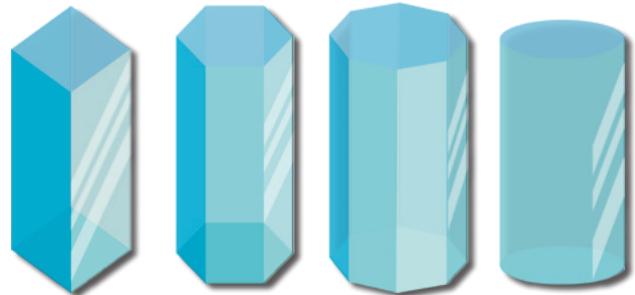
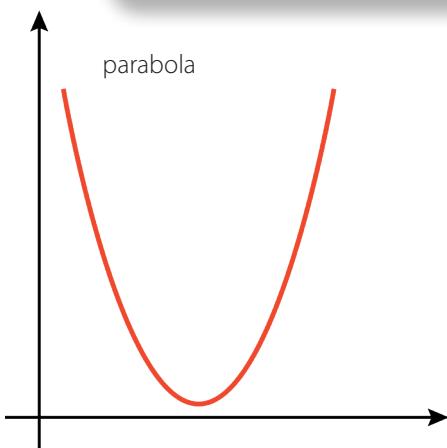




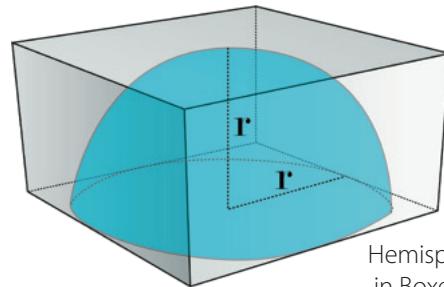
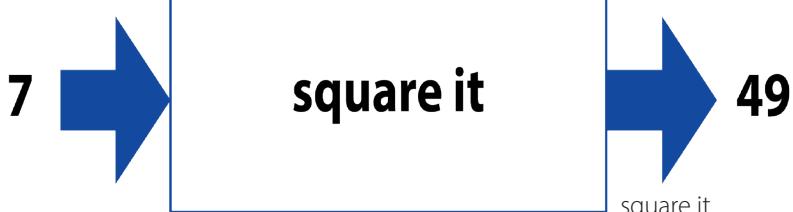
# Functions and Volume



## Teacher Guide



figures



Hemisphere  
in Boxes



Recycling

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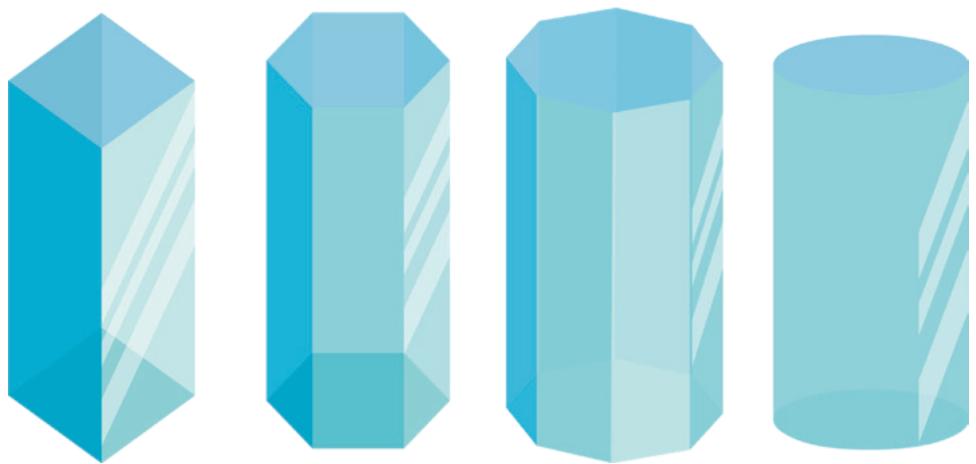
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# Functions and Volume

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**Functions and Volume**  
**Teacher Guide**  
Core Knowledge Mathematics™



# Functions and Volume

## Unit Narrative

In this unit, students are introduced to the concept of a function as a relationship between “inputs” and “outputs” in which each allowable input determines exactly one output. In the first three sections of the unit, students work with relationships that are familiar from previous grades or units (perimeter formulas, proportional relationships, linear relationships), expressing them as functions. In the remaining three sections of the unit, students build on their knowledge of the formula for the volume of a right rectangular prism from grade 7, learning formulas for volumes of cylinders, cones, and spheres. Students express functional relationships described by these formulas as equations. They use these relationships to reason about how the volume of a figure changes as another of its measurements changes, transforming algebraic expressions to get the information they need (MP1).

The first section begins with examples of “input–output rules” such as “divide by 3” or “if even, then . . . ; if odd, then . . . ” In these examples, the inputs are (implicitly) numbers, but students note that some inputs are not allowable for some rules, e.g.,  $\frac{1}{2}$  is not even or odd. Next, students work with tables that list pairs of inputs and outputs for rules specified by “input–output diagrams,” noting that a finite list of pairs does not necessarily determine a unique input–output rule (MP6). Students are then introduced to the term “function” as describing a relationship that assigns exactly one output to each allowable input.

In the second section, students connect the terms “independent variable” and “dependent variable” (which they learned in grade 6) with the inputs and outputs of a function. They use equations to express a dependent variable as a function of an independent variable, viewing formulas from earlier grades (e.g.,  $C = 2\pi r$ ), as determining functions. They work with tables, graphs, and equations for functions, learning the convention that the independent variable is generally shown on the horizontal axis. They work with verbal descriptions of a function arising from a real-world situation, identifying tables, equations, and graphs that represent the function (MP1), and interpreting information from these representations in terms of the real-world situation (MP2).

The third section of the unit focuses on linear and piecewise linear functions. Students use linear and piecewise linear functions to model relationships between quantities in real-world situations (MP4), interpreting information from graphs and other representations in terms of the situations (MP2). The lessons on linear functions provide an opportunity for students to coordinate and synthesize their understanding of new and old terms that describe aspects of linear and piecewise functions. In working with proportional relationships in grade 7, students learned the term “constant of proportionality,” and that any proportional relationship can be represented by an equation of the form  $y = kx$  where  $k$  is the constant of proportionality, that its graph lies on a line through the origin that passes through Quadrant I, and that the constant of proportionality indicates the steepness of the line. In an earlier grade 8 unit, students were introduced to “rate of change” as a way to describe the rate per 1 in a linear relationship and noted that its numerical value is the same as that of the slope of the line that represents the relationship. In this section, students connect their understanding of “increasing” and “decreasing” from the previous section

with their understanding of linear functions, noting, for example, that if a linear function is increasing, then its graph has positive slope, and that its rate of change is positive. Similarly, they connect their understanding of  $y$ -intercept (learned in an earlier unit) with the new term “initial value,” noting, for example, when the numerical part of an initial value of a function is given by the  $y$ -intercept of its graph (MP1).

In the remaining three sections of the unit, students work with volume, using abilities developed in earlier work with geometry and geometric measurement.

Students’ work with geometry began in kindergarten, where they identified and described two- and three-dimensional shapes that included cones, cylinders, and spheres. They continued this work in grades 1 and 2, composing, decomposing, and identifying two- and three-dimensional shapes.

Students’ work with geometric measurement began with length and continued with area and volume. Students learned to “structure two-dimensional space,” that is, to see a rectangle with whole-number side lengths as composed of an array of unit squares or composed of iterated rows or iterated columns of unit squares. In grade 3, students connected rectangle area with multiplication, understanding why (for whole-number side lengths) multiplying the side lengths of a rectangle yields the number of unit squares that tile the rectangle. They used area models to represent instances of the distributive property. In grade 4, students used area and perimeter formulas for rectangles to solve real-world and mathematical problems. In grade 5, students extended the formula for the area of rectangles to rectangles with fractional side lengths. They found volumes of right rectangular prisms by viewing them as layers of arrays of cubes and used formulas to calculate these volumes as products of edge lengths or as products of base area and height. In grade 6, students extended the formula for the volume of a right rectangular prism to right rectangular prisms with fractional side lengths and used it to solve problems. They extended their reasoning about area to include shapes not composed of rectangles and combined their knowledge of geometry and geometric measurement to produce formulas for the areas of parallelograms and triangles, using these formulas to find surface areas of polyhedra. In grade 7, students analyzed and described cross-sections of prisms (including prisms with polygonal but non-rectangular bases), pyramids, and polyhedra, and used the formula for the volume of a right rectangular prism (volume is area of base times height of prism) to solve problems involving area, surface area, and volume.

In this grade 8 unit, students extend their understanding of volume from right rectangular prisms to right cylinders, right cones, and spheres. They begin by investigating the volume of water in a graduated cylinder as a function of the height of the water, and vice versa. They examine depictions of a cylinder, prism, sphere, and cone, in order to develop their abilities to identify radii, bases, and heights of these objects. They estimate volumes of prisms, cylinders, cones, and spheres, in order to reinforce the idea that a measurement of volume indicates the amount of space within an object. Students use their abilities to identify radii, bases, and heights, together with the geometric abilities developed in earlier grades, to perceive similar structure (MP7) in formulas for the volume of a rectangular prism and the volume of a cylinder—both are the product of base and height. After gaining familiarity with a formula for the volume of a cylinder by using it to solve problems, students perceive similar structure (MP7) in a formula for the volume of a cone.

The fifth section of the unit begins with an examination of functional relationships between two quantities that are illustrated by changes in scale for three-dimensional figures. For example, if the radius of a cylinder triples, its volume becomes nine times larger. This work combines grade 7 work on scale and proportional relationships. In grade 7, students studied scaled copies of two-dimensional figures, recognizing lengths are scaled by a scale factor and areas by the square of the scale factor, and applied their knowledge to scale drawings, e.g., maps and floor plans. In their study of proportional relationships, grade 7 students solved problems set in contexts commonly associated with proportional relationships such as constant speed, unit pricing, and measurement conversions, and learned that any proportional relationship can be represented by an equation of the form  $y = kx$  where  $k$  is the constant of proportionality. In this section, students use their knowledge of scale, proportional relationships, and volume to reason about how the volume of a prism, cone, or cylinder changes as another measurement changes.

In the last section of the unit, students reason about how the volume of a sphere changes as its radius changes. They consider a situation in which water flows into a cylinder, cone, and sphere at the same constant rate. Information about the height of the water in each container is shown in an equation, graph, or table, allowing students to use it strategically (MP5) to compare water heights and capacities for the containers.

### **Progression of Disciplinary Language**

In this unit, teachers can anticipate students using language for mathematical purposes such as generalizing, justifying, and comparing. Throughout the unit, students will benefit from routines designed to grow robust disciplinary language, both for their own sense-making and for building shared understanding with peers. Teachers can formatively assess how students are using language in these ways, particularly when students are using language to:

#### **Generalize**

- about what happens to inputs for each rule (Lesson 1)
- about dimensions of cylinders (Lesson 14)
- about the relationship between the volumes of cylinders and cones (Lesson 15)
- about dimensions of cones (Lesson 16)
- about volumes of spheres, cones, and cylinders as functions of their radii (Lesson 21)

#### **Justify**

- claims about what can be determined from given information (Lesson 2)
- claims about volumes of cubes and spheres based on graphs (Lesson 7)
- claims about approximately linear relationships (Lesson 10)
- reasoning about the volumes of spheres and cones (Lesson 21)

#### **Compare**

- different representations of functions (Lesson 3)

- features of graphs, equations, and situations (Lesson 4)
- features of a situation with features of a graph (Lesson 6)
- temperatures shown on a graph with different temperatures given in a table (Lesson 7)
- the volumes of cones with the volumes of cylinders (Lesson 16)
- methods for finding and approximating the volume of a sphere as function of its radius (Lesson 20)

In addition, students are expected to interpret representations of volume functions of cylinders, cones, and spheres; describe quantities in a situation; describe volume measurements and features of three dimensional figures; describe the effects of varying dimensions of rectangular prisms and cones on their volumes; and describe and represent approximately linear relationships. Students are also expected to use language to represent relationships between volume and variable side length of a rectangular prism and relationships between volume and variable height of a cylinder; explain and represent how height and volume of cylinders are related; and explain reasoning about finding the volume of a cylinder and about the relationship between volumes of hemispheres and volumes of boxes, cylinders, and cones.

The table shows lessons where new terminology is first introduced, including when students are expected to understand the word or phrase receptively and when students are expected to produce the word or phrase in their own speaking or writing. Terms from the glossary appear bolded. Teachers should continue to support students' use of a new term in the lessons that follow the one in which it was first introduced.

# Learning Targets

## Functions and Volume

### Lesson 1: Inputs and Outputs

- I can write rules when I know input-output pairs.
- I know how an input-output diagram represents a rule.

### Lesson 2: Introduction to Functions

- I know that a function is a rule with exactly one output for each allowable input.
- I know that if a rule has exactly one output for each allowable input, then the output depends on the input.

### Lesson 3: Equations for Functions

- I can find the output of a function when I know the input.
- I can name the independent and dependent variables for a given function and represent the function with an equation.

### Lesson 4: Tables, Equations, and Graphs of Functions

- I can identify graphs that do, and do not, represent functions.
- I can use a graph of a function to find the output for a given input and to find the input(s) for a given output.

### Lesson 5: More Graphs of Functions

- I can explain the story told by the graph of a function.

### Lesson 6: Even More Graphs of Functions

- I can draw the graph of a function that represents a real-world situation.

### Lesson 7: Connecting Representations of Functions

- I can compare inputs and outputs of functions that are represented in different ways.

## **Lesson 8: Linear Functions**

- I can determine whether a function is increasing or decreasing based on whether its rate of change is positive or negative.
- I can explain in my own words how the graph of a linear function relates to its rate of change and initial value.

## **Lesson 9: Linear Models**

- I can decide when a linear function is a good model for data and when it is not.
- I can use data points to model a linear function.

## **Lesson 10: Piecewise Linear Functions**

- I can create graphs of non-linear functions with pieces of linear functions.

## **Lesson 11: Filling containers**

- I can collect data about a function and represent it as a graph.
- I can describe the graph of a function in words.

## **Lesson 12: How Much Will Fit?**

- I know that volume is the amount of space contained inside a three-dimensional figure.
- I recognize the 3D shapes cylinder, cone, rectangular prism, and sphere.

## **Lesson 13: The Volume of a Cylinder**

- I can find the volume of a cylinder in mathematical and real-world situations.
- I know the formula for volume of a cylinder.

## **Lesson 14: Finding Cylinder Dimensions**

- I can find missing information about a cylinder if I know its volume and some other information.

## **Lesson 15: The Volume of a Cone**

- I can find the volume of a cone in mathematical and real-world situations.
- I know the formula for the volume of a cone.

## **Lesson 16: Finding Cone Dimensions**

- I can find missing information of about a cone if I know its volume and some other information.

## **Lesson 17: Scaling One Dimension**

- I can create a graph the relationship between volume and height for all cylinders (or cones) with a fixed radius.
- I can explain in my own words why changing the height by a scale factor changes the volume by the same scale factor.

## **Lesson 18: Scaling Two Dimensions**

- I can create a graph representing the relationship between volume and radius for all cylinders (or cones) with a fixed height.
- I can explain in my own words why changing the radius by a scale factor changes the volume by the scale factor squared.

## **Lesson 19: Estimating a Hemisphere**

- I can estimate the volume of a hemisphere by calculating the volume of shape I know is larger and the volume of a shape I know is smaller.

## **Lesson 20: The Volume of a Sphere**

- I can find the volume of a sphere when I know the radius.

## **Lesson 21: Cylinders, Cones, and Spheres**

- I can find the radius of a sphere if I know its volume.
- I can solve mathematical and real-world problems about the volume of cylinders, cones, and spheres.

## **Lesson 22: Volume As a Function of . . .**

- I can compare functions about volume represented in different ways.

lesson	new terminology	
	receptive	productive
8.5.1	input output	
8.5.2	<b>function</b>	input output depends on
8.5.3	<b>independent variable</b> <b>dependent variable</b> <b>radius</b>	
8.5.5	<b>prediction</b>	
8.5.7	<b>volume</b> cube	
8.5.8	functional relationship linear function	<b>function</b>
8.5.9	<b>mathematical model</b>	<b>prediction</b>
8.5.10	<b>piecewise linear function</b>	linear function constant rate
8.5.11	<b>cylinder</b> three-dimensional	
8.5.12	<b>cone</b> <b>sphere</b> dimension	<b>cylinder</b> cube cubic centimeter rectangular prism
8.5.13	base (of a cylinder or cone) approximation for $\pi$	
8.5.14		<b>radius</b> base (of a cylinder or cone)
8.5.16		<b>cone</b>
8.5.19	<b>hemisphere</b>	

8.5.20		<b>sphere</b>
8.5.21	spherical	<b>volume</b>
8.5.22	approximate range	

## Required Materials

**Colored pencils**

**Graduated cylinders**

**Pre-printed slips, cut from copies of the**

**Instructional master**

**Spherical objects**

**Straightedges**

A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to

use an unruled straightedge, like a blank index card.

**Tools for creating a visual display**

Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

8.5.20		<b>sphere</b>
8.5.21	spherical	<b>volume</b>
8.5.22	approximate range	

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# Section: Inputs and Outputs

## Lesson 1: Inputs and Outputs

### Goals

- Describe (orally) how input-output diagrams represent rules.
- Identify a rule that describes the relationship between input-output pairs and explain (orally) a strategy used for figuring out the rule.

### Learning Targets

- I can write rules when I know input-output pairs.
- I know how an input-output diagram represents a rule.

### Lesson Narrative

This is the first of two lessons introducing students to functions, developing the idea of a function as a rule that assigns to each allowable input exactly one output. The word function is not introduced until the second lesson. In future lessons, students will expand on this definition as they work with different representations of functions.

In the first classroom activity students take turns guessing each other's rules from input-output pairs. In the second activity students use rules represented by input-output diagrams to fill out a table with inputs and associated outputs. In each table, the first input-output pair is identical, illustrating that a single pair is insufficient for determining a rule. The last table returns to the topic of the warm-up and introduces the idea that not all inputs are possible for a rule.

### Alignments

#### Addressing

- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.

#### Building Towards

- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.

#### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR3: Clarify, Critique, Correct
- MLR7: Compare and Connect
- Think Pair Share

## Required Materials

Pre-printed slips, cut from copies of the

Instructional master

## Required Preparation

Print and cut up slips from the Guess my Rule Instructional master. Prepare 1 copy for every 4 students. During the activity, you will want students to place the slips in a pile, face down, with rule A on top and rule D on the bottom.

### Student Learning Goals

Let's make some rules.

## 1.1 Dividing by 0

### Warm Up: 5 minutes

The purpose of this activity is for students to see why an expression that contains dividing by zero can't be evaluated. They use their understanding of related multiplication and division equations to make sense of this.

### Building Towards

- 8.F.A.1

### Instructional Routines

- Think Pair Share

### Launch

Arrange students in groups of 2. Tell students to consider the statements and try to find a value for  $x$  that makes the second statement true.

Give students 1 minute of quiet think time followed by 1 minute to share their thinking with their partner. Finish with a whole-class discussion.

### Student Task Statement

Study the statements carefully.

- $12 \div 3 = 4$  because  $12 = 4 \cdot 3$
- $6 \div 0 = x$  because  $6 = x \cdot 0$

What value can be used in place of  $x$  to create true statements? Explain your reasoning.

## Student Response

None. Explanations vary. Sample explanation: There is no number that can be multiplied by zero to get something other than zero. Therefore,  $x \cdot 0 = 6$  is never a true statement for any  $x$ .

## Activity Synthesis

Select 2–3 groups to share their conclusions about  $x$ .

As a result of this discussion, we want students to understand that any expression where a number is divided by zero can't be evaluated. Therefore, we can state that there is no value for  $x$  that makes both equations true.

# 1.2 Guess My Rule

**15 minutes (there is a digital version of this activity)**

The purpose of this activity is to introduce the idea of *input-output rules*. One student chooses inputs to tell a partner who uses a rule written on a card only they can see to respond with the corresponding output. The first student then guesses the rule on the card once they think they have enough input-output pairs to know what it is. Partners then reverse roles.

Monitor for students who:

- Appear to have a strategy for choosing which numbers to give, such as always starting with 0 or 1, or choosing a sequence of consecutive whole numbers.
- Notice the difference between the result for odd numbers and the result for even numbers for the card with a different rule for each.

## Addressing

- 8.F.A.1

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

## Launch

Arrange students in groups of 2.

For students using the print version: Have a student act as your partner and demonstrate the game using a simple rule that does not match one of the cards (like “divide by 2” or “subtract 4”).

Ask groups to decide who will be Player 1 and who will be Player 2. Give each group the four rule cards, making sure that the simplest rules are at the top of the deck when face down. Tell students to be careful not to let their partner see what the rule is as they pick up the rule cards. If necessary, tell students that all numbers are allowed, including negative numbers.

For students using the digital version: Demonstrate how to use the applet by choosing one input value, clicking the appropriate button, and noting where the output is recorded. Prompt partners to discuss which input values to select for the rule and alternate who guesses the rule.

---

### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity by beginning with fewer cards. For example, give students a subset of the cards to start with and introduce the remaining cards once students have completed their initial set of rules.

*Supports accessibility for: Conceptual processing; Organization*

---

### Anticipated Misconceptions

Students may have difficulty with the last rule because it involves conditional statements. Encourage them to focus on a single value at a time to help them understand how this kind of rule works.

#### Student Task Statement

Keep the rule cards face down. Decide who will go first.

1. Player 1 picks up a card and silently reads the rule without showing it to Player 2.
2. Player 2 chooses an integer and asks Player 1 for the result of applying the rule to that number.
3. Player 1 gives the result, without saying how they got it.
4. Keep going until Player 2 correctly guesses the rule.

After each round, the players switch roles.

#### Student Response

$$x + 7, 3x, 2x - 5, \frac{x}{2}, 3x + 1$$

#### Are You Ready for More?

If you have a rule, you can apply it several times in a row and look for patterns. For example, if your rule was "add 1" and you started with the number 5, then by applying that rule over and over again you would get 6, then 7, then 8, etc., forming an obvious pattern.

Try this for the rules in this activity. That is, start with the number 5 and apply each of the rules a few times. Do you notice any patterns? What if you start with a different starting number?

#### Student Response

1. 5, 12, 19, 26, 33, ...

2. 5, 15, 45, 135, . . .

3. 5, 5, 5, 5, . . .

4. 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1

If you change the starting number from 5, the first two rules give very similar patterns, just adding 7 and tripling each time. The third pattern has a much harder pattern to recognize when you don't start with a 5. The fourth pattern turns out to be incredibly difficult! Most mathematicians suspect that no matter *what* value you start with, you will always eventually get back to 1, and starting cycling 1, 4, 2, 1, 4, 2, 1, . . . . Computers have checked that this pattern appears for any starting number up to one billion billion, but no one knows for sure that it will *always* appear! For more information, research the Collatz Conjecture.

### **Activity Synthesis**

The goal of this discussion is for students to understand what an input-output rule is and share strategies they used for figuring out the rule. Tell students we start with a number, called the input, and apply a rule to that number which results in a number called the output. We say the output *corresponds* to that input.

To highlight these words, ask:

- "What is an example of an input from this activity?" (The input is the number the player without a card chose, or the number placed in the 'input' box in the applet.)
- "Where in the activity was the rule applied?" (The rule was applied when the player with the card applied the rule to the input their partner told them, or it's what happened in the black box in the applet.)
- "Where in the activity was the output?" (The output is the number the player said after applying the rule to the given input, or the number the black box in the applet gave.)

Select previously identified students who appeared to have a strategy for figuring out a rule to share it. Sequence students starting with the most common strategies to the least. Make connections between the successful aspects of each strategy (for example, if a student does not say it, stating clearly why using a sequence of integers or using 0 and 1 are helpful for determining the rule).

Students might think the last rule isn't allowable because there were two "different" rules. Explain that a rule can be anything that always produces an output for a given input. Consider the rule "flip" where the input is "coin." The output may be "heads" or "tails." We will consider several different types of rules in the following activities and lessons.

---

### Access for English Language Learners

*Listening, Speaking, Conversing: MLR7 Compare and Connect.* Use this routine when students present their strategies for determining the rules for each card. Ask students to identify "What is similar and what is different" about each approach. Draw students' attention to the different ways the rules were represented in each approach. For instance, for the rule  $4x + 1$ , students may have expressed it as either  $3 + 3 + 3 + 3 + 1$ , or  $2(3) + 2(3) + 1$ , or  $5(3) - (3) + 1$ , or  $4(3) + 1$ . In this discussion, emphasize the mathematical language used to make sense of the different ways students generalized and represented the rule. These exchanges strengthen students' mathematical language use and reasoning when comparing strategies for identifying and representing functions.

