triangle? Explain your reasoning.

Lesson 11: Finding Distances in the Coordinate Plane

Cool Down: Lengths of Line Segments

Here are two line segments with lengths e and f. Calculate the exact values of e and f. Which is larger?
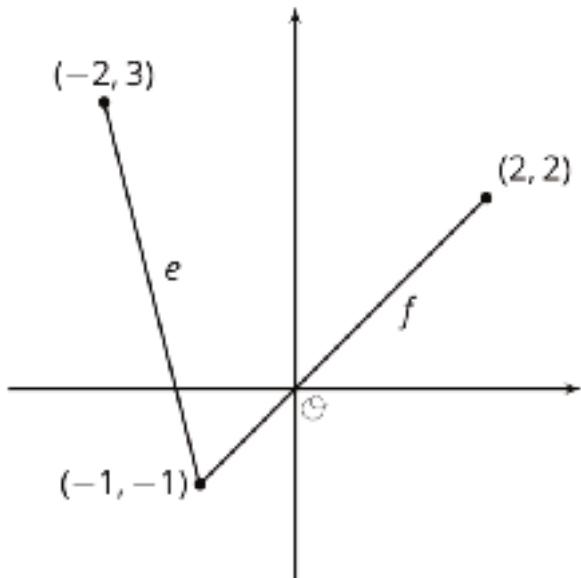
Lesson 12: Edge Lengths and Volumes

Cool Down: Roots of 36

Plot $\sqrt{36}$ and $\sqrt[3]{36}$ on the number line.
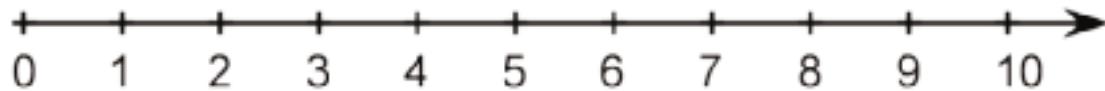
Lesson 13: Cube Roots

Cool Down: Different Types of Roots

Lin is asked to place a point on a number line to represent the value of $\sqrt[3]{49}$ and she writes:
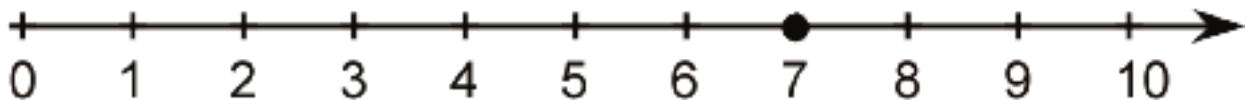
Where should $\sqrt[3]{49}$ actually be on the number line? How do you think Lin got the answer she did?

Lesson 14: Decimal Representations of Rational Numbers

Cool Down: An Unknown Rational Number

Explain how you know that -3.4 is a rational number.

Lesson 15: Infinite Decimal Expansions

Cool Down: Repeating in Different Ways

Let $x = 0.147$ and let $y = 0.\overline{147}$.

- Is x a rational number?

- Is y a rational number?

- Which is larger, x or y?

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Instructional
Masters

Instructional Masters for Pythagorean Theorem and Irrational Numbers

address	title	students per copy	written on?	requires cutting?	card stock recommended?	color paper recommended?
Activity Grade8.8.1.4	Making Squares	1	no	yes	no	no
Activity Grade8.8.15.2	Some Numbers Are Rational	1	no	yes	no	no
Activity Grade8.8.12.3	Card Sort: Rooted in the Number Line	1	no	yes	no	no
Activity Grade8.8.7.4	A Transformational Proof	1	no	yes	no	no

Rooted Number Line	Rooted Number Line	Rooted Number Line
A $\sqrt{50}$	Number line 0 to 10, point at ~7	$x^2 = 50$
B $\sqrt{18}$	Number line 0 to 10, point at ~4.2	$x^2 = 18$
C $\sqrt{5.5}$	Number line 0 to 10, point at ~2.3	$x^2 = 5.5$
D $\sqrt{90}$	Number line 0 to 10, point at ~9.5	$x^2 = 90$
E $\sqrt[3]{22}$	Number line 0 to 10, point at ~2.8	$x^3 = 22$
F $\sqrt[3]{100}$	Number line 0 to 10, point at ~4.6	$x^3 = 100$
G $\sqrt[3]{957}$	Number line 0 to 10, point at ~9.8	$x^3 = 957$
H 8	Number line 0 to 10, point at ~8	$x^3 = 512$
I $\sqrt[3]{50}$	Number line 0 to 10, point at ~3.7	$x^3 = 50$

Some Numbers are Rational

$$0.4\overline{85}$$

Some Numbers are Rational

I want to turn this repeating decimal into a fraction. I can see this decimal number has a two-digit repeating pattern.

Some Numbers are Rational

$$x = 0.4\overline{85}$$

Some Numbers are Rational

First I'll set x equal to this number.

Some Numbers are Rational

$$100x = 48.5\overline{85}$$

Some Numbers are Rational

Since the repeating pattern is 2 digits long, I'm going to multiply by 100 and write out a few more digits so I can still see the pattern.

Some Numbers are Rational

$$100x = 48.5\overline{85}$$
$$-x \quad -0.48\overline{85}$$

Some Numbers are Rational

Now I'll subtract the value of the decimal from each side. By lining the subtraction up vertically, it's easier to see what the left side will equal.

Some Numbers are Rational

$$99x = 48.1$$
$$990x = 481$$

Some Numbers are Rational

If I multiply each side by 10, I can re-write my equation without any decimal numbers.

Some Numbers are Rational

$$x = \frac{481}{990}$$

Some Numbers are Rational

Dividing each side by 990, I now know

$$0.4\overline{85} = \frac{481}{990}$$

Credits

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