*Design Principle(s): Cultivate conversation; Maximize meta-awareness*

---

## 1.3 Making Tables

**15 minutes**

The purpose of this activity is for students to think of rules more broadly than simple arithmetic operations in preparation for the more abstract idea of a function, which is introduced in the next lesson.

Each problem begins with a diagram representing a rule followed by a table for students to complete with input-output pairs that follow the rule. Monitor for students who notice that even though the rules are different, each one starts out with the same input-output pair:  $\frac{3}{4}$  and 7. An important conclusion is that different rules can determine the same input-output pair.

If using digital activities, there is a rule generator as an extension. Students give an input and the generator gives an output, after a few inputs students can guess a potential rule, the generator indicates if the rule is "reasonable but not my rule", "correct! how did you know", or "does not match data."

### Addressing

- 8.F.A.1

### Instructional Routines

- MLR3: Clarify, Critique, Correct

### Launch

Arrange students in groups of 2. Display the following diagram for all to see:



Tell students that this diagram is one way to think about input-output rules. For example, if the rule was "multiply by 2" and the input  $\frac{3}{2}$ , then the output would be 3. Tell students they will use diagrams like this one during the activity to complete tables of input and output values.

Give students 3–5 minutes of quiet work time to complete the first three tables. Then give them time to share their responses with their partner and to resolve any differences.

Give partners 1–2 minutes of quiet work time for the final rule followed by a whole-class discussion. Depending on time, have students add only one additional input-output pair instead of two.

### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Activate or supply background knowledge about the similarities between this activity and the previous activity identifying input-output rules. Allow students to use calculators to ensure inclusive participation in the activity.

*Supports accessibility for: Memory; Conceptual processing*

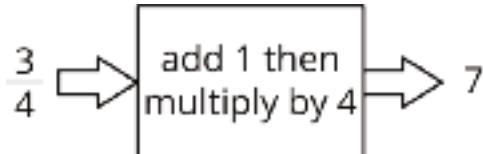
### Anticipated Misconceptions

Students may have trouble thinking of "write 7" as a rule. Emphasize that a rule can be anything that produces a well-defined output, even if it ignores the value of the input. Students who know about infinite decimal expansions might wonder about the second rule, because  $0.999\ldots = 1.0$ , so the same number could have two outputs. If this comes up, discuss how you might refine the rule in this case.

### Student Task Statement

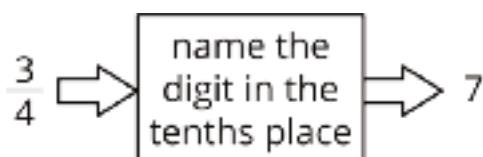
For each input-output rule, fill in the table with the outputs that go with a given input. Add two more input-output pairs to the table.

1.



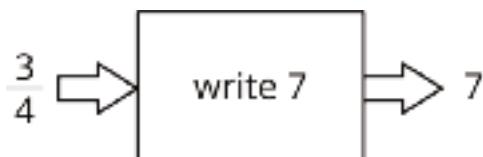
input	output
$\frac{3}{4}$	7
2.35	
42	

2.



input	output
$\frac{3}{4}$	7
2.35	
42	

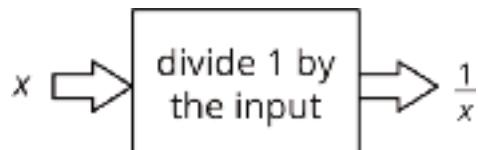
3.



input	output
$\frac{3}{4}$	7
2.35	
42	

Pause here until your teacher directs you to the last rule.

4.



input	output
$\frac{3}{7}$	$\frac{7}{3}$
1	
0	

### Student Response

Answers vary for the last two rows in each table.

1.

input	output
$\frac{3}{4}$	7
2.35	13.4
42	172
9	40
1	8

2.

input	output
$\frac{3}{4}$	7
2.35	3
42	0
7.31	3
4.95	9

3.

input	output
$\frac{3}{4}$	7
2.35	7
42	7
511	7
91	7

4.

input	output
$\frac{3}{7}$	$\frac{7}{3}$
1	1
0	no output
2	$\frac{1}{2}$

### Activity Synthesis

The purpose of this discussion is for students to see rules as more than arithmetic operations on numbers and consider how sometimes not all inputs are possible.

Display a rule diagram with input 2, output 6, and a blank space for the rule for all to see.



Select 2-3 previously identified students and ask what the rule for the input-output pair might be. Display these possibilities next to the diagram for all to see. For example, students may suggest the rules such as "add 4", "multiply by 3", or "add 1 then multiply by 2."

The last rule, "1 divided by the input," calls back to the warm-up. Explain to students that not all inputs are possible for a rule. To highlight this idea, ask:

- "Why was 0 not a valid input for the last rule?" (1 divided by 0 does not exist)
- "What are some other situations when a rule might not have a valid input?" (Any time an operation requires you to divide by 0, or when the input must be non-negative, such as a side length of a square when you know the area.)

- "How does using a variable  $x$  to denote the input and  $\frac{1}{x}$  to denote the output help us understand the function rule?" (You can clearly see the relationship between the input and output.)

---

### Access for English Language Learners

*Writing: MLR3 Clarify, Critique, Correct.* Present an incorrect statement that reflects a possible misunderstanding about what a rule can be. For example, Bryce said, "Write 7' isn't a rule because you just write the number 7." Prompt students to critique the statement (e.g., ask students whether they agree, and why or why not), and then write feedback to the author that explains why "Write 7" is a rule. Improved statements should describe rules as more than arithmetic operations on numbers. This helps students evaluate, and improve on, the written mathematical arguments of others.

*Design Principle(s): Maximize meta-awareness*

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## Lesson Synthesis

The main focus of this lesson are the parallel ideas of using sets of inputs and outputs to identify the rule describing the relationship between them and using a rule to create a set of inputs and outputs. When we have an input-output table that represents only some of the input-output pairs, we can guess a rule but we won't know the rule for sure until we see it. For some rules, there are some numbers that are not allowed as inputs because the rule does not make sense.

To highlight some things to remember about input-output rules, ask:

- "Which rule would you rather make an input output table for: 'divide 10 by the input' or 'write the current year'?" (I would rather make a table for the second rule since the outputs would all just be the current year. The table for the first rule will have fractions and you can't input 0 since 0 divided by 10 does not exist.)
- "What input can you not use with the rule 'divide 10 by the input added to 3'?" (We cannot use -3 since  $-3 + 3 = 0$  and 0 divided by 10 does not exist.)

To conclude the discussion, poll the class to find out how many input-output pairs students think they would need to figure out a rule and record their answers for all to see. While much of the work students do in grade 8 is linear making "two" a sufficient number of pairs, if students think 2 is always enough point out that both the first and second rule in this activity share the pair  $\frac{3}{4}$  and 7 and the pair -1 and 0. Depending on the context, 2 input-output pairs is not always sufficient to determine the rule.

## 1.4 What's the Rule?

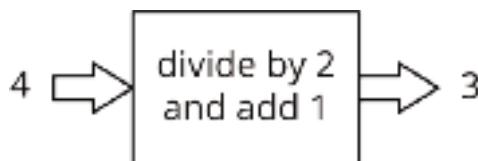
Cool Down: 5 minutes

## Addressing

- 8.F.A.1

### Student Task Statement

Fill in the table for this input-output rule:



input	output
0	
2	
-8	
100	

### Student Response

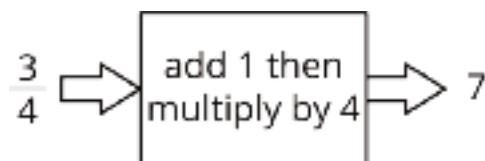
In each row, the output should be one more than half of the input.

input	output
0	1
2	2
-8	-3
100	51

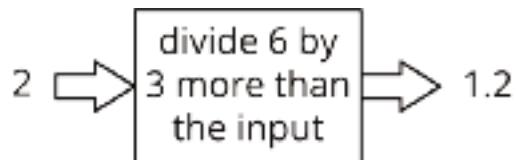
### Student Lesson Summary



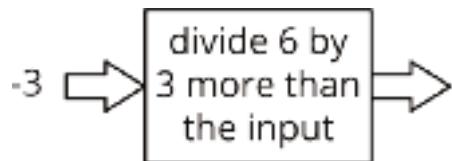
An *input-output rule* is a rule that takes an allowable input and uses it to determine an output. For example, the following diagram represents the rule that takes any number as an input, then adds 1, multiplies by 4, and gives the resulting number as an output.



In some cases, not all inputs are allowable, and the rule must specify which inputs will work. For example, this rule is fine when the input is 2:



But if the input is -3, we would need to evaluate  $6 \div 0$  to get the output.



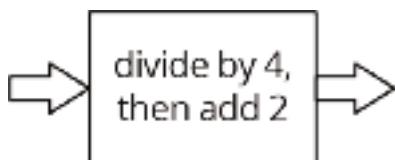
So, when we say that the rule is "divide 6 by 3 more than the input," we also have to say that -3 is not allowed as an input.

# Lesson 1 Practice Problems

## Problem 1

### Statement

Given the rule:



Complete the table for the function rule for the following input values:

input	0	2	4	6	8	10
output						

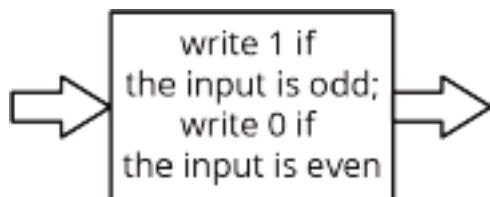
### Solution

2, 2.5, 3, 3.5, 4, 4.5

## Problem 2

### Statement

Here is an input-output rule:



Complete the table for the input-output rule:

input	-3	-2	-1	0	1	2	3
output							

### Solution

1, 0, 1, 0, 1, 0, 1, respectively

## Problem 3

### Statement

Andre's school orders some new supplies for the chemistry lab. The online store shows a pack of 10 test tubes costs \$4 less than a set of nested beakers. In order to fully equip the lab, the school orders 12 sets of beakers and 8 packs of test tubes.

- a. Write an equation that shows the cost of a pack of test tubes,  $t$ , in terms of the cost of a set of beakers,  $b$ .
- b. The school office receives a bill for the supplies in the amount of \$348. Write an equation with  $t$  and  $b$  that describes this situation.
- c. Since  $t$  is in terms of  $b$  from the first equation, this expression can be substituted into the second equation where  $t$  appears. Write an equation that shows this substitution.
- d. Solve the equation for  $b$ .
- e. How much did the school pay for a set of beakers? For a pack of test tubes?

### Solution

- a.  $t = b - 4$
- b.  $8t + 12b = 348$
- c.  $8(b - 4) + 12b = 348$
- d.  $b = 19$
- e. \$19 and \$15

(From Unit 4, Lesson 15.)

## Problem 4

### Statement

$$\text{Solve: } \begin{cases} y = x - 4 \\ y = 6x - 10 \end{cases}$$

### Solution

$\left(\frac{6}{5}, \frac{-14}{5}\right)$ . Substituting  $x - 4$  for  $y$  into the second equation, we get  $x - 4 = 6x - 10$ . Solving this equation gives  $x = \frac{6}{5}$ . Substituting  $x = \frac{6}{5}$  into  $y = x - 4$ , we get  $y = \frac{-14}{5}$ .

(From Unit 4, Lesson 14.)

## Problem 5

### Statement

For what value of  $x$  do the expressions  $2x + 3$  and  $3x - 6$  have the same value?

### Solution

$$x = 9$$

(From Unit 4, Lesson 9.)

# Lesson 2: Introduction to Functions

## Goals

- Comprehend the structure of a function as having one and only one output for each allowable input.
- Describe (orally and in writing) a context using function language, e.g., "the [output] is a function of the [input]" or "the [output] depends on the [input]."
- Identify (orally) rules that produce exactly one output for each allowable input and rules that do not.

## Learning Targets

- I know that a function is a rule with exactly one output for each allowable input.
- I know that if a rule has exactly one output for each allowable input, then the output depends on the input.

## Lesson Narrative

In this second introductory lesson, students learn the term **function** for a rule that produces a single output for a given input. They also start to connect function language to language they learned in earlier grades about independent and dependent variables.

We can say, "the output is a function of the input," and we also say, "the output depends on the input." In the optional activity, students see how it is possible to use different words to describe the same function as long as all input-output pairs are the same. This helps solidify the notion of a function as something different from the method of calculating its values.

## Alignments

### Addressing

- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.

### Building Towards

- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.

## Instructional Routines

- MLR1: Stronger and Clearer Each Time

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Think Pair Share

### **Student Learning Goals**

Let's learn what a function is.

## **2.1 Square Me**

### **Warm Up: 5 minutes**

The purpose of this warm-up is to remind students that two different numbers can have the same square. This is an example of two inputs having the same output for a given rule—in this case "square the number." Later activities in the lesson explore rules that have multiple outputs for the same input.

### **Building Towards**

- 8.F.A.1

### **Launch**

Give students 1–2 minutes of quiet work time followed by a whole-class discussion.

### **Student Task Statement**

Here are some numbers in a list:

$$1, -3, -\frac{1}{2}, 3, 2, \frac{1}{4}, 0.5$$

1. How many different numbers are in the list?
2. Make a new list containing the squares of all these numbers.
3. How many different numbers are in the new list?
4. Explain why the two lists do not have the same number of different numbers.

### **Student Response**

1. 7
2.  $1, 9, \frac{1}{4}, 9, 4, \frac{1}{16}, 0.25$
3. 5
4. Answers vary. Sample response: Some numbers in the list are different but have the same square. This can happen because a negative times a negative is a positive. For example,  $-3$  squared is 9. 3 squared is also 9.

## Activity Synthesis

The focus of this discussion should be on the final question, which, even though the language isn't used in the problem, helps prepare students for thinking about the collection of values that make up the input and output of rules. Here, the input is a list of 7 unique values while the output has only 5 unique values.

Invite students to share their responses to the second problem and display the list of numbers for all to see along with the original list. Next, invite different students to share their explanations from the forth problem. Emphasize the idea that when we square a negative number, we get a positive number. This means two different numbers can have the same square, or, using the language of inputs and outputs, two different inputs can have the same output for a rule.

If time allows, ask "Can you think of other rules where different inputs can have the same output?" After 30 seconds of quiet think time, select students to share their rules. They may recall the previous lesson where they encountered the rule "write 7," which has only one output for all inputs, and the rule "extract the digit in the tenths place," which has only 10 unique outputs for all inputs.

## 2.2 You Know This, Do You Know That?

**15 minutes**

In this activity students are presented with a series of questions like, "A person is 60 inches tall. Do you know their height in feet?" For some of the questions the answer is "yes" (because you can convert from inches to feet by dividing by 12). In other cases the answer is "no" (for example, "A person is 14 years old. Do you know their height?"). The purpose is to develop students' understanding of the structure of a function as something that has one and only one output for each allowable input. In cases where the answer is yes, students draw an input-output diagram with the rule in the box. In cases where the answer is no, they give examples of an input with two or more outputs. In the Activity Synthesis, the word function is introduced to students for the first time.

Identify students who use different ways to describe the rules and different notation for the input and outputs of the final two problems.

### Addressing

- 8.F.A.1

### Instructional Routines

- MLR8: Discussion Supports

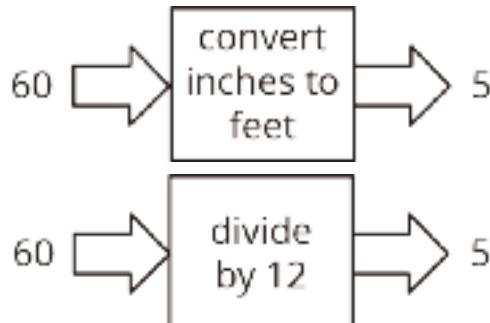
### Launch

Display the example statement (but not the input-output diagram) for all to see.

Example: A person is 60 inches tall. Do you know their height in feet?

Give students 30 seconds of quiet think time, and ask them to be prepared to justify their response. Select students to share their answers, recording and displaying different justifications for all to see.

Display the following input-output diagrams for all to see. Ask students if the rules in the diagrams match the justifications they just heard:



or

Tell students that they will draw input-output diagrams like these as part of the task. Answer any questions students might have around the input-output diagrams. Be sure students understand that if they answer yes to the question they will need to draw the input-output diagram and if they answer no they need to give an example of why the question does not have one answer.

Give students 8–10 minutes of quiet work time for the problems followed by a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Access for Perception.* Read all statements aloud. Students who both listen to and read the information will benefit from extra processing time.

*Supports accessibility for: Language*

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### Student Task Statement

Say yes or no for each question. If yes, draw an input-output diagram. If no, give examples of two different outputs that are possible for the same input.

1. A person is 5.5 feet tall. Do you know their height in inches?
2. A number is 5. Do you know its square?
3. The square of a number is 16. Do you know the number?

4. A square has a perimeter of 12 cm. Do you know its area?

5. A rectangle has an area of  $16 \text{ cm}^2$ . Do you know its length?

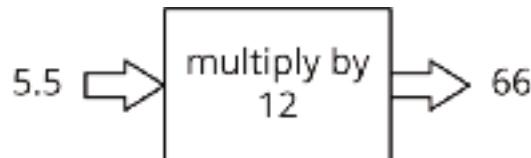
6. You are given a number. Do you know the number that is  $\frac{1}{5}$  as big?

7. You are given a number. Do you know its reciprocal?

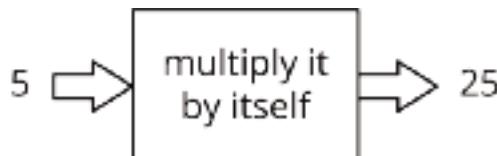
### Student Response

Answers vary. Sample responses:

1. Yes, multiply a person's height in feet by 12 to get their height in inches. Since  $5.5(12) = 66$ , this person is 66 inches tall.

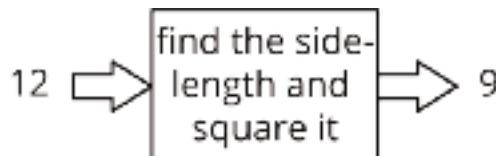


2. Yes, the square of 5 is 25.



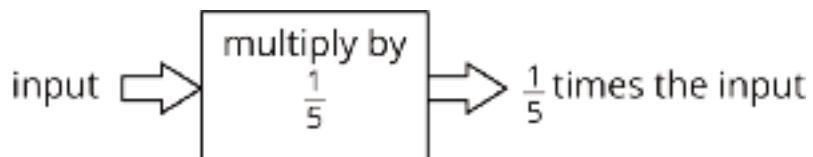
3. No, there are two different numbers whose square is 16, namely 4 and -4.

4. Yes, a square with perimeter 12 cm must have a side length of 3 cm, and so an area of  $9 \text{ cm}^2$ .

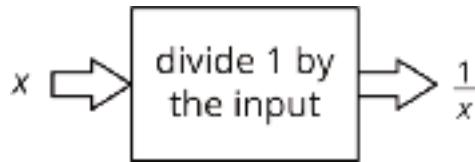


5. No, a rectangle with length 8 cm and width 2 cm has area  $16 \text{ cm}^2$ , as does a rectangle with length 16 cm and width 1 cm.

6. Yes, for any number, we can simply divide that number by 5.



7. Yes, as long as the number isn't 0 (since 0 doesn't have a reciprocal).



## Activity Synthesis

The goal of this discussion is for students to understand that functions are rules that have one distinct output for each input. For several problems, select previously identified students to share their rule descriptions. For example, some students might write the rule for finding the area of a square from its perimeter as "find the side length then square it" and others might write, "divide by four and then square it." Compare the different rules and ask students if they agree (or disagree) that the statements represent the same rule.

For the final two problems where the input is not a specific value, select previously identified students to share what language they used. For example, in the final problem some students may use a letter to stand for the input while others may just write "input" or "a number." Ask, "How can using a letter sometimes make it easier to represent the output?" (Writing  $\frac{1}{5}n$  is shorter than writing  $\frac{1}{5}$  of a number.)

Tell students that each time they answered one of the questions with a yes, the sentence defined a **function**, and that one way to represent a function is by writing a rule to define the relationship between the input and the output. Functions are special types of rules where each input has only one possible output. Because of this, functions are useful since once we know an input, we can find the single output that goes with it. Contrast this with something like a rolling a number cube where the input "roll" has many possible outputs. For the questions students responded to with no, these are not functions because there is no single output for each input.

To highlight how rules that are not functions do not determine outputs in a unique way, end the discussion by asking:

- "Was the warm-up, where you have to square numbers, an example of a function?" (Yes, each input has only one output, even though some inputs have the same output.)
- "Is the reverse, that is knowing what number was squared to get a specific number, a function?" (No, if a number squared is 16, we don't know if the number was 4 or -4.)

---

### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support whole-class discussion. Call on students to use mathematical language to restate and/or revoice the rule descriptions that are presented. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer accurately restated their rule. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language to describe functions.

*Design Principle(s): Support sense-making; Maximize meta-awareness*

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## 2.3 Using Function Language

**15 minutes**

In this activity students revisit the questions in the previous activity and start using the language of functions to describe the way one quantity depends on another. For the "yes" questions students write a statement like, "[the output] depends on [the input]" and "[the output] is a function of [the input]." For the "no" questions, they write a statement like, "[the output] does not depend on [the input]." Students will use this language throughout the rest of the unit and course when describing functions.

Depending on the time available and students' needs, you may wish to assign only a subset of the questions, such as just the odds.

### Addressing

- 8.F.A.1

### Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

### Launch

Display the example statement from the previous activity ("A person is 60 inches tall. Do you know their height in feet?") for all to see. Tell students that since the answer to this question is yes, we can write a statement like, "height in feet depends on the height in inches" or "height in feet is a function of height in inches."

Arrange students in groups of 2. Give students 5–8 minutes of quiet work time and then additional time to share their responses with their partner. If they have a different response than their partner, encourage them to explain their reasoning and try to reach agreement. Follow with a whole-class discussion.

## Student Task Statement

Here are the questions from the previous activity. For the ones you said yes to, write a statement like, “The height a rubber ball bounces to depends on the height it was dropped from” or “Bounce height is a **function** of drop height.” For all of the ones you said no to, write a statement like, “The day of the week does not determine the temperature that day” or “The temperature that day is not a function of the day of the week.”

1. A person is 5.5 feet tall. Do you know their height in inches?
2. A number is 5. Do you know its square?
3. The square of a number is 16. Do you know the number?
4. A square has a perimeter of 12 cm. Do you know its area?
5. A rectangle has an area of  $16 \text{ cm}^2$ . Do you know its length?
6. You are given a number. Do you know the number that is  $\frac{1}{5}$  as big?
7. You are given a number. Do you know its reciprocal?

## Student Response

1. Yes, height in feet depends on the height in inches.
2. Yes, the square of a number depends on the number.
3. No, knowing the square of a number does not determine the number.
4. Yes, the area of a square is determined by the perimeter of the square.
5. No, knowing the area of a rectangle does not determine its length.
6. Yes, every number determines the number which is  $\frac{1}{5}$  as large.
7. Yes, every (non-zero) number determines its reciprocal.

## Activity Synthesis

The goal of this discussion is for students to use the language like “[the output] depends on [the input]” and “[the output] is a function of [the input]” while recognizing that a function means each input gives exactly one output.

Begin the discussion by asking students if any of them had a different response from their partner that they were not able to reach agreement on. If any groups say yes, ask both partners to share their responses. Next, select groups to briefly share their responses for the other questions and address any questions. For example, students may have a correct answer but be unsure since they used different wording than the person who shared their answer verbally with the class.

If time permits, give groups 1–2 minutes to invent a new question like the ones in the task that is *not* a function. Select 2–3 groups to share their question and ask a different group to explain why it is not a function using language like, "the input does not determine the output because. . . ."

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### Access for English Language Learners

*Speaking, Writing: MLR2 Collect and Display.* While pairs are working, circulate and listen to student talk as they use the language like "[the output] depends on [the input]" and "[the output] is a function of [the input]." Capture student language that reflect a variety of ways to describe that a function means each input gives exactly one output. Display the language collected visually for the whole class to use as a reference during further discussions throughout the lesson and unit. Invite students to suggest revisions, updates, and connections to the display as they develop new mathematical ideas and new ways of communicating ideas. This will help students increase awareness of use of math language as they progress through the unit.

*Design Principle(s): Support sense-making; Maximize meta-awareness*

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## 2.4 Same Function, Different Rule?

**Optional:** 5 minutes

The activity calls back to a previous lesson where students filled out tables of values from input-output diagrams. Here, students determine if a rule is describing the same function but with different words, giving them an opportunity to look for and make use of the structure of a function (MP7).

Students are given 3 different input-output diagrams and need to determine which rules could describe the same function. A key point in this activity is that context plays an important role. For example, if the first rule is limited to positive inputs and the second rule is about sides of squares (which also has only positive inputs), then the two input-output rules describe the same function.

### Addressing

- 8.F.A.1

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

### Launch

Give students 1–2 minutes of quiet work time followed by a whole-class discussion.

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### Access for Students with Disabilities

*Action and Expression: Develop Expression and Communication.* To help get students started, display sentence frames such as "Is it always true that..." or "That could/couldn't be the same function because..."

*Supports accessibility for: Language; Organization*

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### Access for English Language Learners

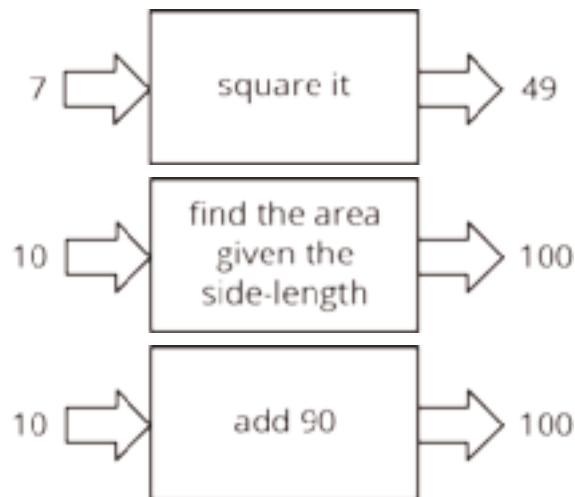
*Writing, Listening, Speaking: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise and refine their explanations for "Which input-output rules could describe the same function (if any)?" Give students time to meet with 2–3 partners, to share and get feedback on their responses. Provide prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., "Can you say more about why?", "Can you give an example?", "How could you say that another way?", etc.). Give students 1–2 minutes to revise their writing based on the feedback they received.

*Design Principle(s): Optimize output (for explanation)*

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### Student Task Statement

Which input-output rules could describe the same function (if any)? Be prepared to explain your reasoning.



### Student Response

The first two rules define the same function if we are talking about the area of a square. Since the area of a square is the square of its side length, the second rule is the same as taking the input and

squaring it to find the output. On the other hand, even though adding 90 to 10 gives the same result as squaring 10, this third rule does not define the same function as the first two. For example, adding 90 to 20 gives 110, which is not the same as squaring it, or finding the area of the square with side length 20.

### Are You Ready for More?

The phrase “is a function of” gets used in non-mathematical speech as well as mathematical speech in sentences like, “The range of foods you like is a function of your upbringing.” What is that sentence trying to convey? Is it the same use of the word “function” as the mathematical one?

### Student Response

Answers vary.

### Activity Synthesis

The goal of this discussion is for students to explain how two different rules can describe the same function and that two functions are the same if and only if all of their input-output pairs are the same.

Consider asking some of the following questions:

- “Do the latter two input-output rules describe the same function since they both take an input of 10 to an output of 100?” (No, every input-output pair needs to match in order for the two rules to describe the same function not just a few input-output pairs.)
- “Do any of the input-output rules describe the same function?” (Yes, if the first is restricted to positive inputs and the second is about areas of squares, then they share the same input-output pairs and describe the same function.)

### Lesson Synthesis

The purpose of this lesson was to define functions as rules that assign exactly one output to each allowable input. We say things like “the output is a **function** of the input,” and “the output depends on the input” when talking about the relationship between inputs and outputs of functions.

To highlight the language and definition of functions from the lesson, ask:

- “How else could we describe the function ‘double the input?’” (Multiply the input by 2 or, if the input is  $x$ ,  $2x$ .)
- “Is the rule ‘the radius of a circle with circumference  $C$ ’ a function? Why or why not?” (Yes, this rule is a function because the radius of circle depends on the circumference and each radius gives only one circumference.)
- “Why does the description ‘A person’s age is 14 years old. What is their height in inches?’ not define a function?” (The age of a person does not determine what height they are. Different 14

year olds are different heights. The same 14 year old can be different heights depending on how long ago they turned 14.)

## 2.5 Wait Time

Cool Down: 5 minutes

### Addressing

- 8.F.A.1

#### Student Task Statement

You are told that you will have to wait for 5 hours in a line with a group of other people. Determine whether:

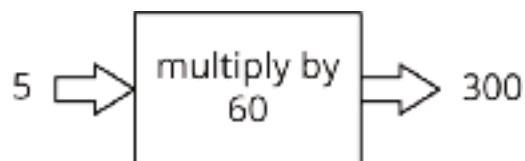
1. You know the number of minutes you have to wait.
2. You know how many people have to wait.

For each statement, if you answer yes draw an input-output diagram and write a statement that describes the way one quantity depends on another.

If you answer no give an example of 2 outputs that are possible for the same input.

#### Student Response

1. Yes, if you know how many hours you have to wait in line, then you can determine the number of minutes you have to wait in line. Answers vary. Sample response: You will have to wait 300 minutes, since each hour is 60 minutes, and  $5 \cdot 60 = 300$ .

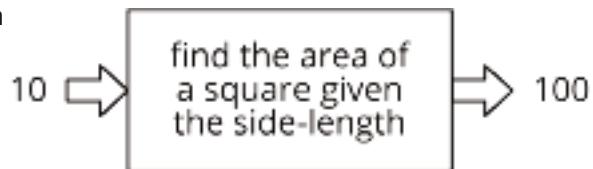


2. No, if you know how many hours you have to wait in line, you do not necessarily know how many people are in line. Answers vary. Sample response: The number of people who have to wait cannot be determined by the amount of time you have to wait. For example, there could be 50 people waiting, or there could be 100 people waiting.

#### Student Lesson Summary

Let's say we have an input-output rule that for each allowable input gives exactly one output. Then we say the output *depends* on the input, or the output is a **function** of the input.

For example, the area of a square is a function of the side length, because you can find the area from the side length by squaring it. So when the input is 10 cm, the output is 100  $\text{cm}^2$ .



Sometimes we might have two different rules that describe the same function. As long as we always get the same, single output from the same input, the rules describe the same function.

## Glossary

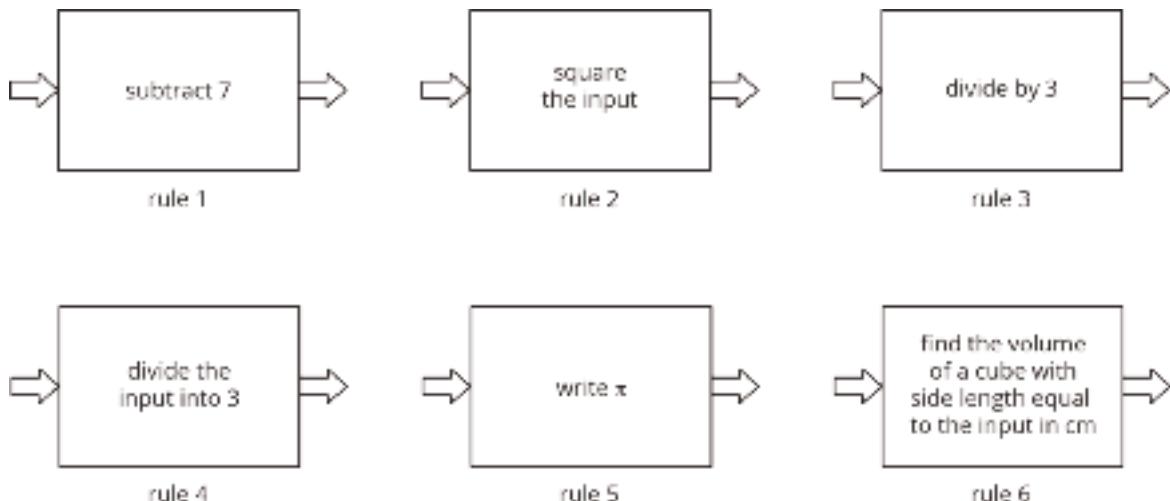
- function

# Lesson 2 Practice Problems

## Problem 1

### Statement

Here are several function rules. Calculate the output for each rule when you use  $-6$  as the input.



### Solution

- Rule 1:  $-13$
- Rule 2:  $36$
- Rule 3:  $-2$
- Rule 4:  $-\frac{1}{2}$
- Rule 5:  $\pi$
- Rule 6:  $-6$  is not a valid input for this rule since it doesn't make sense to express a side length with a negative number.

## Problem 2

### Statement

A group of students is timed while sprinting 100 meters. Each student's speed can be found by dividing 100 m by their time. Is each statement true or false? Explain your reasoning.

- a. Speed is a function of time.
- b. Time is a function of distance.
- c. Speed is a function of number of students racing.

d. Time is a function of speed.

## Solution

- a. True. For each time, one speed is generated.
- b. False. For each distance (100 m), many times are generated.
- c. False. The number of students racing does not affect any student's speed, and the same speed may be reached for more than one student in a group of the same size.
- d. True. For each speed calculated, there is only one possible time.

## Problem 3

### Statement

Diego's history teacher writes a test for the class with 26 questions. The test is worth 123 points and has two types of questions: multiple choice worth 3 points each, and essays worth 8 points each. How many essay questions are on the test? Explain or show your reasoning.

## Solution

9 essay questions. Explanations vary. Sample response: Use  $m$  to represent multiple choice questions and  $e$  for essay questions. Write the system as  $m + e = 26$  and  $3m + 8e = 123$ , and solve it by substituting  $m = 26 - e$  into the second equation.

(From Unit 4, Lesson 15.)

## Problem 4

### Statement

These tables correspond to inputs and outputs. Which of these input and output tables could represent a function rule, and which ones could not? Explain or show your reasoning.

Table A:

input	output
-2	4
-1	1
0	0
1	1
2	4

Table B:

input	output
4	-2
1	-1
0	0
1	1
4	2

Table C:

input	output
1	0
2	0
3	0

Table D:

input	output
0	1
0	2
0	3

## Solution

Table A and Table C represent functions, but Table B and Table D do not. Explanations vary. Sample response: Tables B and D have multiple outputs for the same input, but functions take each input to only one output. On the other hand, it is okay for a function rule to take different inputs to the same output.

# Section: Representing and Interpreting Functions

## Lesson 3: Equations for Functions

### Goals

- Calculate the output of a function for a given input using an equation in two variables, and interpret (orally and in writing) the output in context.
- Create an equation that represents a function rule.
- Determine (orally and in writing) the independent and dependent variables of a function, and explain (orally) the reasoning.

### Learning Targets

- I can find the output of a function when I know the input.
- I can name the independent and dependent variables for a given function and represent the function with an equation.

### Lesson Narrative

So far we have used input-output diagrams and descriptions of the rules to describe functions. This is the first of five lessons that introduces and connects the different ways in which we represent functions in mathematics: verbal descriptions, equations, tables, and graphs. In this lesson students transition from input-output diagrams and descriptions of rules to equations.

This lesson also introduces the use of **independent** and **dependent variables** in the context of functions. For an equation that relates two quantities, it is sometimes possible to write either of the variables as a function of the other. For example, in the activity Dimes and Quarters, we can choose either the number of quarters or the number of dimes to be the independent variable. If we know the number of quarters and have questions about the number of dimes, then this would be a reason to choose the number of quarters as the independent variable.

### Alignments

#### Addressing

- 8.F.A: Define, evaluate, and compare functions.
- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.

#### Building Towards

- 8.F.B.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from

two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

### Instructional Routines

- MLR6: Three Reads
- MLR8: Discussion Supports
- Think Pair Share

### Student Learning Goals

Let's find outputs from equations.

## 3.1 A Square's Area

**Warm Up:** 5 minutes

The purpose of this warm-up is for students to use repeated reasoning to write an algebraic expression to represent a rule of a function (MP8). The whole-class discussion should focus on the algebraic expression in the final row, however the numbers in the table give students an opportunity to also practice calculating the square of numbers written in fraction and decimal form.

### Addressing

- 8.F.A.1

### Launch

Arrange students in groups of 2. Give students 1–2 minutes of quiet work time and then time to share their algebraic expression with their partner. Follow with a whole-class discussion.

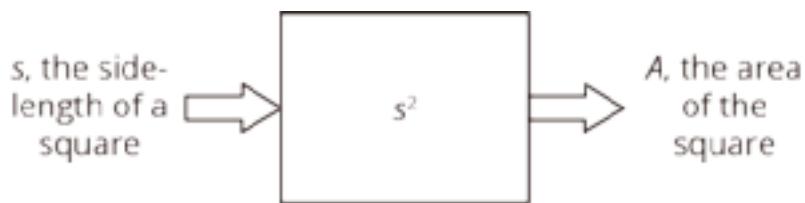
### Student Task Statement

Fill in the table of input-output pairs for the given rule. Write an algebraic expression for the rule in the box in the diagram.



input	output
8	
2.2	
$12\frac{1}{4}$	
$s$	

### Student Response



input	output
8	64
2.2	4.84
$12\frac{1}{4}$	$150\frac{1}{16}$
$s$	$s^2$ or $A$

### Activity Synthesis

Select students to share how they found each of the outputs. After each response, ask the class if they agree or disagree. Record and display responses for all to see. If both responses are not mentioned by students for the last row, tell students that we can either put  $s^2$  or  $A$  there. Tell students we can write the equation  $A = s^2$  to represent the rule of this function.

End the discussion by telling students that while we've used the terms input and output so far to talk about specific values, when a letter is used to represent any possible input we call it the **independent variable** and the letter used to represent all the possible outputs is the **dependent variable**. Students may recall these terms from earlier grades. In this case,  $s$  is the independent variable and  $A$  the dependent variable, and we say " $A$  depends on  $s$ ."

## 3.2 Diagrams, Equations, and Descriptions

15 minutes

The purpose of this activity is for students to make connections between different representations of functions and start transitioning from input-output diagrams to other representations of functions. Students match input-output diagrams to descriptions and come up with equations for each of those matches. Students then calculate an output given a specific input and determine the independent and dependent variables.

### Addressing

- 8.F.A

### Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

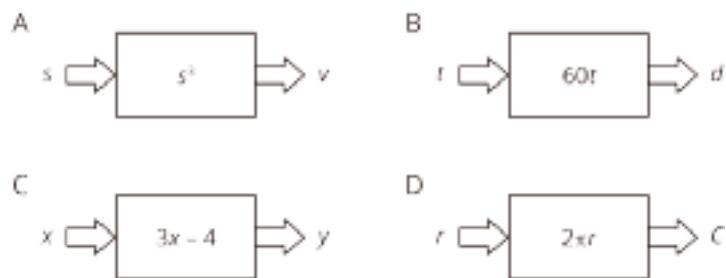
### Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and time to share their responses with their partner and come to agreement on their answers. Follow with whole-class discussion.

#### Student Task Statement

Record your answers to these questions in the table provided.

1. Match each of these descriptions with a diagram:
  - a. the circumference,  $C$ , of a circle with **radius**,  $r$
  - b. the distance in miles,  $d$ , that you would travel in  $t$  hours if you drive at 60 miles per hour
  - c. the output when you triple the input and subtract 4
  - d. the volume of a cube,  $v$  given its edge length,  $s$
2. Write an equation for each description that expresses the output as a function of the input.
3. Find the output when the input is 5 for each equation.
4. Name the **independent** and **dependent variables** of each equation.



description	a	b	c	d
diagram				
equation				
input = 5 output = ?				
independent variable				
dependent variable				

### Student Response

description	a	b	c	d
diagram	D	B	C	A
equation	$C = 2\pi r$	$d = 60t$	$y = 3x - 4$	$v = s^3$
input = 5 output = ?	$10\pi \approx 31.4$	300	11	125
independent variable	$r$	$t$	$x$	$s$
dependent variable	$C$	$d$	$y$	$v$

### Are You Ready for More?

Choose a 3-digit number as an input.

Apply the following rule to it, one step at a time:

- Multiply your number by 7.
- Add one to the result.
- Multiply the result by 11.
- Subtract 5 from the result.
- Multiply the result by 13
- Subtract 78 from the result to get the output.

Can you describe a simpler way to describe this rule? Why does this work?

### Student Response

If we apply the steps to a generic 3-digit number  $x$ , the result is

$$13(11(7x + 1) - 5) - 78 = 1,001x$$

For any 3-digit number  $x$ , the number  $1,001x$  is just that number repeated twice. This works since  $1,001x = 1,000x + x$ , so for example,

$$\begin{aligned}1,001 \cdot 314 &= 1,000 \cdot 314 + 1 \cdot 314 \\&= 314,000 + 314 \\&= 314,314\end{aligned}$$

### Activity Synthesis

The goal of this discussion is for students to describe the connections they see between the different entries for the 4 descriptions. Display the table for all to see and select different groups to share the answers for a column in the table. As groups share their answers, ask:

- “How did you know that this diagram matched with this description?” (We remembered the formula for the circumference of a circle, so we knew description A went with diagram D.)
- “Where in the equation do you see the rule that is in the diagram?” (The equation is the dependent variable set equal to the rule describing what happens to the independent variable in the diagram.)
- “Explain why you chose those quantities for your independent and dependent variables.” (We know the independent variable is the input and the dependent variable is the output, so we matched them up with the input and output shown in the diagram.)

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* As students describe the connections they noticed in the table across from the different entries for the four descriptors, revoice student ideas to demonstrate mathematical language use. In addition, press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example. This will help students to produce and make sense of the language needed to communicate their own ideas about functions and independent and dependent variables.

*Design Principle(s): Support sense-making; Optimize output (for explanation)*

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## 3.3 Dimes and Quarters

15 minutes

The purpose of this activity is for students to work with a function where either variable could be the independent variable. Knowing the total value for an unknown number of dimes and quarters, students are first asked to consider if the number of dimes could be a function of the number of quarters and then asked if the reverse is also true. Since this isn't always the case when students are working with functions, the discussion should touch on reasons for choosing one variable vs. the other, which can depend on the types of questions one wants to answer.

Identify students who efficiently rewrite the original equation in the third problem and the last problem to share during the discussion.

### Addressing

- 8.F.A.1

### Building Towards

- 8.F.B.4

### Instructional Routines

- MLR6: Three Reads

### Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time followed by partner discussion for students to compare their answers and resolve any differences. Follow with a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems and other text-based content.

*Supports accessibility for: Language; Conceptual processing*

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### Access for English Language Learners

*Reading: MLR6 Three Reads.* Use this routine to support reading comprehension and to set students up to interpret the representations of the situation provided in the task statement (an equation) and discussion (function diagrams). In the first read, students read the information with the goal of comprehending the situation (e.g., Jada has dimes and quarters). In the second read, ask students to identify important quantities. Listen for, and amplify, naming of the quantities that vary in relation to each other in this situation: number of dimes, total value of dimes, number of quarters, total value of quarters. After the third read, ask students to discuss possible strategies to answer the questions that follow, paying attention to the different coin values, and how each question is phrased. This will help students comprehend the problem and make sense of important quantities and variables when working with a function in which either variable could be the independent variable.

*Design Principle(s): Support sense-making*

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### Anticipated Misconceptions

Some students may be unsure how to write rules for the number of dimes as a function of the number of quarters and vice versa. Prompt them to use the provided equation and what they know about keeping equations equal to create the new equations.

#### Student Task Statement

Jada had some dimes and quarters that had a total value of \$12.50. The relationship between the number of dimes,  $d$ , and the number of quarters,  $q$ , can be expressed by the equation  $0.1d + 0.25q = 12.5$ .

1. If Jada has 4 quarters, how many dimes does she have?
2. If Jada has 10 quarters, how many dimes does she have?
3. Is the number of dimes a function of the number of quarters? If yes, write a rule (that starts with  $d = \dots$ ) that you can use to determine the output,  $d$ , from a given input,  $q$ . If no, explain why not.
4. If Jada has 25 dimes, how many quarters does she have?

5. If Jada has 30 dimes, how many quarters does she have?

6. Is the number of quarters a function of the number of dimes? If yes, write a rule (that starts with  $q = \dots$ ) that you can use to determine the output,  $q$ , from a given input,  $d$ . If no, explain why not.

### Student Response

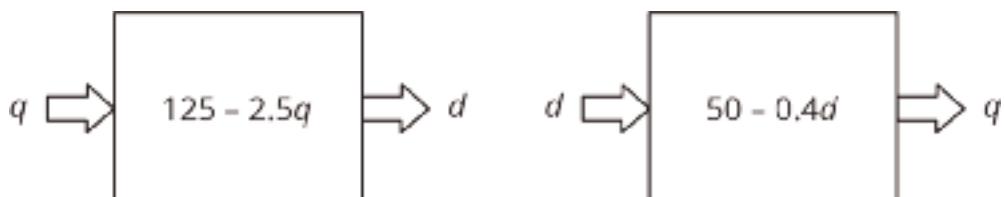
115. If  $q = 4$ , then the equation tells us that  $0.1d + (0.25) \cdot 4 = 12.5$ . Subtracting 1 from both sides gives  $0.1d = 11.5$ , so  $d = 115$ .
100. If  $q = 10$ , then the equation tells us that  $0.1d + (0.25) \cdot 10 = 12.5$ . Subtracting 2.5 from both sides gives  $0.1d = 10$ , so  $d = 100$ .
- Yes. If you know the number of quarters, then you can determine the number of dimes from the equation. We can even write the equation in a way that shows this:  $d = 125 - 2.5q$ . The expression  $125 - 2.5q$  represents the output—it is the rule that determines the output  $d$  from a given input  $q$ .
40. If  $d = 25$ , then the equation tells us that  $0.1(25) + (0.25)q = 12.5$ . Subtracting 2.5 from both sides gives  $.25q = 10$ , so  $q = 40$ .
38. If  $d = 30$ , then the equation tells us that  $0.1(30) + (0.25)q = 12.5$ . Subtracting 3 from both sides gives  $.25q = 9.5$ , so  $q = 38$ .
- Yes. If you know the number of dimes, then you can determine the number of quarters from the equation. We can even write the equation in a way that shows this:  $q = 50 - 0.4d$ . The expression  $50 - 0.4d$  represents the output—it is the rule that determines the output  $q$  from a given input  $d$ .

### Activity Synthesis

Select previously identified students to share their rules for dimes as a function of the number of quarters and quarters as a function of the number of dimes, including the steps they used to rewrite the original equation.

Tell students that if we write an equation like  $d = 125 - 2.5q$ , this shows that  $d$  is a function of  $q$  because it is clear what the output (value for  $d$ ) should be for a given input (value for  $q$ ).

Display the diagrams for all to see:



When we have an equation like  $0.1d + 0.25q = 12.5$ , we can choose either  $d$  or  $q$  to be the independent variable. That means we are viewing one as depending on the other. If we know the number of quarters and want to answer a question about the number of dimes, it is helpful to write

$d$  as a function of  $q$ . If we know the number of dimes and want to answer a question about the number of quarters, it is helpful to write  $q$  as a function of  $d$ .

Ensure students understand that we can't always do this type of rearranging with equations and have it make sense because sometimes only one variable is a function of the other, and sometimes neither is a function of the other. For example, students saw earlier that while squaring values is a function, the reverse—that is, identifying what value was squared—is not. We will continue to explore when these different things happen in future lessons.

## Lesson Synthesis

Tell students that we often use **independent** and **dependent variables** to represent the inputs and outputs of functions. For some functions, we can describe the relationship between the variables with an equation. Sometimes we can choose, depending on the situation, which variable should be the independent and which should be the dependent variable. To help students think more about what independent and dependent variables represent and their use with functions, ask:

- “How can we describe the area of square  $A$  of side length  $s$  with an equation? Which is the independent and which is the dependent variable?” (We can write  $s^2 = A$ , where  $s$  is the independent variable and  $A$  is the dependent variable.)
- “The relationship between the number of dimes,  $d$ , and the number of nickels,  $n$ , that total \$5 can be expressed by the equation  $0.1d + 0.05n = 5$ . When would it be useful to choose the number of dimes as the independent variable and rewrite the equation?” (If we knew the number of dimes and wanted to know the number of nickels, it would be useful to rewrite the equation so it looked like  $n = \dots$ .)

## 3.4 The Value of Some Quarters

Cool Down: 5 minutes

### Addressing

- 8.F.A.1

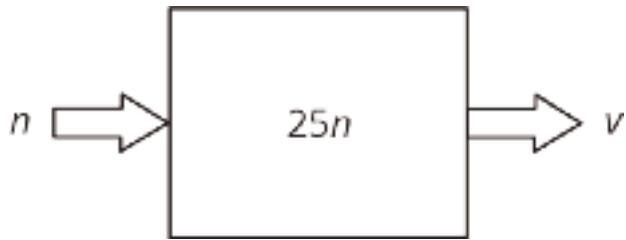
#### Student Task Statement

The value  $v$  of your quarters (in cents) is a function of  $n$ , the number of quarters you have.

1. Draw an input-output diagram to represent this function.
2. Write an equation that represents this function.
3. Find the output when the input is 10.
4. Identify the independent and dependent variables.

#### Student Response

1. Here is the diagram:



2.  $v = 25n$ . This reflects the statement that the value (in cents) of your collection of quarters is always 25 times the number of quarters you have.
3. When the input is 10, the output is 250 (since  $250 = 25 \cdot 10$ ).
4.  $n$  is the independent variable, and  $v$  is the dependent variable.

## Student Lesson Summary

We can sometimes represent functions with equations. For example, the area,  $A$ , of a circle is a function of the radius,  $r$ , and we can express this with an equation:

$$A = \pi r^2$$

We can also draw a diagram to represent this function:



In this case, we think of the radius,  $r$ , as the input, and the area of the circle,  $A$ , as the output. For example, if the input is a radius of 10 cm, then the output is an area of  $100\pi \text{ cm}^2$ , or about 314 square cm. Because this is a function, we can find the area,  $A$ , for any given radius,  $r$ .

Since it is the input, we say that  $r$  is the **independent variable** and, as the output,  $A$  is the **dependent variable**.

Sometimes when we have an equation we get to choose which variable is the independent variable. For example, if we know that

$$10A - 4B = 120$$

then we can think of  $A$  as a function of  $B$  and write

$$A = 0.4B + 12$$

or we can think of  $B$  as a function of  $A$  and write

$$B = 2.5A - 30$$

## Glossary

- dependent variable
- independent variable
- radius

## Lesson 3 Practice Problems

### Problem 1

#### Statement

Here is an equation that represents a function:  $72x + 12y = 60$ .

Select **all** the different equations that describe the same function:

A.  $120y + 720x = 600$

B.  $y = 5 - 6x$

C.  $2y + 12x = 10$

D.  $y = 5 + 6x$

E.  $x = \frac{5}{6} - \frac{y}{6}$

F.  $7x + 2y = 6$

G.  $x = \frac{5}{6} + \frac{y}{6}$

#### Solution

["A", "B", "C", "E"]

### Problem 2

#### Statement

a. Graph a system of linear equations with no solutions.

b. Write an equation for each line you graph.



## Solution

Answers vary. The graph could be any two lines that are parallel.

(From Unit 4, Lesson 13.)

## Problem 3

### Statement

Brown rice costs \$2 per pound, and beans cost \$1.60 per pound. Lin has \$10 to spend on these items to make a large meal of beans and rice for a potluck dinner. Let  $b$  be the number of pounds of beans Lin buys and  $r$  be the number of pounds of rice she buys when she spends all her money on this meal.

- Write an equation relating the two variables.
- Rearrange the equation so  $b$  is the independent variable.
- Rearrange the equation so  $r$  is the independent variable.

## Solution

a.  $2r + 1.6b = 10$

b.  $r = 5 - 0.8b$

c.  $b = 6.25 - 1.25r$

## Problem 4

### Statement

Solve each equation and check your answer.

$$2x + 4(3 - 2x) = \frac{3(2x+2)}{6} + 4$$

$$4z + 5 = -3z - 8$$

$$\frac{1}{2} - \frac{1}{8}q = \frac{q-1}{4}$$

### Solution

a.  $x = 1$ .

b.  $z = \frac{-13}{7}$

c.  $q = 2$

(From Unit 4, Lesson 6.)

# Lesson 4: Tables, Equations, and Graphs of Functions

## Goals

- Determine whether a graph represents a function, and explain (orally) the reasoning.
- Identify the graph of an equation that represents a function, and explain (orally and in writing) the reasoning.
- Interpret (orally and in writing) points on a graph, including a graph of a function and a graph that does not represent a function.

## Learning Targets

- I can identify graphs that do, and do not, represent functions.
- I can use a graph of a function to find the output for a given input and to find the input(s) for a given output.

## Lesson Narrative

In this lesson, students work with graphs and tables that represent functions in addition to the equations and descriptions used previously. They learn the conventions of graphing the independent variable (input) on the horizontal axis and the dependent variable (output) on the vertical axis and that each coordinate point represents an input-output pair of the function.

By matching contexts and graphs and reading information about functions from graphs and tables, students become familiar with the different representations and draw connections between them.

## Alignments

### Addressing

- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.
- 8.F.A.3: Interpret the equation  $y = mx + b$  as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function  $A = s^2$  giving the area of a square as a function of its side length is not linear because its graph contains the points  $(1, 1)$ ,  $(2, 4)$  and  $(3, 9)$ , which are not on a straight line.

## Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR5: Co-Craft Questions

- Notice and Wonder

### **Student Learning Goals**

Let's connect equations and graphs of functions.

## **4.1 Notice and Wonder: Doubling Back**

### **Warm Up: 5 minutes**

The purpose of this warm-up is to familiarize students with one of the central graphical representations they will be working with in the lesson. As students notice and wonder, they have the opportunity to reason abstractly and quantitatively if they consider the situation the graph represents (MP2).

### **Addressing**

- 8.F.A.1

### **Instructional Routines**

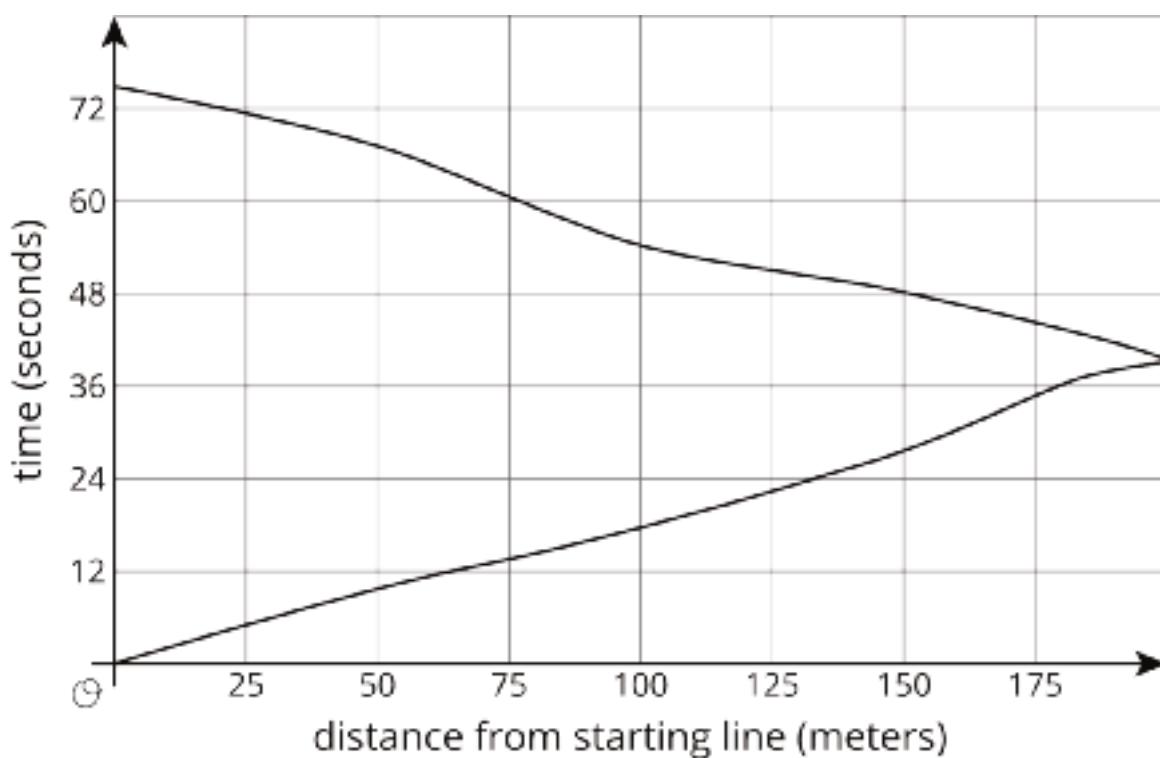
- Notice and Wonder

### **Launch**

Tell students they will look at a graph, and their job is to think of at least one thing they notice and at least one thing they wonder about the picture. Display the graph for all to see and give 1 minute of quiet think time. Ask students to give a signal when they have noticed or wondered about something.

### **Student Task Statement**

What do you notice? What do you wonder?



### Student Response

Students may notice things such as:

- The graph looks like a sideways V.
- The graph doubles back on itself.
- The graph is made of pieces that are almost, but not quite, straight.
- Distances go between 0 and 200 meters.
- Times go from 0 to a little more than 72.
- The horizontal axis tells us the distance from the starting line in meters.
- The vertical axis tells us the time in seconds.
- The person gets farther from the starting line and then comes back.
- The person was 75 meters from the starting line at about 14 seconds and 60 seconds.
- The person got back to the starting line in about 75 seconds.
- The furthest distance the person got from the start line was 200 meters.

Students may wonder things such as:

- Is this about a person, or more than one person?
- Why does the graph double back?
- Who is the graph about?
- Why are they coming back?
- Are they running or walking?
- Did they go out and come back in the exact same time?
- What is the title of this graph?

### Activity Synthesis

Ask students to share things they noticed and wondered about the graph. Record and display these ideas for all to see. For each of the things students notice and wonder, ask them to reference the graph in their explanation. If no one notices that at every distance from the starting line, there are two associated times (except at 200 meters), bring that to their attention.

If there is time, ask students to make some estimations or guesses for the wonders that refer to the information in the graph. For example, if someone wonders what the title of the graph may be, ask them to create a title that would make sense for this context.

## 4.2 Equations and Graphs of Functions

15 minutes

The purpose of this activity is for students to connect different function representations and learn the conventions used to label a graph of a function. Students first match function contexts and equations to graphs. They next label the axes and calculate input-output pairs for each function. The focus of the discussion should be on what quantities students used to label the axes and recognizing the placement of the independent or dependent variables on the axes.

Monitor for students who recognize that there is one graph that is not linear and match that graph with the equation that is not linear.

### Addressing

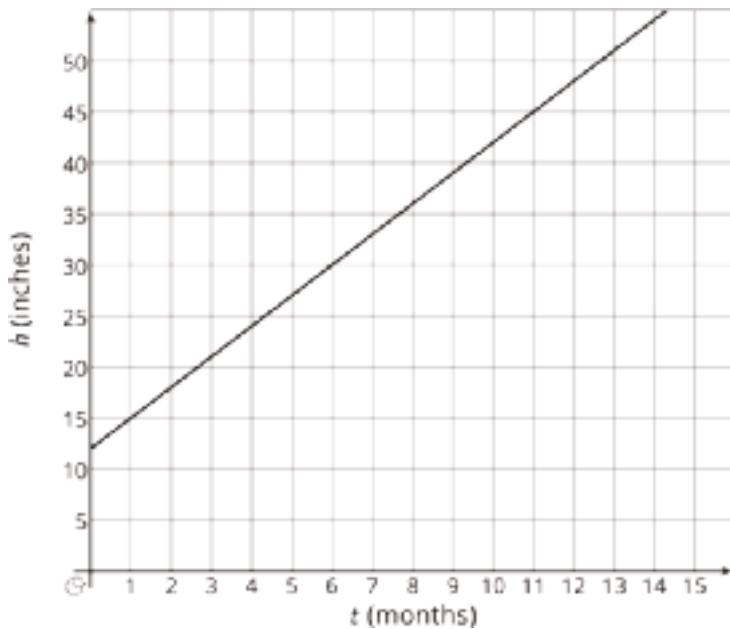
- 8.F.A.1
- 8.F.A.3

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

### Launch

Arrange students in groups of 2. Display the graph for all to see. Ask students to consider what the graph might represent.



After brief quiet think time, select 1–2 students to share their ideas. (For example, something starts at 12 inches and grows 15 inches for every 5 months that pass.)

Remind students that axes labels help us determine what quantities are represented and should always be included. Let them know that in this activity the graphs of three functions have been started, but the labels are missing and part of their work is to figure out what those labels are meant to be.

Give students 3–5 minutes of quiet work time and then time to share responses with their partner. Encourage students to compare their explanations for the last three problems and resolve any differences. Follow with a whole-class discussion.

---

#### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example ask students to use the same color to denote inputs and another color for outputs. Students can use the colors when calculating values, labeling axes, and plotting points in the graphs.

*Supports accessibility for: Visual-spatial processing*

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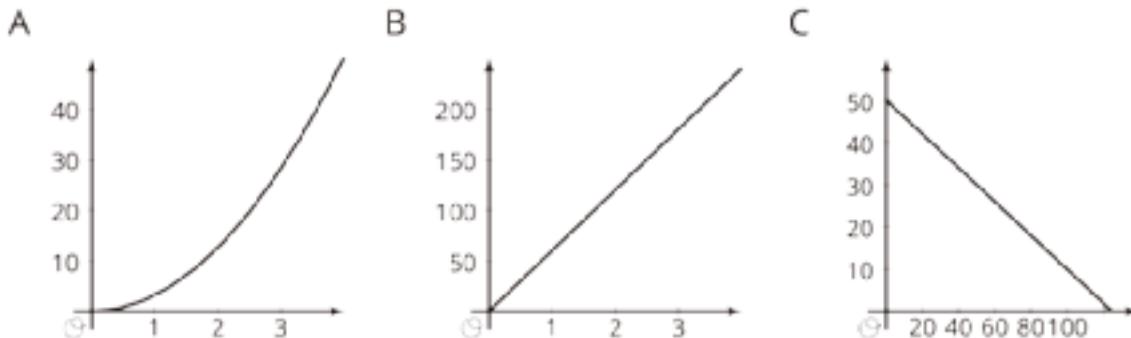
## Access for English Language Learners

*Writing, Speaking: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise and refine their explanation of how they matched the equations and graphs in the first problem. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help teams strengthen their ideas and clarify their language (e.g., “What did you do first?”, “How did you verify your match?”, etc.). Students can borrow ideas and language from each partner to strengthen their final version.

*Design Principle(s): Optimize output (for explanation)*

## Student Task Statement

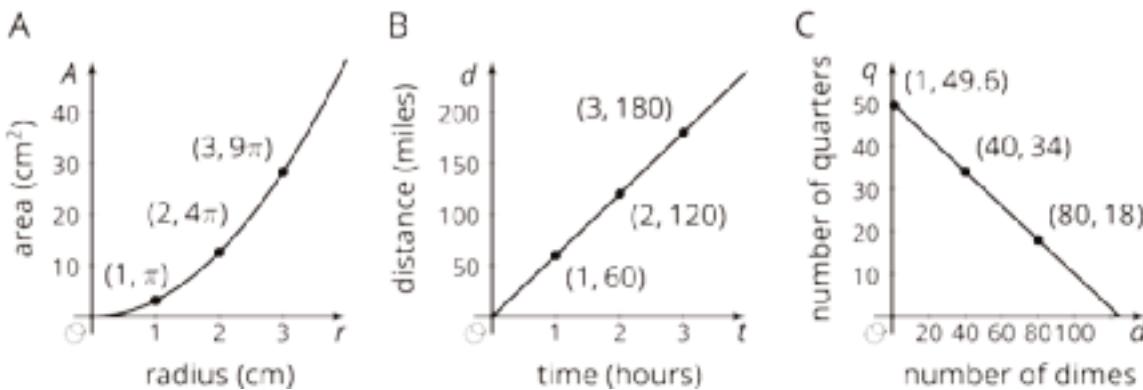
The graphs of three functions are shown.



1. Match one of these equations to each of the graphs.
  - a.  $d = 60t$ , where  $d$  is the distance in miles that you would travel in  $t$  hours if you drove at 60 miles per hour.
  - b.  $q = 50 - 0.4d$ , where  $q$  is the number of quarters, and  $d$  is the number of dimes, in a pile of coins worth \$12.50.
  - c.  $A = \pi r^2$ , where  $A$  is the area in square centimeters of a circle with radius  $r$  centimeters.
2. Label each of the axes with the independent and dependent variables and the quantities they represent.
3. For each function: What is the output when the input is 1? What does this tell you about the situation? Label the corresponding point on the graph.
4. Find two more input-output pairs. What do they tell you about the situation? Label the corresponding points on the graph.

## Student Response

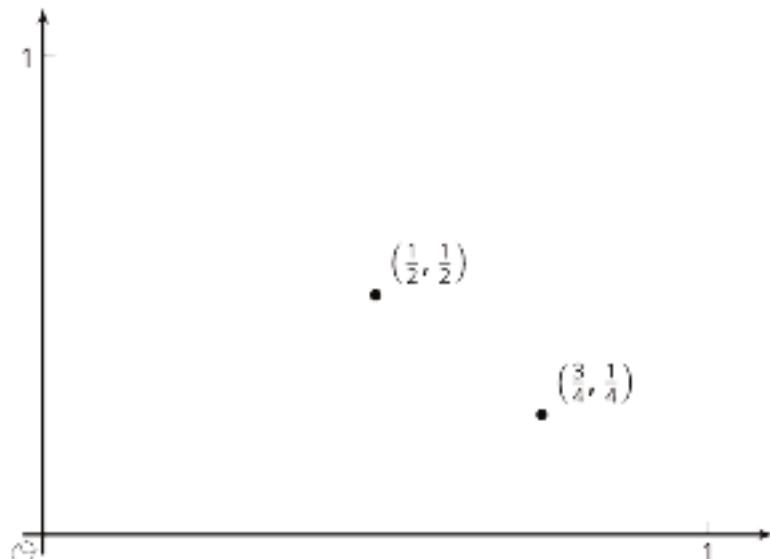
- B. This graph represents a proportional relationship with a constant positive slope.
  - C. This is the only one of the graphs, and the only equation, with a negative slope and vertical intercept of 50.
  - A. This is the only non-linear relationship.
- See answer to part 4.
- In figure A, we have point  $(1, \pi)$ , representing the fact that a circle of radius 1 cm has area  $\pi$   $\text{cm}^2$ .  
In figure B, we have point  $(1, 60)$ , representing the fact that after traveling for 1 hour at 60 miles per hour, you would travel 60 miles.  
In figure C, we have point  $(1, 49.6)$ . This does not have a concrete interpretation in terms of the context, as it says that if you had only one dime, you would have 49.6 quarters.  
See answer to part 4 for the graphs.
- In figure A, we mark the points  $(2, 4\pi)$  and  $(3, 9\pi)$ , representing the fact that circles of radius 2 cm and 3 cm have respective areas  $4\pi \text{ cm}^2$  and  $9\pi \text{ cm}^2$ .  
In figure B, we mark the points  $(2, 120)$  and  $(3, 180)$ , representing the fact that after traveling for 2 and 3 hours at 60 miles per hour, you would respectively travel 120 and 180 miles.  
In figure C, we mark the points  $(40, 34)$  and  $(80, 18)$ , representing the fact that if there were 40 or 80 dimes in the pile, there would have to be 34 and 18 quarters, respectively.



## Are You Ready for More?

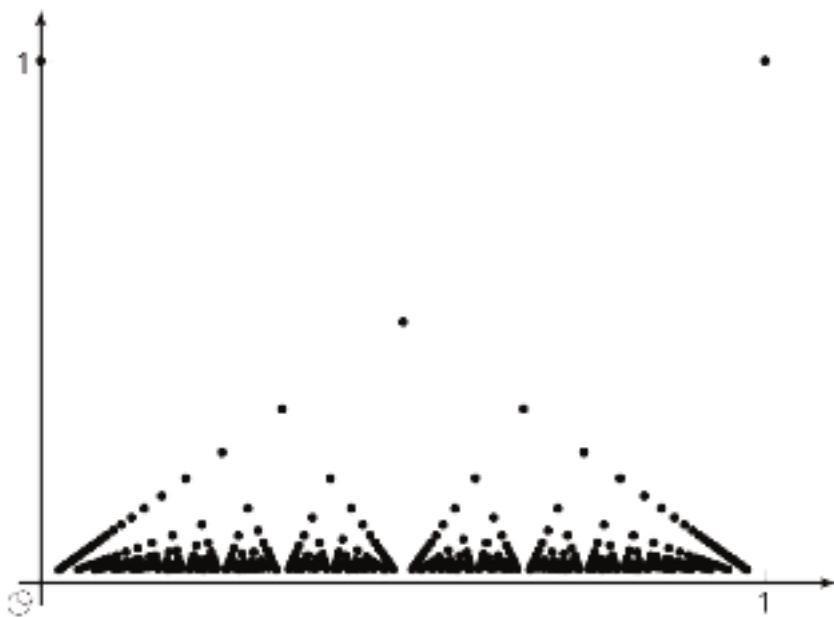
A function inputs fractions  $\frac{a}{b}$  between 0 and 1 where  $a$  and  $b$  have no common factors, and outputs the fraction  $\frac{1}{b}$ . For example, given the input  $\frac{3}{4}$  the function outputs  $\frac{1}{4}$ , and to the input  $\frac{1}{2}$  the function outputs  $\frac{1}{2}$ . These two input-output pairs are shown on the graph.

Plot at least 10 more points on the graph of this function. Are most points on the graph above or below a height of 0.3? Of height 0.01?



### Student Response

This is a very complicated graph! Here is a computer-generated plot of several hundred inputs with denominators  $b < 50$ . Only very few inputs have height above 0.3. The only ones above are  $\frac{0}{1}$ ,  $\frac{1}{1}$ , and  $\frac{1}{2}$ . Every other fraction between 0 and 1 has a denominator of  $b \geq 4$ , so  $\frac{1}{b} \leq \frac{1}{4}$ . Less obvious is that the same is true for height 0.01. Having an output less than 0.01 is the same as having  $b > 100$ . Since more fractions have  $b > 100$  than  $b < 100$ , there are more points on the graph with height under 0.01 than over.



For more information, do some research on the Thomae function.

## Activity Synthesis

The purpose of this discussion is for students to understand the conventions of constructing a graph of a function and where input and outputs are found on a graph. Select students previously identified to share how they figured out  $A = \pi r^2$  matched the non-linear graph.

Ask students:

- “Where are the independent variables labeled on the graphs?” (The horizontal axis)
- “Where are the dependent variables labeled on the graphs?” (The vertical axis)

Tell students that by convention, the independent variable is on the horizontal axis and the dependent variable is on the vertical axis. This means that when we write coordinate pairs, they are in the form of (input, output). For some functions, like the one with quarters and dimes, we can choose which variable is the independent and which is the dependent, which means the graph could be constructed either way based on our decisions.

Conclude the discussion by asking students to share their explanations for the point (1, 49.6) for graph C. (There is no such thing as 0.6 of a quarter.) Remind students that sometimes we have to restrict inputs to only those that make sense. Since it's not possible to have 49.6 quarters, an input of 1 dime does not make sense. Similarly, 2, 3, or 4 dimes result in numbers of quarters that do not make sense. 0 dimes or 5 dimes, however, do produce outputs that make sense. Sometimes it is easier to sketch a graph of the line even when graphing discrete points would be more accurate for the context. Keeping the context of a function in mind is important when making sense of the input-output pairs associated with the function.

## 4.3 Running around a Track

**15 minutes**

The purpose of this activity is for students to interpret coordinates on graphs of functions and non-functions as well as understand that context does not dictate the independent and dependent variables.

In the first problem time is a function of distance, and the graph and table help determine how long it takes for Kiran to run a specific distance. In the second problem, however, time is not a function of Priya's distance from the starting line. This results in a graph where each input does not give exactly one output.

### Addressing

- 8.F.A.1

### Instructional Routines

- MLR5: Co-Craft Questions

## Launch

Give students 3–5 minutes of quiet work time followed by whole-class discussion.

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### Access for English Language Learners

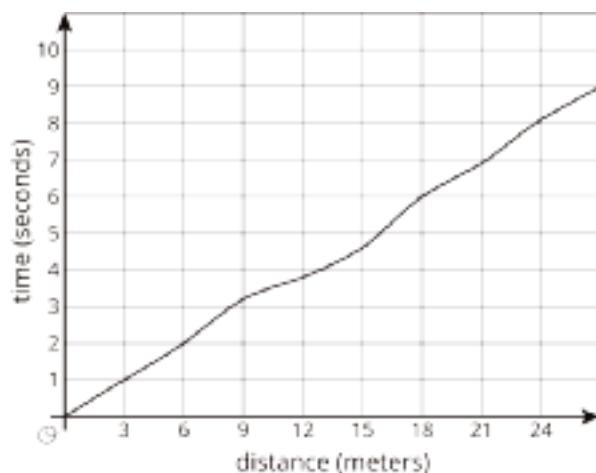
*Writing, Conversing: MLR5 Co-Craft Questions.* Display only the graph and context (i.e., “Kiran was running around the track. The graph shows the time,  $t$ , he took to run various distances,  $d$ .”). Ask pairs of students to write possible questions that could be answered by the graph. Invite pairs to share their questions with the class. Look for questions that ask students to interpret quantities represented in the graph. Next, reveal the questions of the activity. This helps students produce the language of mathematical questions and talk about the relationships between the two quantities in this task (e.g., distance and time).

*Design Principle(s): Maximize meta-awareness; Support sense-making*

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### Student Task Statement

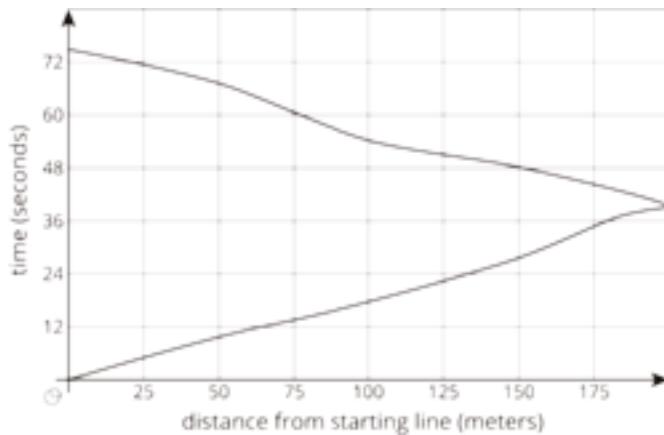
1. Kiran was running around the track. The graph shows the time,  $t$ , he took to run various distances,  $d$ . The table shows his time in seconds after every three meters.



$d$	0	3	6	9	12	15	18	21	24	27
$t$	0	1.0	2.0	3.2	3.8	4.6	6.0	6.9	8.09	9.0

- How long did it take Kiran to run 6 meters?
- How far had he gone after 6 seconds?
- Estimate when he had run 19.5 meters.
- Estimate how far he ran in 4 seconds.
- Is Kiran's time a function of the distance he has run? Explain how you know.

2. Priya is running once around the track. The graph shows her time given how far she is from her starting point.



- What was her farthest distance from her starting point?
- Estimate how long it took her to run around the track.
- Estimate when she was 100 meters from her starting point.
- Estimate how far she was from the starting line after 60 seconds.
- Is Priya's time a function of her distance from her starting point? Explain how you know.

### Student Response

- 2 seconds, since from the table, when  $d = 6$ , we have  $t = 2$ .
  - 18 meters, since from the table, when  $t = 6$ , we have  $d = 18$ .
  - Answers vary, but 6.45 seconds is a reasonable estimate. It took Kiran 6 seconds to run 18 meters, and 6.9 seconds to run 21 meters. Since 19.5 is halfway between 18 and 21, it is reasonable to estimate halfway between 6 seconds and 6.9 seconds. This estimate is further supported by the graph.
  - Answers vary, but 12.75 meters is a reasonable estimate. He runs 12 meters in 3.8 seconds and 15 meters in 4.6 seconds. Since 4 is a quarter of the way from 3.8 to 4.6, a reasonable estimate for distance would be a quarter of the way from 12 to 15, which is 12.75. This estimate is further supported by the graph.
  - Kiran's time is a function of the distance he has traveled. By reading the graph, you can use the distance he has traveled to find the time it took him to travel it.
- 200 meters, reflected by the rightmost point on the provided graph.
  - 75 seconds. From the top-left point on the graph, we can see that the first time after departing that Priya returns to the start line after her initial departure occurs at about 75 seconds.

- c. 18 seconds and 54 seconds. There are two points on the graph representing a distance of 100 meters from the starting line, at heights 18 and 54.
- d. 75 meters. There is only one point on the graph corresponding to a time of 60 seconds, and it occurs at distance 75 meters.
- e. No. The time is not determined by the distance from the starting line, as the example of 100 meters above shows. There are two different times corresponding to the distance of 100 meters.

### Activity Synthesis

The purpose of this discussion is for students to understand that independent and dependent variables are not determined by the context (and specifically that time is not always a function of distance). Select students to share their strategies for calculating the answers for the first set of problems. For each problem, ask students whether the graph or table was more useful. Further the discussion by asking:

- “When are tables useful for answering questions?” (The exact answers that are listed in table while with the graph we can only approximate.)
- “When are graphs more useful for answering questions?” (A graph shows more input-output pairs than a table can list easily.)
- “Why does it make sense to have time be a function of distance in this problem?” (The farther Kiran runs, the longer it will take so it makes sense to represent time as a function of distance.)
- “Does time always have to be a function of distance?” (No, this graph could be made the other way with time on the horizontal and distance on the vertical and then it would show distance as a function of time, which makes sense since the longer Kiran runs, the further he will travel.)

For the second graph, ask students to indicate if they think it represents a function or not. If there are students who say yes and no, invite students from each side to say why they think it is or is not a function and try to persuade the rest of the class to their side. If all students are not persuaded that the graph is not a function, remind students that functions can only have one output for each input, and ask students to look back at their answer to the question “Estimate when she was 100 meters from her starting point.” Since that question has two responses, the graph cannot be a function.

If time allows, ask students “If the axes of the second graph were switched, that is, time was the independent variable and distance from the starting line was the dependent variable, then would the graph be a representation of a function?” (Yes, because each input (time) would have only one output (distance).)

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### Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.

*Supports accessibility for: Language; Social-emotional skills; Attention*

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## Lesson Synthesis

Each representation of a function presents information about the input and output of the function in different ways. Tell students to imagine we have a function with independent variable  $x$  and dependent variable  $y$ .

- “How do we find input-output pairs from a graph of the function?” (Any coordinate on the graph gives an input-output pair where the input is the  $x$ -value and the output is the  $y$ -value.)
- “What is something you won’t see on the graph of the function?” (The graph will never “double-back” or have two  $y$ -values for the same  $x$ -value, because each input will have only one output.)
- “If the graph of the function contains the point  $(18, 6)$ , what else do we know about the function?” (If we input 18 into the function we will get 6 as an output. An equation for the function could be  $y = \frac{1}{3}x$ , but we would need to know more points on the graph to be sure.)

## 4.4 Subway Fare Card

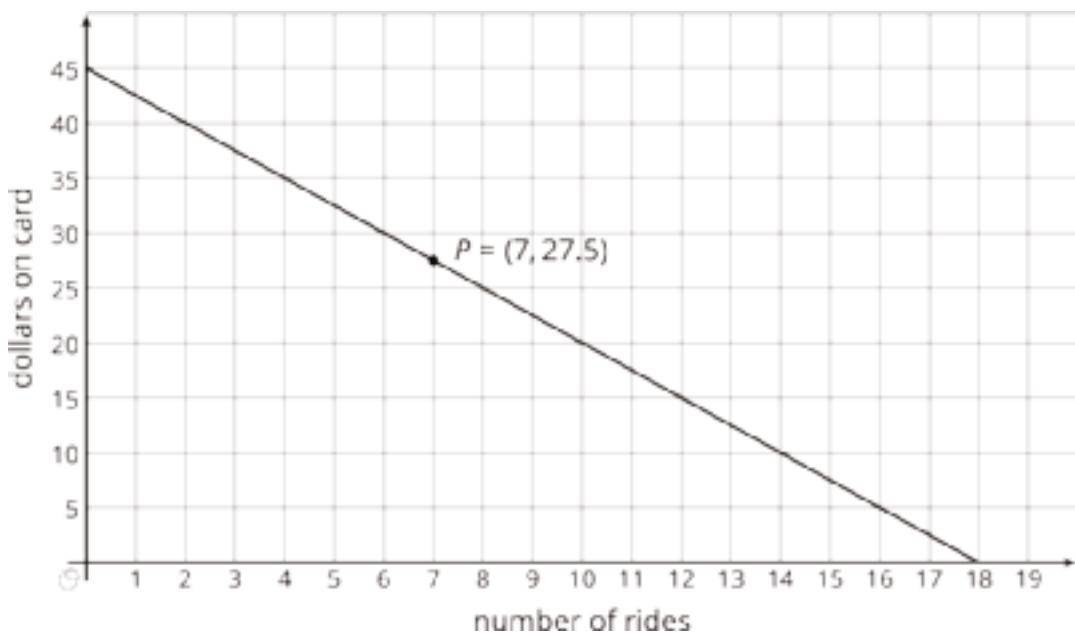
Cool Down: 5 minutes

### Addressing

- 8.F.A.1

#### Student Task Statement

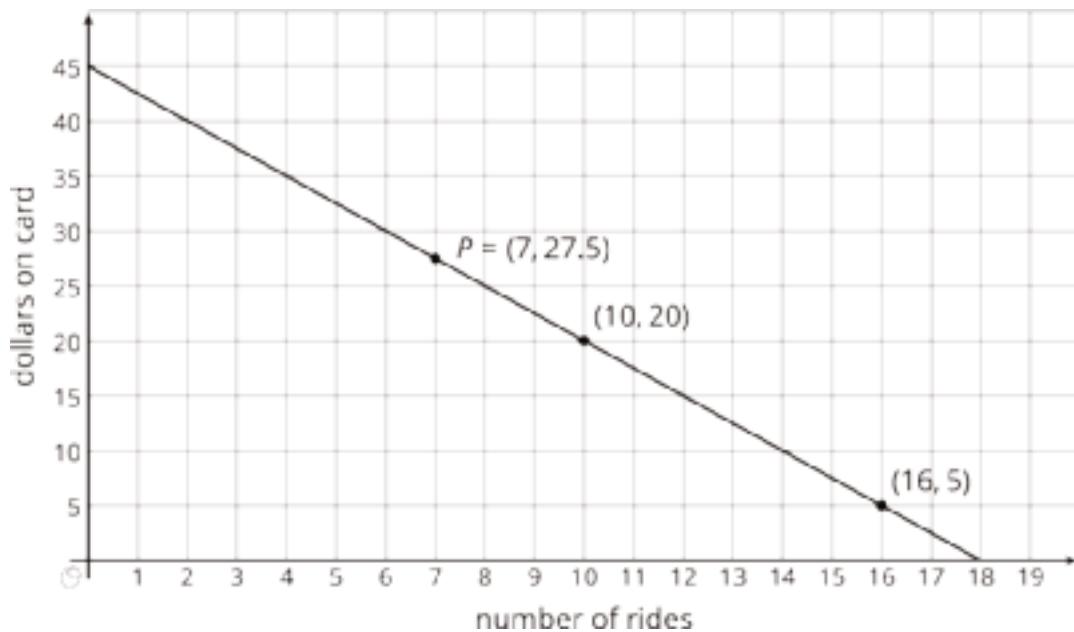
Here is the graph of a function showing the amount of money remaining on a subway fare card as a function of the number of rides taken.



1. What is the output of the function when the input is 10? On the graph, plot this point and label its coordinates.
2. What is the input to the function when the output is 5? On the graph, plot this point and label its coordinates.
3. What does point  $P$  tell you about the situation?

### Student Response

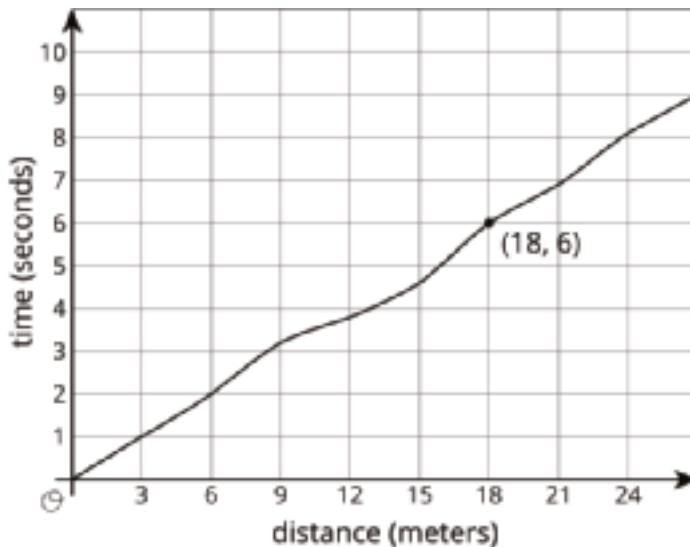
1. 20. See graph in part 2.
2. 16



3. After taking 7 rides, there will be \$27.50 remaining on the card.

## Student Lesson Summary

Here is the graph showing Noah's run.



The time in seconds since he started running is a function of the distance he has run. The point  $(18, 6)$  on the graph tells you that the time it takes him to run 18 meters is 6 seconds. The input is 18 and the output is 6.

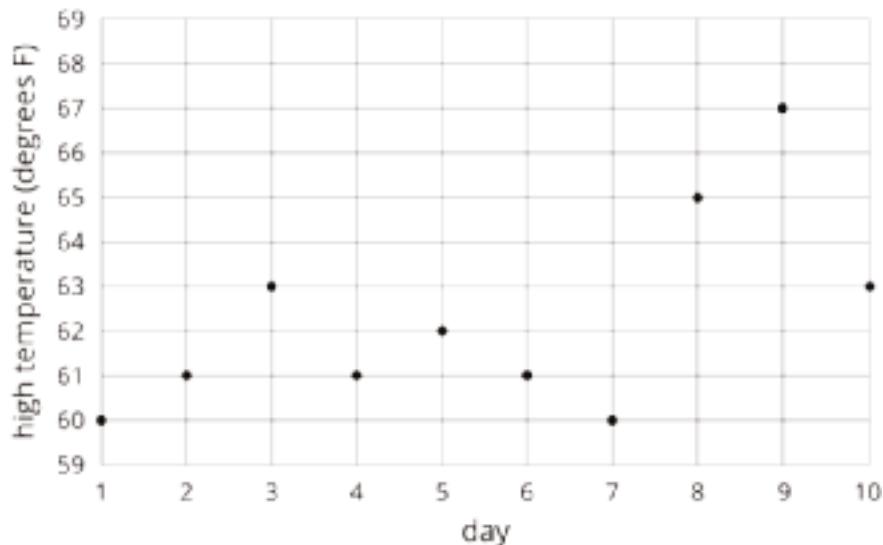
The graph of a function is all the coordinate pairs, (input, output), plotted in the coordinate plane. By convention, we always put the input first, which means that the inputs are represented on the horizontal axis and the outputs, on the vertical axis.

# Lesson 4 Practice Problems

## Problem 1

### Statement

The graph and the table show the high temperatures in a city over a 10-day period.



day	1	2	3	4	5	6	7	8	9	10
temperature (degrees F)	60	61	63	61	62	61	60	65	67	63

- What was the high temperature on Day 7?
- On which days was the high temperature 61 degrees?
- Is the high temperature a function of the day? Explain how you know.
- Is the day a function of the high temperature? Explain how you know.

### Solution

- 60 degrees F
- Days 2, 4, 6
- The high temperature is a function of the day. There are no different outputs for the same input. That is, there is no day with two different high temperatures.
- Day could not be a function of temperature as there are multiple days that have the same high temperature. There would be the different outputs for the same input.

## Problem 2

### Statement

The amount Lin's sister earns at her part-time job is proportional to the number of hours she works. She earns \$9.60 per hour.

- Write an equation in the form  $y = kx$  to describe this situation, where  $x$  represents the hours she works and  $y$  represents the dollars she earns.
- Is  $y$  a function of  $x$ ? Explain how you know.
- Write an equation describing  $x$  as a function of  $y$ .

## Solution

- $y = 9.6x$ , where  $x$  is number of hours worked and  $y$  is amount earned in dollars
- $y$  is a function of  $x$  because there is only one output for each input.
- $x = \frac{1}{9.6}y$

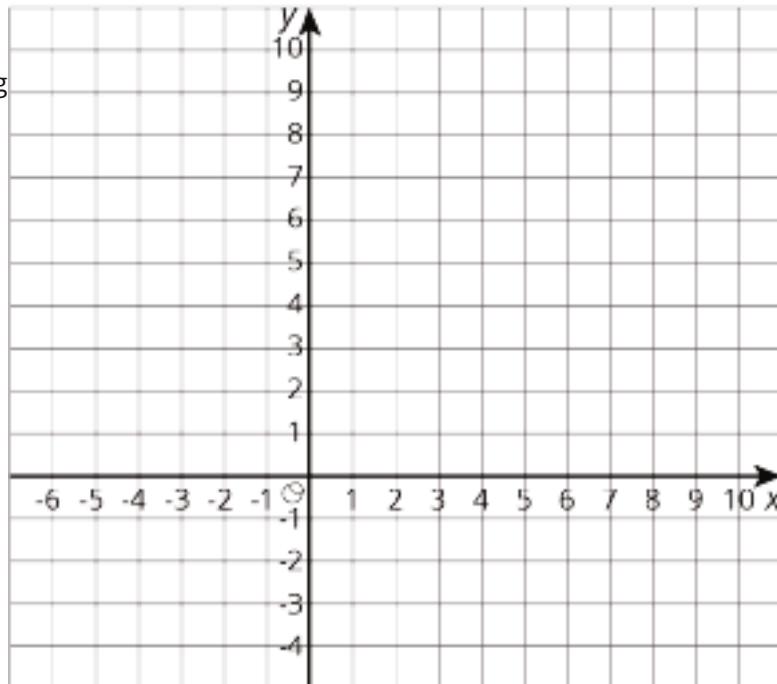
## Problem 3

### Statement

Use the equation

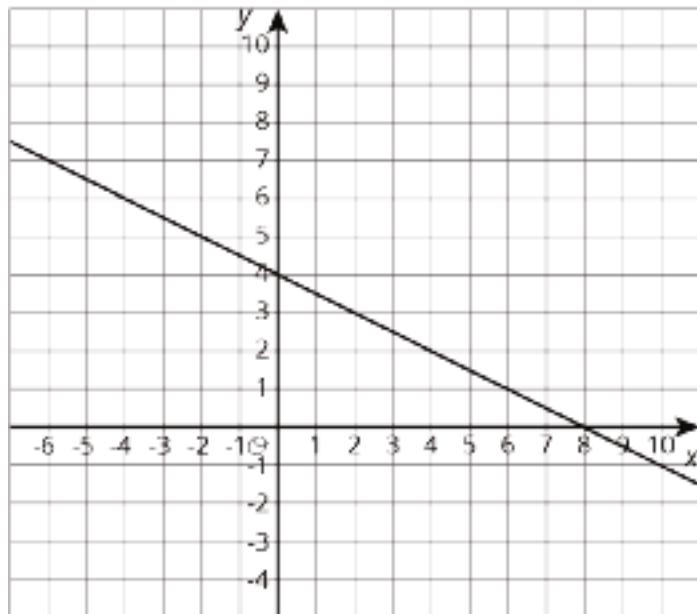
$2m + 4s = 16$  to complete the table, then graph the line using  $s$  as the dependent variable.

$m$	0		-2	
$s$		3		0



## Solution

$m$	0	2	-2	8
$s$	4	3	5	0



## Problem 4

### Statement

Solve the system of equations:  $\begin{cases} y = 7x + 10 \\ y = -4x - 23 \end{cases}$

## Solution

(-3, -11)

(From Unit 4, Lesson 13.)

# Lesson 5: More Graphs of Functions

## Goals

- Describe (orally and in writing) a graph of a function as “increasing” or “decreasing” over an interval, and explain (orally) the reasoning.
- Interpret (orally and in writing) a graph of temperature as a function of time, using language such as “input” and “output”.

## Learning Targets

- I can explain the story told by the graph of a function.

## Lesson Narrative

In this lesson, students begin to analyze graphs of functions and use them to answer questions about a context. Students also look at what happens over intervals of input values and learn that graphs can be viewed as dynamic objects that tell stories.

In the temperature activity, students connect specific features of the graph, such as the highest point, with specific features of the contextual situation, i.e., the highest temperature of the day and when it was attained. In the activity about garbage production, students investigate what happens over ranges of input values. The graph tells us how much garbage was produced at certain times and we can also determine if the amount of garbage was increasing or decreasing over time.

As students learn to interpret graphs in terms of a context and use them to answer questions, they learn an important skill in mathematical modeling (MP4).

## Alignments

### Addressing

- 8.F.A.1:  
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.
- 8.F.B.5: Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

### Building Towards

- 8.F.B: Use functions to model relationships between quantities.
- 8.F.B.5: Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR8: Discussion Supports
- Think Pair Share
- Which One Doesn't Belong?

## Student Learning Goals

Let's interpret graphs of functions.

# 5.1 Which One Doesn't Belong: Graphs

**Warm Up: 10 minutes**

The purpose of this warm-up is for students to notice and describe the features of graphs using their own language. Students will encounter a variety of graphs over the next several lessons and throughout these lessons they will gradually develop more precise language around graphs as the needs of activities dictate.

## Addressing

- 8.F.A.1

## Building Towards

- 8.F.B.5

## Instructional Routines

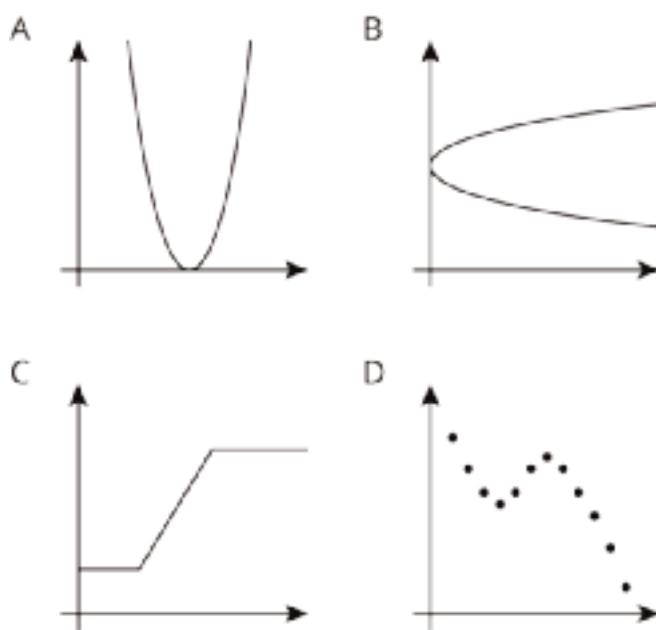
- Which One Doesn't Belong?

## Launch

Arrange students in groups of 2–4. Display the image of the four graphs for all to see. Ask students to indicate when they have noticed one figure that does not belong and can explain why. Give students 1 minute of quiet think time and then time to share their thinking with their group. Follow with a whole-class discussion.

## Student Task Statement

Which graph doesn't belong?



### Student Response

Answers vary. Sample response:

A doesn't belong because it is the only graph that touches the horizontal axis.

B doesn't belong because it is the only one that is not a function.

C doesn't belong because it is the only one made of straight line segments or because it is the only graph with no interval where it is decreasing.

D doesn't belong because it is the only one made of discrete points or because it is the only graph with two distinct intervals where it is decreasing.

### Activity Synthesis

After students have conferred in groups, invite each group to share one reason why a particular graph might not belong. Record and display the responses for all to see. After each response, ask the rest of the class if they agree or disagree. Since there is no single correct answer to the question of which graph does not belong, attend to students' explanations and ensure the reasons given are correct.

Try to highlight the two facts that there are points with the same first coordinate and different second coordinates in graph B (which means it is not a function), the straight segments of C vs. curves of the others, and the discrete nature of D during this discussion, using whatever language students bring to it. Avoid introducing the traditional  $x$  and  $y$  names for the axes into the discussion unless students use them first. More formal vocabulary will be developed in later activities, lessons, and grades, and much of the motivation of this added vocabulary is to improve upon the somewhat clunky language we are led to use without it.

## 5.2 Time and Temperature

15 minutes

The purpose of this activity is for students to begin using a graph of a functional relationship between two quantities to make quantitative observations about their relationship. For some questions students must identify specific input-output pairs while in others they can use the shape of the graph. For example, when asked for which time the temperature was warmer, students need only compare the relative height of the graph at the two different times (MP2). Similarly, students can identify another time the temperature was the same as 4:00 p.m. without actually knowing the temperature at 4:00 p.m.

Identify students who reason about the graph without identifying specific values to share during the discussion. For example, a student can identify that the temperature was highest at about 5:45 p.m. by finding the highest point on the graph without stating that that highest temperature was approximately 59°F.

### Building Towards

- 8.F.B

### Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

### Launch

Arrange students in groups of 2. Give students 4–6 minutes of quiet work time and then time to share their responses with their partner. Follow with a whole-class discussion.

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### Access for Students with Disabilities

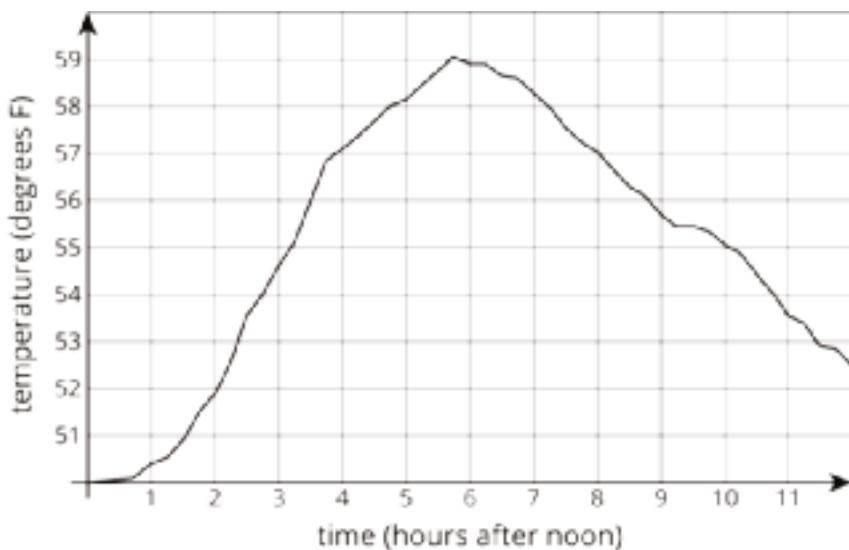
*Representation: Internalize Comprehension.* Represent the same information as the graph through different modalities by using tables and sentences. If students are unsure where to begin, suggest they create a table to represent the time and temperature.

*Supports accessibility for: Conceptual processing; Visual-spatial processing*

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### Student Task Statement

The graph shows the temperature between noon and midnight in one day in a certain city.



1. Was it warmer at 3:00 p.m. or 9:00 p.m.?
2. Approximately when was the temperature highest?
3. Find another time that the temperature was the same as it was at 4:00 p.m.
4. Did the temperature change more between 1:00 p.m. and 3:00 p.m. or between 3:00 p.m. and 5:00 p.m.?
5. Does this graph show that temperature is a function of time, or time is a function of temperature?
6. When the input for the function is 8, what is the output? What does that tell you about the time and temperature?

### Student Response

1. The temperature in the city was warmer at 9:00 p.m.
2. The temperature in the city was highest at approximately 5:45 p.m.
3. At 8:00 p.m. the temperature in the city was the same as at 4:00 p.m.
4. The temperature changed approximately 4 degrees between 1:00 p.m. and 3:00 p.m. but only approximately 3.5 degrees between 3:00 p.m. and 5:00 p.m.
5. Temperature is a function of time.
6. 57. At 8:00 p.m., the temperature is 57°F.

### Activity Synthesis

Display the graph for all to see during the discussion. Select a few previously identified students per problem to model how they found their answers on the displayed graph for the first five questions. If not mentioned by students, demonstrate how to find the solution to the fourth problem by either

identifying the temperature values at each time and subtracting or by measuring the vertical change for each time interval.

For the final question, ask students to plot the point on their graphs if they did not do so already. Invite students to describe what the point means in the context.

If time allows, give 1 minute quite think time for groups to come up with their own question that someone else could answer using the graph. Invite groups to share their question and ask a different group to give the answer.

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support whole-class discussion. Call on students to use mathematical language to restate and/or revoice the response presented. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language based on what they can interpret from a graph.

*Design Principle(s): Support sense-making; Maximize meta-awareness*

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## 5.3 Garbage

**15 minutes**

The purpose of this activity is for students to identify where a function is increasing or decreasing from a graphical representation. In the previous activity students focused more on single points. In this activity they focus on collections of points within time intervals and what the overall shape of the graph says about the relationship between the two quantities.

As students work, monitor for strategies for identifying increasing or decreasing intervals. Some strategies might be:

- finding the amount of garbage that corresponds to different years and comparing their values
- drawing line segments between discrete points and observing whether the line segment slants up or down as you move from left to right on the graph
- using a finger to show that, as you move from left to right on the graph, the function trends upward or downward

### Addressing

- 8.F.B.5

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time

### Launch

Tell students to close their books or devices (or to keep them closed). Arrange students in groups of 2. Display the first graph for all to see. If students are not familiar with time plots, explain that each point represents the value for one year starting with the point for 1991. Ask students “Does the graph show the amount of garbage produced as a function of time, or the time as a function of the amount of garbage produced?” (Amount of garbage produced as a function of time.)

Give groups 1 minute to decide on a question that the information in the graph can answer. For example, “About how much garbage was produced in 2010?” (About 250,000 thousand or 250 billion tons.) Invite groups to share their question and ask a different group to give the answer.

Tell students to reopen their books or devices and read the first problem. Ask:

- “What do the words increase and decrease mean?” (Increase means a value is going up and decrease means a value is going down.)
- “From 1999 to 2000, did the amount of garbage produced *increase* or *decrease*?” (It increased.)
- “How can you tell it increased?”

One way to answer this last question is to find the amount of garbage (about 235,000 thousand tons in 1999 and about 245,000 in 2000) and compare the values, but there are easier ways. Call upon students to articulate other methods, such as tracing from left to right with your finger and noting that your finger is traveling upwards.

Give students 3–5 minutes work time for the remaining problems. Encourage students to discuss the last question pertaining to the second graph. If partners do not agree have them work together until they come to agreement. Follow with a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Develop Language and Symbols.* Create a display of important terms and vocabulary, and maintain the display for reference throughout the unit. Invite students to suggest language or diagrams to include that will support their understanding of: increase, decrease, steep, independent, dependent, function, and variable.

*Supports accessibility for: Memory; Language*

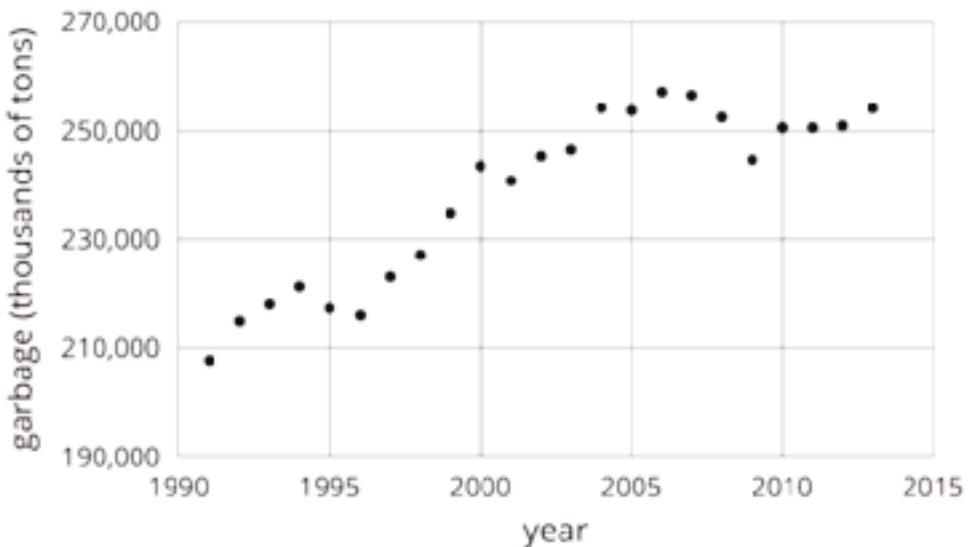
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## Anticipated Misconceptions

Students may not answer with a range of dates, they might instead list each year it increased. A list is acceptable, but be sure students see the connection between, for example, the list 1996, 1997, and 1998 and the same years stated as “from 1996 to 1998.”

### Student Task Statement

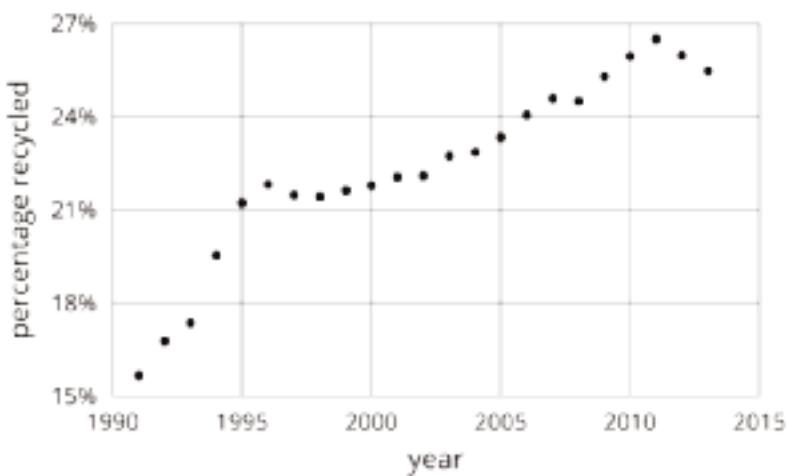
1. The graph shows the amount of garbage produced in the US each year between 1991 and 2013.



- a. Did the amount of garbage increase or decrease between 1999 and 2000?
- b. Did the amount of garbage increase or decrease between 2005 and 2009?
- c. Between 1991 and 1995, the garbage increased for three years, and then it decreased in the fourth year. Describe how the amount of garbage changed in the years between 1995 and 2000.



2. The graph shows the percentage of garbage that was recycled between 1991 and 2013.



- When was it increasing?
- When was it decreasing?
- Tell the story of the change in the percentage of garbage recycled in the US over this time period.

### Student Response

- Increase. Based on the graph, from 1999 to 2000 the amount of garbage produced increased from about 235,000 to 245,000 thousand tons.
  - Decrease. Based on the graph, from 2005 to 2009 the amount of garbage produced decreased from about 255,000 to 245,000 thousand tons.
  - The amount of garbage decreased for one year, then increased for four years in a row.
- The graph is increasing from 1991 until 1996, and from 1998 to 2011.
  - The graph is decreasing from 1996 to 1998, and 2011 to 2013.
  - Answers vary. Sample response: The percentage of garbage recycled generally increased from 1991 until a peak at 2011, and then began to decrease. There were brief periods of decrease from 1996 to 1998 and 2007 to 2008.

### Are You Ready for More?

Refer to the graph in the first part of the activity.

- Find a year where the amount of garbage produced increased from the previous year, but not by as much it increased the following year.
- Find a year where the amount of garbage produced increased from the previous year, and then increased by a smaller amount the following year.
- Find a year where the amount of garbage produced decreased from the previous year, but not by as much it decreased the following year.

4. Find a year where the amount of garbage produced decreased from the previous year, and then decreased by a smaller amount the following year.

### Student Response

1. 2003
2. 1997
3. 2008
4. 1995

### Activity Synthesis

While discussing each graph, display for all to see. For the first graph, ask previously selected students to share their responses and their strategies for finding years where the amount of garbage increased or decreased. Sequence student responses in the order listed in the Activity Narrative.

Ask students to share their responses for the second graph. Close the discussion by asking "How might you describe this graph in general to someone who couldn't see it?" Invite student to share their descriptions. For example, the percentage of garbage that was recycled increased overall from 1990 to 2011, but began decreasing from 2011 to 2013.

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### Access for English Language Learners

*Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time.* Use this routine to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. Give students time to meet with 2–3 partners, to share their response to, "Tell the story of the change in the percentage of garbage produced in the US over this time period." Provide prompts for feedback that will help students strengthen their ideas and clarify their language. For example, "What happens first in your story?", "How do the points in your story match the graph?", "What happened in 2011?", etc. Give students 1–2 minutes to revise their writing based on the feedback they received.

*Design Principle(s): Optimize output (for explanation)*

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### Lesson Synthesis

The graph of a function tells us a story about the context it represents. Specific points on the graph connect to specific features of the situation.

Consider asking some of the following questions about the graphs from the activities to reinforce these ideas:

- "On the temperature graph, how do we find the time when it was the coolest?" (By finding the point on the graph that is the lowest.)

- “On the temperature graph, how do we find the difference between the hottest and the coolest temperatures?” (We can find the lowest and the highest points on the graph and count the difference between them using the grid or we can find the hottest temperature and subtract from it the coolest temperature. Either way the difference is about 9 degrees.)
- “Looking at the garbage production graph, how does the production before 2005 compare with the production since 2005?” (From 1990 to 2005, production was increasing. After 2005 production leveled off and seemed to decrease slightly.)

## 5.4 Diego's 10K Race

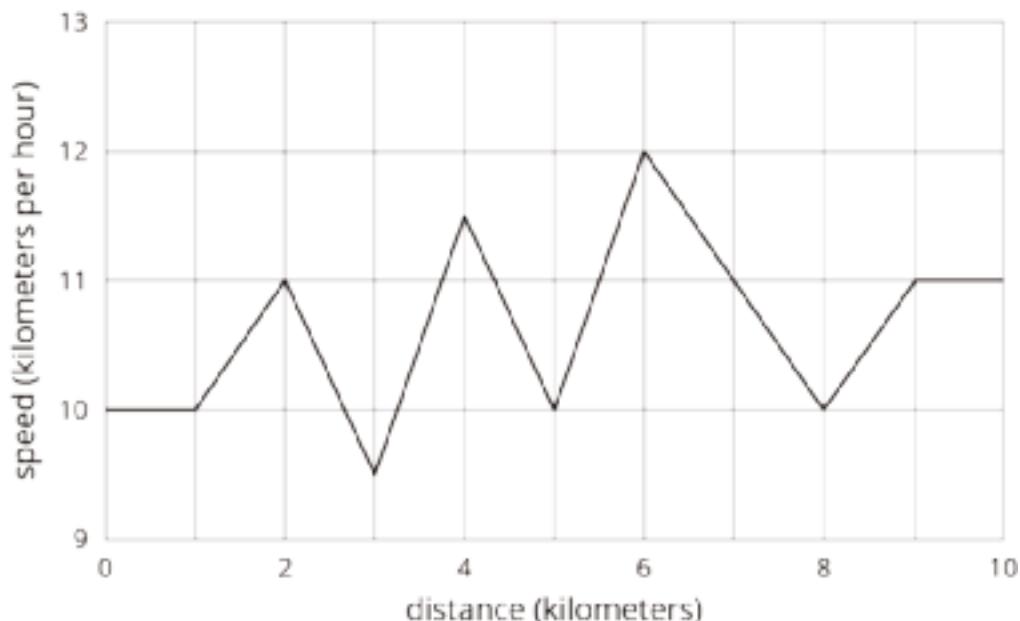
Cool Down: 5 minutes

### Building Towards

- 8.F.B

#### Student Task Statement

Diego runs a 10 kilometer race and keeps track of his speed.



- What was Diego's speed at the 5 kilometer mark in the race?
- According to the graph, where was Diego when he was going the slowest during the race?
- Describe what happened to Diego's speed in the second half of the race (from 5 km to 10 km).

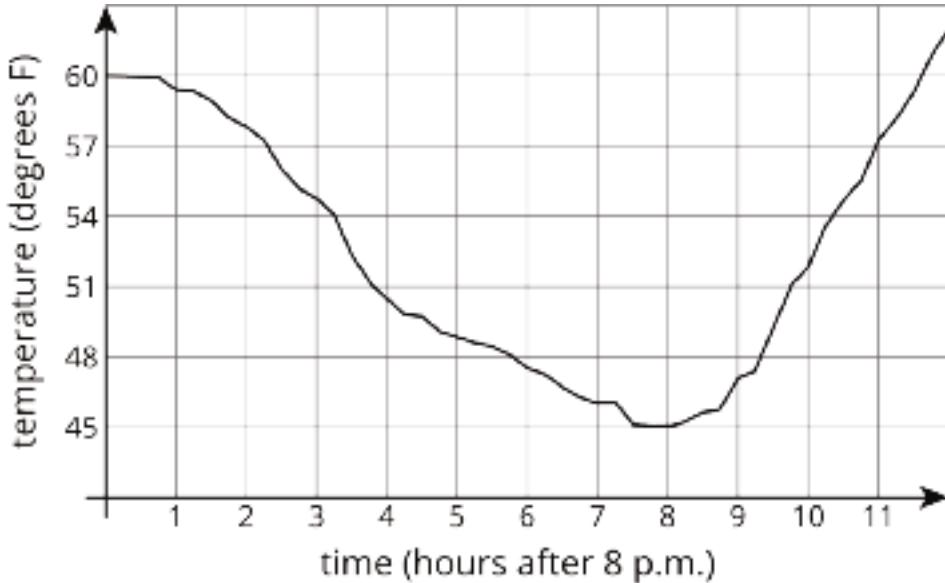
#### Student Response

- 10 kilometers per hour. This is the second coordinate of the point with first coordinate representing 5 km.

2. 3 kilometers into the race. This is the first coordinate of the lowest point on the graph, representing the slowest speed.
3. From 5 km to 6 km, Diego went faster but slowed down from 6 km to 8 km. He sped up again from 8 km to 9 km and finished the last kilometer at the same speed.

## Student Lesson Summary

Here is a graph showing the temperature in a town as a function of time after 8:00 p.m.



The graph of a function tells us what is happening in the context the function represents. In this example, the temperature starts out at 60° F at 8:00 p.m. It decreases during the night, reaching its lowest point at 8 hours after 8:00 p.m., or 4:00 a.m. Then it starts to increase again.

# Lesson 5 Practice Problems

## Problem 1

### Statement

The solution to a system of equations is  $(6, -3)$ . Choose two equations that might make up the system.

- A.  $y = -3x + 6$
- B.  $y = 2x - 9$
- C.  $y = -5x + 27$
- D.  $y = 2x - 15$
- E.  $y = -4x + 27$

### Solution

["C", "D"]

(From Unit 4, Lesson 13.)

## Problem 2

### Statement

A car is traveling on a small highway and is either going 55 miles per hour or 35 miles per hour, depending on the speed limits, until it reaches its destination 200 miles away. Letting  $x$  represent the amount of time in hours that the car is going 55 miles per hour, and  $y$  being the time in hours that the car is going 35 miles per hour, an equation describing the relationship is:

$$55x + 35y = 200$$

- a. If the car spends 2.5 hours going 35 miles per hour on the trip, how long does it spend going 55 miles per hour?
- b. If the car spends 3 hours going 55 miles per hour on the trip, how long does it spend going 35 miles per hour?
- c. If the car spends no time going 35 miles per hour, how long would the trip take? Explain your reasoning.

### Solution

- a. About 2.05 hours
- b. 1 hour

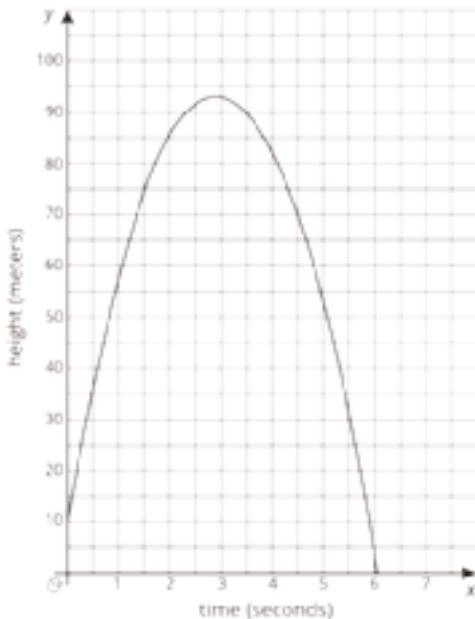
c. About 3.64 hours. If the car spent the entire trip going 55 mph, the trip would be completed in about 3.64 hours.

(From Unit 5, Lesson 3.)

## Problem 3

### Statement

The graph represents an object that is shot upwards from a tower and then falls to the ground. The independent variable is time in seconds and the dependent variable is the object's height above the ground in meters.



- How tall is the tower from which the object was shot?
- When did the object hit the ground?
- Estimate the greatest height the object reached and the time it took to reach that height. Indicate this situation on the graph.

### Solution

- 10 meters
- 6 seconds after it was shot
- Approximately 93 meters high at 2.9 seconds. A point should be plotted at (2.9, 93).

# Lesson 6: Even More Graphs of Functions

## Goals

- Compare and contrast (orally) peers' graphs that represent the same context.
- Comprehend that graphs representing the same context can appear different, depending on the variables chosen.
- Draw the graph of a function that represents a context, and explain (orally) which quantity is a function of which.

## Learning Targets

- I can draw the graph of a function that represents a real-world situation.

## Lesson Narrative

This lesson focuses on qualitative aspects of graphs, so there are no units or scale on the axes. In the warm up, students analyze two different graphs that represent the same situation (based on a series of photos). Depending on which quantities are chosen as the dependent and independent variable, both graphs describe different aspects of the same story. The two functions represented have the same independent variable (time), but different dependent variables (distance from edge of lawn vs. distance from the camera).

In the following activity, students identify independent and dependent variables from contexts and select an appropriate graph to match their choices. Different choices are possible, so students must be precise about which choice they are making and explain how the choice relates to the graph (MP6). In the final activity, students create a graph from a story. In doing so, students have to make many choices about the aspects of a situation they want to represent with a mathematical object—this is an important part of modeling with mathematics (MP4). Depending on the variables chosen, graphs of the same situation can appear to be different but still tell the same story.

## Alignments

### Addressing

- 8.F.B.5: Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

## Instructional Routines

- Group Presentations
- MLR2: Collect and Display
- MLR8: Discussion Supports

## Required Materials

### Tools for creating a visual display

Any way for students to create work that can be easily displayed to the class. Examples: chart

paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

## Required Preparation

Students are asked to make displays of their work in groups of 2–3. Prepare materials for creating a visual display in this way such as markers, chart paper, board space, etc.

### Student Learning Goals

Let's draw a graph from a story.

## 6.1 Dog Run

### Warm Up: 5 minutes

The purpose of this warm-up is for students to realize there are different dependent variables that can be used when making a model of a context and the choice of which we use affects how a graph of a function looks. In the warm-up, students compare two graphs and then determine what the creator of each graph chose as their dependent variable. During the partner discussion, students should listen to their partner's argument and decide if they agree or disagree with what they are saying (MP3).

### Addressing

- 8.F.B.5

### Launch

Display the 5 pictures of the dog from the print statement for all to see. Ask “What is happening in this succession of pictures?” After a brief quiet think time, invite students to share their answers. (As time elapses, the dog is moving.) Tell students they are going to consider two graphs describing the dog’s movement.

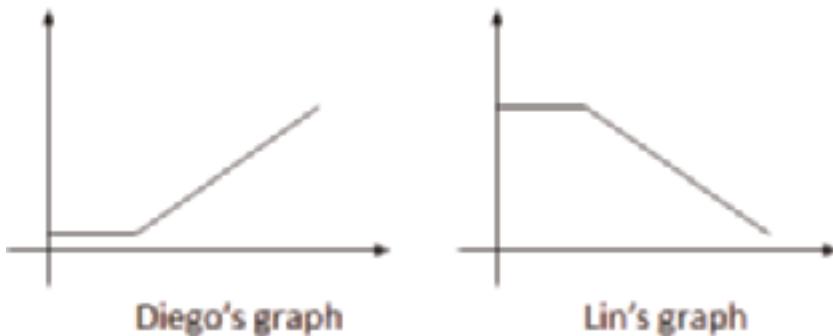
Arrange students in groups of 2. Give 1–2 minutes of quiet work time and then have them share their responses with their partner to see if they agree or disagree with what variables Diego and Lin graphed. If partners do not agree, encourage students to make sense of their partner’s thinking and reach a consensus. Follow with a whole-class discussion.

### Student Task Statement

Here are five pictures of a dog taken at equal intervals of time.



Diego and Lin drew different graphs to represent this situation:



They both used time as the independent variable. What do you think each one used for the dependent variable? Explain your reasoning.

### Student Response

Both graphs are good depictions of the scenario, using different variables as the dependent variables. For the dependent variables, Diego used the distance from the edge of the grass and Lin used the distance from the camera.

### Activity Synthesis

The goal of this discussion is for students to understand that the same situation can be represented in different ways depending on what variables you choose to represent.

Select students to share the different variables that they think Lin and Diego used in their graphs. If any partners disagreed at first, ask those groups to share how they decided on their final response for what variables Diego and Lin were using.

## 6.2 Which Graph is It?

10 minutes

The purpose of this activity is for students to sketch a graph showing the qualitative features of the function described in the problem. Building on the warm-up, in the first problem students decide what independent and dependent variables were used to create a given graph. In the second problem, students choose the variables and make their own sketch of the context. The problems are designed to have multiple correct solutions based on which quantities students identify in the descriptions. For example, Jada's information could be viewed from the perspective that the time it

takes her to swim a lap depends on how much she practices. Alternatively, we could also say that the number of seconds she takes off her time depends on how much she practices.

Watch for groups choosing different graphs for the first problem and groups sketching different graphs for the last problem to share during discussion.

## Addressing

- 8.F.B.5

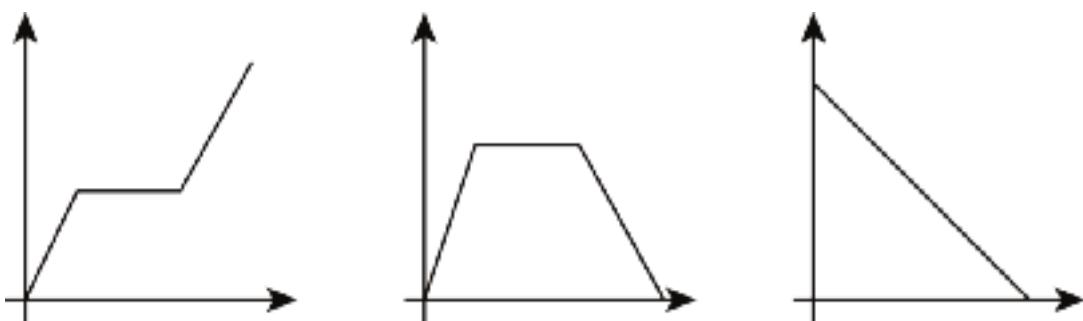
## Instructional Routines

- MLR8: Discussion Supports

## Launch

Tell students to close their books or devices (or to keep them closed). Arrange students in groups of 2. Display the statement and graphs for all to see:

Elena filled up the tub and gave her small dog a bath. Then she let the water out of the tub.



Ask groups to decide which graph best fits the provided context and what independent and dependent variables were used to create it. Give 1–2 minutes for partners to discuss and then select 2–3 groups to share their graph pick and choice for variables. (The second graph shows the amount of water increasing, then staying steady, then decreasing over time. The amount of water in the bath tub is a function of time.)

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## Access for Students with Disabilities

*Representation: Internalize Comprehension.* Represent the same information as the graph through different modalities by using tables. If students are unsure where to begin, suggest they create a table to represent the independent and dependent variables.

*Supports accessibility for: Conceptual processing; Visual-spatial processing*

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* As students explain which graph they think matches the water in Elena's bathtub, revoice student ideas to demonstrate mathematical language use. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example. This will help students to produce and make sense of the language needed to communicate their own ideas.

*Design Principle(s): Support sense-making; Optimize output (for explanation)*

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### Anticipated Misconceptions

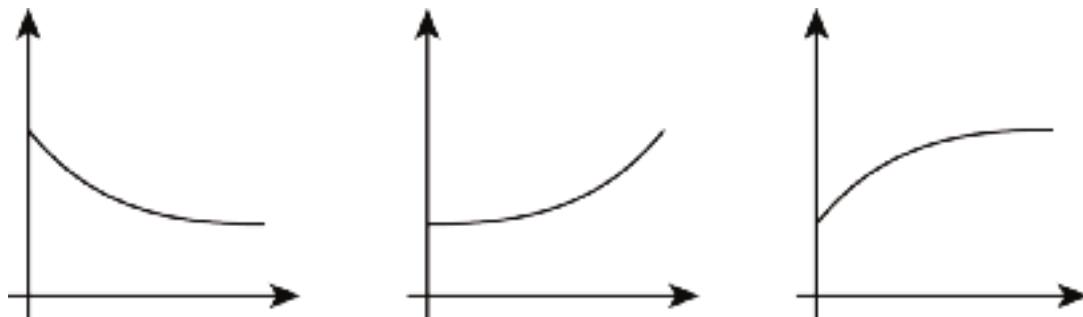
Students may have a difficult time representing the "jump" when money is added to the jar. Remind them of some of the function graphs they have seen in the past, such as with only discrete points plotted, to remind them that graphs do not have to be a single, connected line.

### Student Task Statement

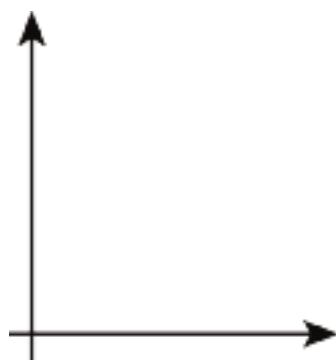
For each situation,

- name the independent and dependent variables
- pick the graph that best fits the situation, or sketch the graph if one isn't provided
- label the axes
- answer the question: which quantity is a function of which? Be prepared to explain your reasoning.

1. Jada is training for a swimming race. The more she practices, the less time it takes for her to swim one lap.



2. Andre adds some money to a jar in his room each week for 3 weeks and then takes some out in week 4.

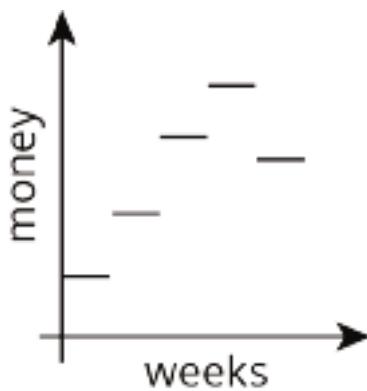


### Student Response

Answers vary. Sample responses:

1. If practice time is the independent variable and time to swim one lap is the dependent variable, then the first graph is the best choice because an increase in practice time corresponds to a decrease in lap completion time. (If the quantities are assigned in the other order, the first graph will also be the best choice.) If practice time is the independent variable and pace is the dependent variable, then the second graph could be a good choice. If practice time is the independent variable and time she takes off her total lap time is the dependent variable, then the third graph is the best choice because increase practice time leads to an increase in how much time she has dropped from her lap time.

2.

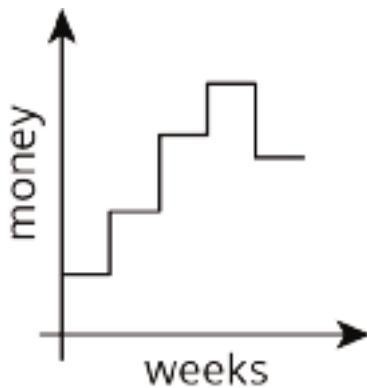


The graph shows money in the jar increasing once per week three times, followed by a decrease in week 4. The amount of money in the jar is a function of the number of weeks.

### Activity Synthesis

Select previously identified groups to share their responses and record and display their graphs for all to see.

If time allows, begin the discussion of the last problem by displaying this graph and asking students what they think of it as a possible representation of the amount of money in Andre's savings jar:



Give students 1 minute to consider the graph. Invite students to explain their thoughts about why it is or is not a good representation. (For example, the vertical lines would mean that at the same time the jar has two different amounts of money in it, which isn't possible, so this is not a good representation.) Remind students that functions only have one input for each output, so relationships whose graphs have vertical lines cannot be functions.

One way to represent a function that steps in this way uses open and closed circles to show that the function has only one value at each particular time. Discuss with students how to redraw this graph with open and closed circles so that the graph represents a function.

## 6.3 Sketching a Story about a Boy and a Bike

**20 minutes (there is a digital version of this activity)**

The purpose of this task is for students to sketch a graph from a story. In order to make the sketch, students must select two quantities from the story to graph, decide which is the independent variable and which is the dependent variable, and create and label their axes based on their decisions (MP4).

Monitor for displays that are correct but different from each other in important ways for students to focus on during the discussion. For example, they may differ by what variables were graphed such as:

- distance from home as a function of time
- distance from park as a function of time
- total distance traveled as a function of time

### Addressing

- 8.F.B.5

### Instructional Routines

- Group Presentations
- MLR2: Collect and Display

## Launch

Arrange students in groups of 2–3. Distribute tools for creating a visual display. Before students begin, it may be necessary to demonstrate how to “create a set of axes” so that a first-quadrant graph can be sketched and is large enough to be seen from a distance. Group work time followed by a whole-class discussion.

If using the digital activity, the Activity Narrative is still valid, and student graphs will be used to drive the class discussion. What varies from the print activity is students will have access to an applet to create their graphs. Students will be able to share their graphs by projecting their screen. They may need to save their graph by either exporting the image or “printscreen” to a word document.

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### Access for Students with Disabilities

*Action and Expression: Provide Access for Physical Action.* Provide access to tools and assistive technologies such as the applet or graphing software. Some students may benefit from a checklist or list of steps to be able to use the applet or software.

*Supports accessibility for: Organization; Conceptual processing; Attention*

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### Access for English Language Learners

*Conversing, Representing, Writing: MLR2 Collect and Display.* As students work in groups, capture the vocabulary and phrases students use to describe the relationship between the variables they selected over time. Listen for students who refer to the story to justify their decisions as they create the shape of the graph. Record their language on a visual display that can be referenced in future discussions. This will help students to produce and make sense of the language needed to communicate about the relationship between quantities represented by functions graphically and in story contexts.

*Design Principle(s): Support sense-making; Maximize meta-awareness*

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## Anticipated Misconceptions

Students may try and start graphing before they have clearly articulated and labeled their axes with their chosen variables. Encourage groups to make sure all are in agreement on what variables they are graphing before they create their display.

### Student Task Statement

Your teacher will give you tools for creating a visual display. With your group, create a display that shows your response to each question.

Here is a story: “Noah was at home. He got on his bike and rode to his friend’s house and stayed there for awhile. Then he rode home again. Then he rode to the park. Then he rode home again.”

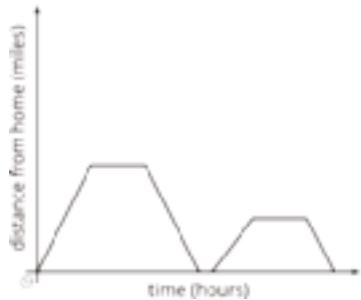
1. Create a set of axes and sketch a graph of this story.
2. What are the two quantities? Label the axes with their names and units of measure. (For example, if this were a story about pouring water into a pitcher, one of your labels might say “volume (liters).”)
3. Which quantity is a function of which? Explain your reasoning.
4. Based on your graph, is his friend’s house or the park closer to Noah’s home? Explain how you know.
5. Read the story and all your responses again. Does everything make sense? If not, make changes to your work.

### Student Response

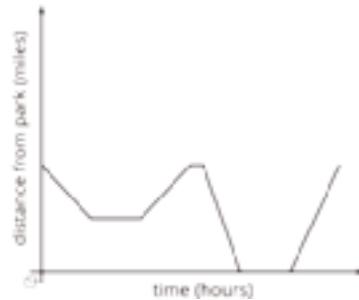
Answers vary. Sample response:

1. There are several ways of choosing which variables to use to tell the story, and once the variables are chosen, several ways to draw a graph to represent the story. These solutions here will reference the graphs A, B, and C, which are all possible graphical interpretations of the story.

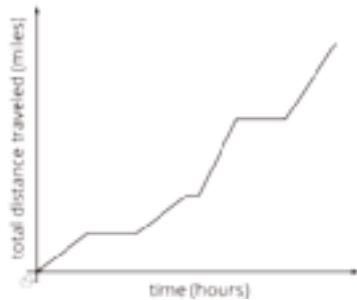
A



B



C



2. In A, the quantities are the time elapsed (in hours), and the distance Noah is from home (in miles).  
In B, the quantities are the time elapsed (in hours), and the distance Noah is from the park (in miles).  
In C, the quantities are the time elapsed (in hours), and the total distance Noah has traveled (in miles).

3. In A, the distance Noah is from home is a function of time.  
In B, the distance Noah is from the park is a function of time.  
In C, the distance Noah has traveled is a function of time.

4. In A, his friend's house is farther than the park, since the graph indicates that while at his friend's house, Noah is further from home than when he is at the park.  
In B, it is impossible to tell from the graph which is farther from Noah's house. The graph indicates that the friend's house is closer to the park than it is to Noah's house, but not enough information about the distance between Noah's house and his friend's house to answer the question.  
In C, the park is closer than his friend's house, since the graph of Noah's total distance traveled changes more when he travels to the park than it does when he travels to his friend's house.

### Are You Ready for More?

It is the year 3000. Noah's descendants are still racing around the park, but thanks to incredible technological advances, now with much more powerful gadgets at their disposal. How might their newfound access to teleportation and time-travel devices alter the graph of stories of their daily adventures? Could they affect whether or not the distance from home is a function of the time elapsed?

### Student Response

Answers vary.

### Activity Synthesis

Select previously identified groups to share their visual displays for all to see throughout the discussion. Give students 2–3 minutes of quiet think time to review the 3 displays and identify differences and similarities. Invite students to share what they identified and record and display the responses for all to see. If not brought up by students, ask:

- “What is the same and different about the two quantities that each group chose to use in their visual display?”
- “Did each group use the same units of measure? Why does it make sense to use the unit of measure seen in any of these examples?”
- “Why does it make sense to have time be the independent variable in this situation?”
- “Do all of these examples make sense in relation to the situation?”

Conclude the discussion by refocusing students on the input-output pairs described by the different graphs. For example, on a graph where distance from home is a function of time, there should be three inputs where the output is zero since he starts at home, returns home from his friend's house, and ends at home.

Alternatively, have the class do a “gallery walk” in which students leave written feedback on sticky-notes for the other groups. Here is guidance for the kind of feedback students should aim to give each other:

- “What is one thing that group did that would have made your project better if you had done it?”
- “What is one thing your group did that would have improved their project if they did it too?”
- “Does their answer make sense in the situation?”
- “Is their answer about which house is closer to the park clear and correct?”
- “If there was a mistake, what could they be more careful about in similar problems?”

## Lesson Synthesis

Keep students in the same groups. Remind them of the multiple representations of the situation in the last activity. Tell students to imagine a situation that could be modeled in at least two different ways, depending upon which variables are chosen for the axes. Give time for students to write a clear explanation of the situation, the variables chosen, and how the choices would affect the appearance of the graph. If time allows, have students sketch an example of each of their graphs and share.

## 6.4 Walking Home From School

Cool Down: 5 minutes

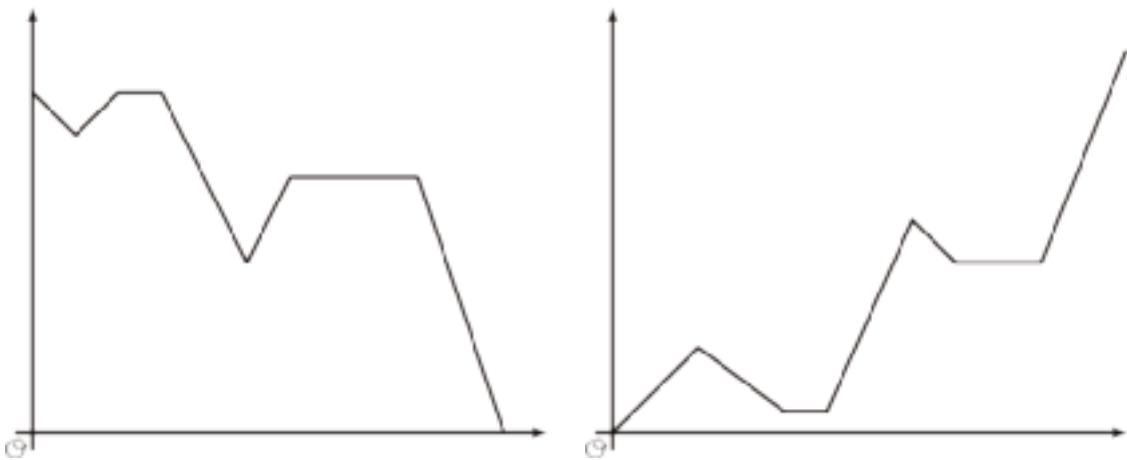
### Addressing

- 8.F.B.5

### Student Task Statement

Elena starts to walk home from school, but has to turn around and go back because she left something in her locker. On her way back home (the second time), she runs into her friend who invites her to the library to do homework with her. She stays at the library and then heads home to do her chores. Determine:

- Which graph fits Elena's story.
- What the two quantities are.
- Which quantity is a function of which.

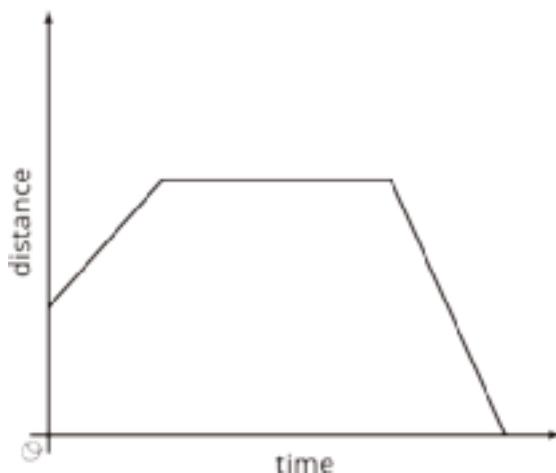


### Student Response

The first graph most directly reflects Elena's story if the vertical axis represents Elena's distance from home and the horizontal axis represents the time since she started to walk home from school the first time. The graph then demonstrates that the distance from home is a function of the time elapsed.

### Student Lesson Summary

Here is a graph showing Andre's distance as a function of time.



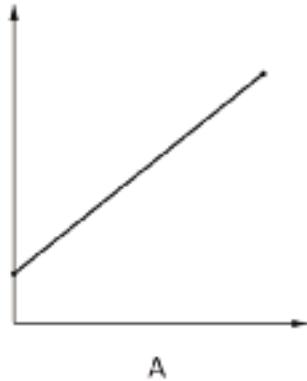
For a graph representing a context, it is important to specify the quantities represented on each axis. For example, if this is showing distance from home, then Andre starts at some distance from home (maybe at his friend's house), moves further away (maybe to a park), then returns home. If instead the graph is showing distance from school, the story may be Andre starts out at home, moves further away (maybe to a friend's house), then goes to school. What could the story be if the graph is showing distance from a park?

# Lesson 6 Practice Problems

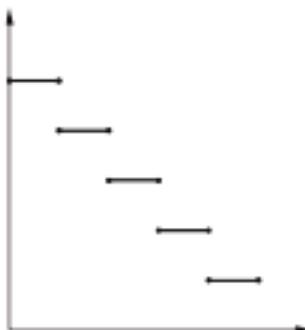
## Problem 1

### Statement

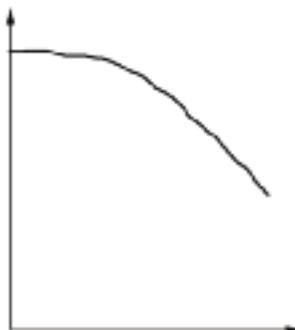
Match the graph to the following situations (you can use a graph multiple times). For each match, name possible independent and dependent variables and how you would label the axes.



A



B



C

- a. Tyler pours the same amount of milk from a bottle every morning.
- b. A plant grows the same amount every week.
- c. The day started very warm but then it got colder.
- d. A carnival has an entry fee of \$5 and tickets for rides cost \$1 each.

### Solution

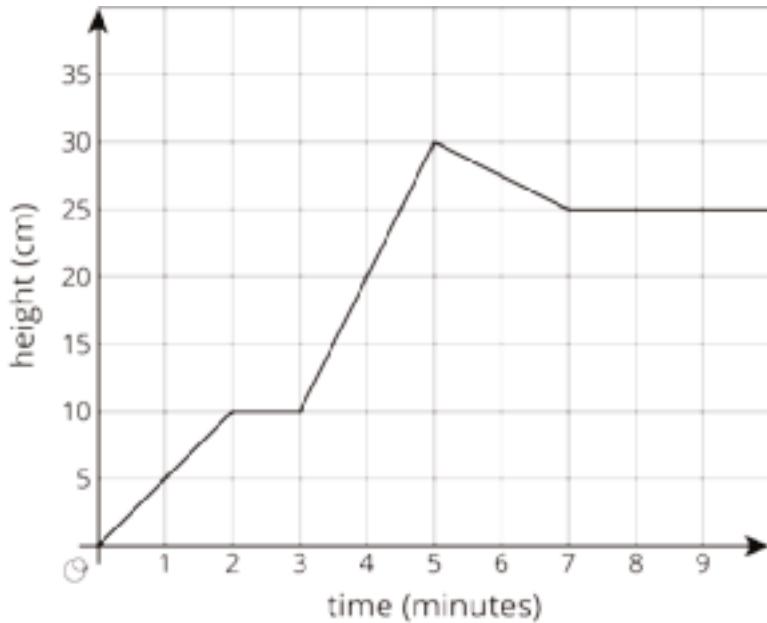
- a. B. Independent variable and horizontal axis label: time (hours). Dependent variable and vertical axis label: amount of milk in bottle (ounces).
- b. A. Independent variable and horizontal axis label: time (weeks). Dependent variable and vertical axis label: height of plant (inches).
- c. C. Independent variable and horizontal axis label: time (hours). Dependent variable and vertical axis label: temperature (degrees Celsius).
- d. A. Independent variable and horizontal axis label: number of rides. Dependent variable and vertical axis label: cost to attend carnival in dollars.

## Problem 2

### Statement

Jada fills her aquarium with water.

The graph shows the height of the water, in cm, in the aquarium as a function of time in minutes. Invent a story of how Jada fills the aquarium that fits the graph.



## Solution

Answers vary. One possible story: Jada turns on the water faucet, and the water in the aquarium is increasing at a constant rate for the first two minutes to a height of 10 cm. Then Jada's mom calls her to take out the trash, so she turns off the faucet for the minute it takes her to take out the trash. After she comes back, she turns on the water higher than before, and the water increases to a height of 30 cm in the next two minutes. This is high enough, and Jada turns off the water. Unfortunately, there is a slow leak, and the water height decreases to 25 cm. After two minutes, Jada notices the leak. She stops it, and the water stays constant after that.

## Problem 3

### Statement

Recall the formula for area of a circle.

- Write an equation relating a circle's radius,  $r$ , and area,  $A$ .
- Is area a function of the radius? Is radius a function of the area?
- Fill in the missing parts of the table.

$r$	3		$\frac{1}{2}$	
$A$		$16\pi$		$100\pi$

## Solution

a.  $A = \pi r^2$

b. Yes for both. Any radius results in one and only one area. Any area results in one and only one radius, assuming that radii have to be positive.

c.

$r$	3	4	$\frac{1}{2}$	10
$A$	$9\pi$	$16\pi$	$\frac{1}{4}\pi$	$100\pi$

(From Unit 5, Lesson 4.)

## Problem 4

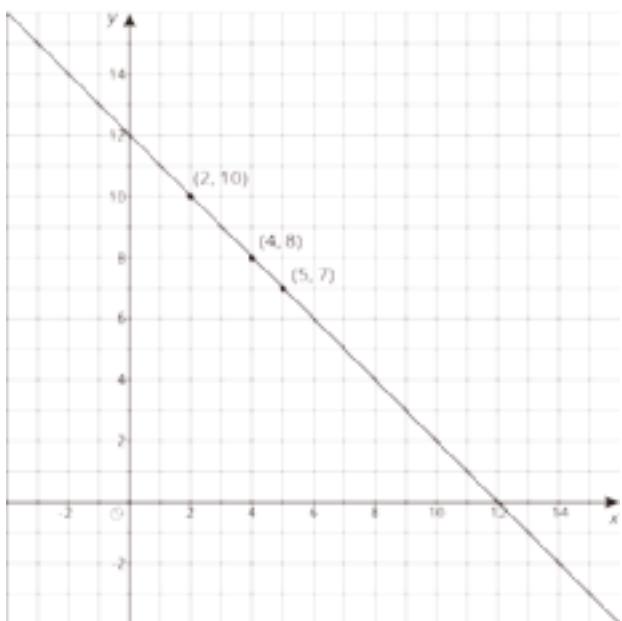
### Statement

The points with coordinates  $(4, 8)$ ,  $(2, 10)$ , and  $(5, 7)$  all lie on the line  $2x + 2y = 24$ .

- Create a graph, plot the points, and sketch the line.
- What is the slope of the line you graphed?
- What does this slope tell you about the relationship between lengths and widths of rectangles with perimeter 24?



## Solution



a.

b. -1

c. A slope of -1 means that for rectangles of perimeter 24, every extra unit of length put into the width is one less unit of length that can be put into the length.

(From Unit 3, Lesson 11.)

# Lesson 7: Connecting Representations of Functions

## Goals

- Compare and contrast (orally) representations of functions, and describe (orally) the strengths and weaknesses of each type of representation.
- Interpret multiple representations of functions, including graphs, tables, and equations, and explain (orally) how to find information in each type of representation.

## Learning Targets

- I can compare inputs and outputs of functions that are represented in different ways.

## Lesson Narrative

In this lesson, students compare two functions represented in different ways (graph and table, graph and equation, and table and verbal description). In each case, students use the different representations to find outputs for different inputs. Even though they use different representations, students are looking for the same information about the contexts and need to interpret each representation appropriately.

In a graph, students identify the input on the horizontal axis, then find the corresponding coordinate point on the graph, which lets them read the associated output. In a table, they find the input value in the first row (or column) and read the output value in the second. For functions given by equations, students substitute the input value into the expression on the right side of the equation and compute the corresponding output value on the left. Students also look for inputs corresponding to a given output by trying to reverse these procedures.

Each representation gives us the ability to find input-output pairs. However, each representation has strengths and weaknesses. Graphs require estimation but easily let us identify important features such as highest point or steepest section. Tables immediately let us find output values but only for limited input values. Equations let us precisely compute outputs for all inputs, but only one at a time. Comparing the different strengths of these representations helps students make decisions about how to use these tools strategically in the future.

Note that this lesson specifically avoids comparisons of linear functions to other linear functions, in order to avoid students associating “function” with only linear relationships. In a later lesson, students revisit some of these ideas and compare linear functions.

## Alignments

### Addressing

- 8.F.A.2: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.

- 8.F.A.3: Interpret the equation  $y = mx + b$  as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function  $A = s^2$  giving the area of a square as a function of its side length is not linear because its graph contains the points  $(1, 1)$ ,  $(2, 4)$  and  $(3, 9)$ , which are not on a straight line.

### Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- MLR5: Co-Craft Questions
- Think Pair Share

### Student Learning Goals

Let's connect tables, equations, graphs, and stories of functions.

## 7.1 Which are the Same? Which are Different?

### Warm Up: 5 minutes

The purpose of this warm-up is for students to identify connections between three different representations of functions: equation, graph, and table. Two of the functions displayed are the same but with different variable names. It is important for students to focus on comparing input-output pairs when deciding how two functions are the same or different.

### Addressing

- 8.F.A.2

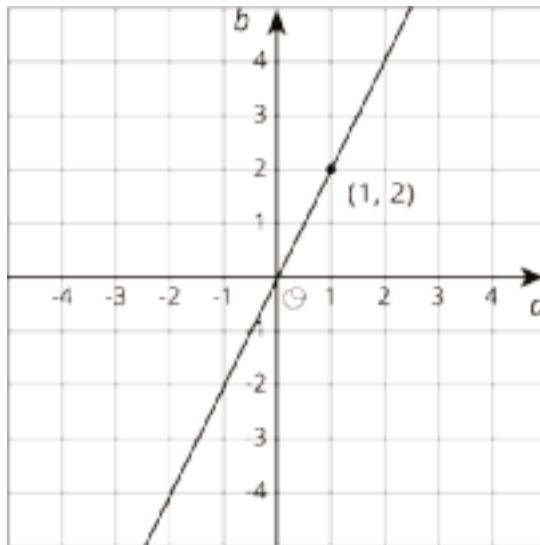
### Launch

Give students 1–2 minutes of quiet work time followed by a whole-class discussion.

### Student Task Statement

Here are three different ways of representing functions. How are they alike? How are they different?

$$y = 2x$$



$p$	-2	-1	0	1	2	3
$q$	4	2	0	-2	-4	-6

### Student Response

Answers vary. Possible responses:

- The equations for the first two would have the same form but different variables.
- The graphs for the first two are identical except for the labels on the axes.
- A table of values for both the equation and graph would have the same ordered pairs but the variables names would be different.
- The third one has opposite outputs for the same input as the first two. The graph would be a line reflected across the  $y$ -axis as compared with the first two.

### Activity Synthesis

Ask students to share ways the representations are alike and different. Record and display the responses for all to see. To help students clarify their thinking, ask students to reference the equation, graph, or table when appropriate. If the relationship between the inputs and outputs in each representation does not arise, ask students what they notice about that relationship in each representation.

## 7.2 Comparing Temperatures

10 minutes

This is the first of three activities where students make connections between different functions represented in different ways. In this activity, students are given a graph and a table of temperatures from two different cities and are asked to make sense of the representations in order to answer questions about the context.

## **Addressing**

- 8.F.A.2
- 8.F.A.3

## **Instructional Routines**

- MLR5: Co-Craft Questions

## **Launch**

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share responses with their partner. Follow with a whole-class discussion.

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### **Access for Students with Disabilities**

*Representation: Internalize Comprehension.* Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, ask students to use the same color to highlight the temperatures for each city that occurred at the same time such as blue for 4:00 pm.

*Supports accessibility for: Visual-spatial processing*

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### **Access for English Language Learners**

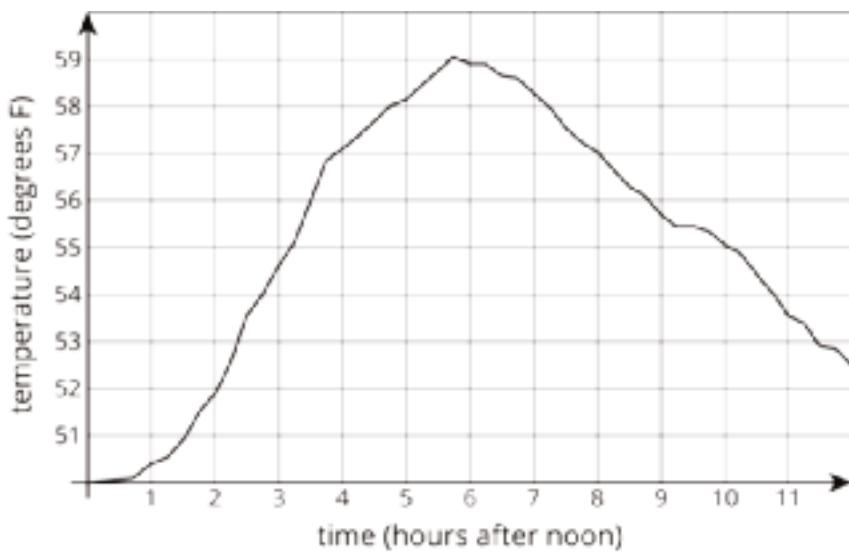
*Writing, Conversing: MLR5 Co-Craft Questions.* Display the graph and table to students without revealing the questions that follow. Invite students to work with their partner to write possible questions that could be answered by the two different representations for the two cities. Select 2–3 groups to share their questions with the class. Highlight questions that invite comparisons between the two cities. This helps students produce the language of mathematical questions and talk about the relationships between the graph and table.

*Design Principle(s): Maximize meta-awareness; Support sense-making*

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## **Student Task Statement**

The graph shows the temperature between noon and midnight in City A on a certain day.



The table shows the temperature,  $T$ , in degrees Fahrenheit, for  $h$  hours after noon, in City B.

$h$	1	2	3	4	5	6
$T$	82	78	75	62	58	59

1. Which city was warmer at 4:00 p.m.?
2. Which city had a bigger change in temperature between 1:00 p.m. and 5:00 p.m.?
3. How much greater was the highest recorded temperature in City B than the highest recorded temperature in City A during this time?
4. Compare the outputs of the functions when the input is 3.

### Student Response

1. City B. From the graph, the temperature in City A is  $57^{\circ}\text{F}$  at 4:00 p.m., and from the table, the temperature in City B is  $62^{\circ}\text{F}$
2. City B. From the graph, the temperature in City A increased about 7.5 degrees, from just under  $50.5^{\circ}\text{F}$  to just over  $58^{\circ}\text{F}$ . From the table, the temperature in B decreased 24 degrees, from  $82^{\circ}\text{F}$  down to  $58^{\circ}\text{F}$
3. About 23 degrees. From the graph, the highest recorded temperature in City A is about  $59^{\circ}\text{F}$ . From the table, the highest recorded temperature in City B is  $82^{\circ}\text{F}$ .  $82 - 59 = 23$ .
4. The first function gives the temperature in City A at 3:00 p.m., which is about  $54.5^{\circ}\text{F}$ . The second function gives the temperature in City B at 3:00 p.m., which is  $75^{\circ}\text{F}$ . City B is hotter than City A at that time by about 20.5 degrees, since  $75 - 54.5 = 20.5$ .

## Activity Synthesis

Display the graph and table for all to see. Select groups to share how they used the two different representations to get their answers for each question. To further student thinking about the advantages and disadvantages of each representation, ask:

- “Which representation do you think is better for identifying the highest recorded temperature in a city?” (The graph, since I just have to find the highest part. In the table I have to read all the values in order to find the highest temperature.)
- “Which representation do you think is quicker for figuring out the change in temperature between 1:00 p.m. and 5:00 p.m.?” (The table was quicker since the numbers are given and I only have to subtract. In the graph I had to figure out the temperature values for both times before I could subtract.)

## 7.3 Comparing Volumes

**10 minutes (there is a digital version of this activity)**

This is the second of three activities where students make connections between different functions represented in different ways. In this activity, students are given an equation and a graph of the volumes of two different objects. Students then compare inputs and outputs of both functions and what those values mean in the context of the shapes.

### Addressing

- 8.F.A.2
- 8.F.A.3

### Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

### Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share their responses with their partner. Follow with a whole-class discussion.

If using the digital activity, students will have an interactive version of the graph that the print statement uses. Using this version, students can click on the graph to determine coordinates, which might be helpful. The focus of the discussion should remain on how and why students used the graph and equation.

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### Access for Students with Disabilities

*Representation: Develop Language and Symbols.* Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example show or provide students with a cube and sphere. Discuss the relationship between the side length or radius and the volume of the object.

*Supports accessibility for: Conceptual processing*

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### Anticipated Misconceptions

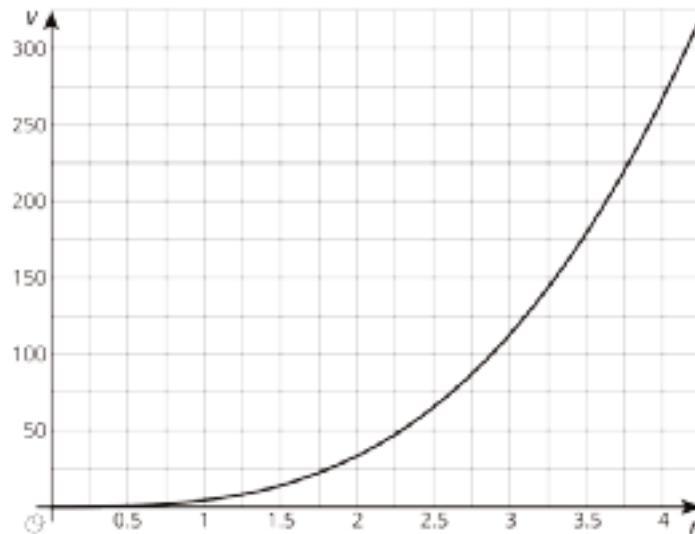
Some students may struggle with the many parts of the second question. These two questions can help scaffold the question for students who need it:

- “What information is given to you and what can you do with it?”
- “What information is the focus of the question and what would you like to know to be able to answer that question.”

### Student Task Statement

The volume,  $V$ , of a cube with edge length  $s$  cm is given by the equation  $V = s^3$ .

The volume of a sphere is a function of its radius (in centimeters), and the graph of this relationship is shown here.



1. Is the volume of a cube with edge length  $s = 3$  greater or less than the volume of a sphere with radius 3?
2. If a sphere has the same volume as a cube with edge length 5, estimate the radius of the sphere.

3. Compare the outputs of the two volume functions when the inputs are 2.

### Student Response

1. Less. The volume of a cube with edge length is  $27 \text{ cm}^3$ , since  $27 = 3^3$ . From the graph, the volume of a sphere of radius  $3 \text{ cm}^3$  is over  $100 \text{ cm}^3$ .
2. About 3.1 cm. The volume of a cube with edge length 5 cm is  $125 \text{ cm}^3$ , since  $125 = 5^3$ . From the graph, the volume of a sphere with radius 3.1 cm is about  $125 \text{ cm}^3$ .
3. The output of the cube volume function is  $8 \text{ cm}^3$  when the input is 2 cm, since  $8 = 2^3$ . From the graph, the output of the sphere volume function when the input is 2 cm is about  $35 \text{ cm}^3$ .

### Are You Ready for More?

Estimate the edge length of a cube that has the same volume as a sphere with radius 2.5.

### Student Response

About 4. From the graph, the volume of a sphere of radius 2.5 is about 65, whereas a cube of side length 4 has volume  $4^3 = 64$ .

### Activity Synthesis

The purpose of this discussion is for students to think about how they used the information from the different representations to answer questions around the context.

Display the equation and graph for all to see. Invite groups to share how they used the representations to answer the questions. Consider asking the following questions to have students expand on their answers:

- “How did you use the given representations to find an answer? How did you use the equation? The graph?”
- “For which problems was it nicer to use the equation? The graph? Explain your reasoning.”

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### Access for English Language Learners

*Conversing, Representing, Writing: MLR2 Collect and Display.* As students work, capture the vocabulary and phrases students use to describe the connections across the two different representations (e.g., equation and graph). Listen for students who justify their ideas by explaining how or why they know something is true based on the equation or graph. Scribe students' language on a visual display that can be referenced in future discussions. This will help students to produce and make sense of the language needed to communicate about the relationships between quantities represented by functions graphically and in equations.

*Design Principle(s): Support sense-making; Maximize meta-awareness*

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## 7.4 It's Not a Race

### Optional: 10 minutes

In this activity, students continue their work comparing properties of functions represented in different ways. Students are given a verbal description and a table to compare and decide whose family traveled farther over the same time intervals. The purpose of this activity is for students to continue building their skill interpreting and comparing functions.

### Addressing

- 8.F.A.2
- 8.F.A.3

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

### Launch

Give students 3–5 minutes of quiet work time, followed with whole-class discussion.

### Anticipated Misconceptions

Students may miss that Elena's family's speed has different units than what is needed to compare with Andre's family. Point out to students that the units are important and possibly ask them to find out how many miles Elena's family travels in 1 minute rather than 1 hour.

### Student Task Statement

Elena's family is driving on the freeway at 55 miles per hour.

Andre's family is driving on the same freeway, but not at a constant speed. The table shows how far Andre's family has traveled,  $d$ , in miles, every minute for 10 minutes.

$t$	1	2	3	4	5	6	7	8	9	10
$d$	0.9	1.9	3.0	4.1	5.1	6.2	6.8	7.4	8	9.1

1. How many miles per minute is 55 miles per hour?
2. Who had traveled farther after 5 minutes? After 10 minutes?
3. How long did it take Elena's family to travel as far as Andre's family had traveled after 8 minutes?
4. For both families, the distance in miles is a function of time in minutes. Compare the outputs of these functions when the input is 3.

## Student Response

1. 0.92 miles per minute.
2. a. Andre had traveled farther after 5 minutes. Elena traveled 4.6 miles, because  $5 \cdot 0.92 = 4.6$ . From the table we see that Andre traveled 5.1 miles in that same time.  
b. Elena had traveled farther after 10 minutes. Elena traveled 9.2 miles, because  $10 \cdot 0.92 = 9.2$ . Andre traveled 9.1 miles in that same time.
3. About 8.04 minutes. After 8 minutes, Andre has traveled 7.4 miles. To find the number of minutes,  $t$ , it takes Elena to travel 7.4 miles at 0.92 miles per minute, we solve the equation  $0.92t = 7.4$  to find that  $t$  is approximately 8.04 minutes.
4. The function for Andre's family gives an output of 3.0 miles on an input of 3 minutes. The function for Elena's family gives an output of 2.76 on an input of 3 minutes, since  $3 \cdot 0.92 = 2.76$ . Therefore, Andre's family has traveled a greater distance after 3 minutes than Elena's family did.

## Activity Synthesis

The purpose of this discussion is for students to think about how they use a verbal description and table to answer questions related to the context. Ask students to share their solutions and how they used the equation and graph. Consider asking some of the following questions:

- “How did you use the table to get information? How did you use the verbal description?”
- “What did you prefer about using the description to solve the problem? What did you prefer about using the table to solve the problem?”

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## Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.

*Supports accessibility for: Language; Social-emotional skills; Attention*

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### Access for English Language Learners

*Writing, Speaking: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to reflect on their problem-solving strategies. Ask students to write an initial response to the prompt, “What did you prefer about using the description to solve the problem? What did you prefer about using the table to solve the problem?” Give students time to meet with 2–3 partners, to share and get feedback on their responses. Provide prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “What do you mean?”, “Can you give an example?”, and “Can you say that another way?”, etc.). Give students 1–2 minutes to revise their writing based on the feedback they received. This will help students produce a written generalization about interpreting and comparing functions.

*Design Principle(s): Optimize output (for generalization)*

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## Lesson Synthesis

We looked at several different ways functions are represented today: graphs, tables, equations, and verbal descriptions. We can use each representation to find outputs for different inputs. In each case we may perform different actions, but we are looking for the same information. Consider discussing these questions to emphasize the strengths and weaknesses of the representations with students:

- “Think about each of the representations of a function that we have used today. What is easy or hard about using each representation? What type of question do you prefer to answer with each?”
- “If we had only used tables in the volume activity, how could that have made it easier? Harder?” (It would have been easier to find the volumes when the inputs were the same value, but it would have been harder if the tables didn’t have exactly the same inputs.)
- “If we had only used graphs to represent the functions in the car activity, how would that have been easier? How would it have been harder?” (It is easier to compare which is farther at a specific time, but it is harder to have an accurate answer, since graphs require estimation.)

## 7.5 Comparing Different Areas

Cool Down: 5 minutes

### Addressing

- 8.F.A.2

#### Student Task Statement

The table shows the area of a square for specific side lengths.

side length (inches)	0.5	1	2	3
area (square inches)	0.25	1	4	9

The area  $A$  of a circle with radius  $r$  is given by the equation  $A = \pi \cdot r^2$ .

Is the area of a square with side length 2 inches greater than or less than the area of a circle with radius 1.2 inches?

### Student Response

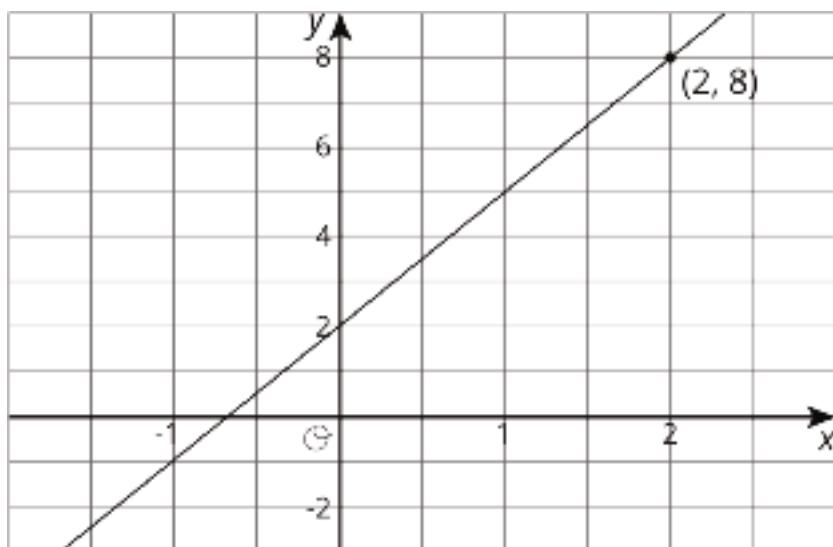
Less than. From the table the area of a square of side length 2 inches is 4 square inches, whereas from the equation the area of a circle with radius 1.2 inches is about 4.52 square inches.

## Student Lesson Summary

Functions are all about getting outputs from inputs. For each way of representing a function—equation, graph, table, or verbal description—we can determine the output for a given input.

Let's say we have a function represented by the equation  $y = 3x + 2$  where  $y$  is the dependent variable and  $x$  is the independent variable. If we wanted to find the output that goes with 2, we can input 2 into the equation for  $x$  and finding the corresponding value of  $y$ . In this case, when  $x$  is 2,  $y$  is 8 since  $3 \cdot 2 + 2 = 8$ .

If we had a graph of this function instead, then the coordinates of points on the graph are the input-output pairs. So we would read the  $y$ -coordinate of the point on the graph that corresponds to a value of 2 for  $x$ . Looking at the graph of this function here, we can see the point  $(2, 8)$  on it, so the output is 8 when the input is 2.



A table representing this function shows the input-output pairs directly (although only for select inputs).

$x$	-1	0	1	2	3
$y$	-1	2	5	8	11

Again, the table shows that if the input is 2, the output is 8.

## Glossary

- volume

# Lesson 7 Practice Problems

## Problem 1

### Statement

The equation and the tables represent two different functions. Use the equation  $b = 4a - 5$  and the table to answer the questions. This table represents  $c$  as a function of  $a$ .

$a$	-3	0	2	5	10	12
$c$	-20	7	3	21	19	45

- When  $a$  is -3, is  $b$  or  $c$  greater?
- When  $c$  is 21, what is the value of  $a$ ? What is the value of  $b$  that goes with this value of  $a$ ?
- When  $a$  is 6, is  $b$  or  $c$  greater?
- For what values of  $a$  do we know that  $c$  is greater than  $b$ ?

### Solution

- $b$
- $a = 5, b = 15$
- There is not enough information to answer this question, since 6 is not in the table for  $a$ .
- 0, 5, and 12

## Problem 2

### Statement

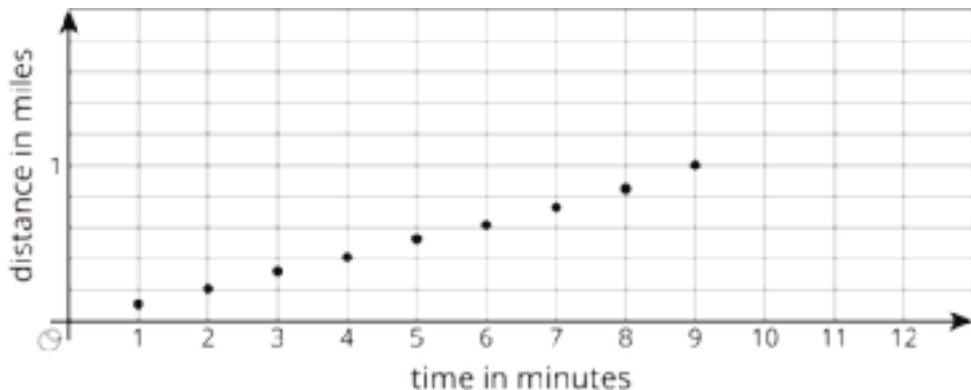
Elena and Lin are training for a race. Elena runs her mile at a constant speed of 7.5 miles per hour.

Lin's total distances are recorded every minute:

time (minutes)	1	2	3	4	5	6	7	8	9
distance (miles)	0.11	0.21	0.32	0.41	0.53	0.62	0.73	0.85	1

- Who finished their mile first?

b. This is a graph of Lin's progress. Draw a graph to represent Elena's mile on the same axes.

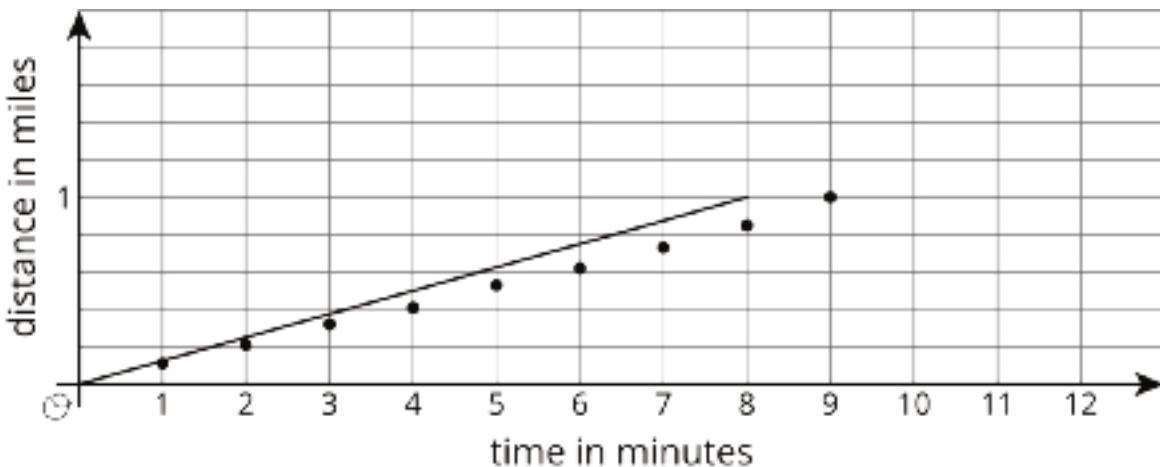


c. For these models, is distance a function of time? Is time a function of distance? Explain how you know.

### Solution

a. Elena finished her mile first. It took her 8 minutes to complete her mile, but took Lin 9 minutes.

b.



c. In both models, distance is a function of time, and time is also a function of distance. Given a time for either runner the distance can be found, and vice versa.

### Problem 3

#### Statement

Match each function rule with the value that could not be a possible input for that function.

A. 3 divided by the input	1. 3
B. Add 4 to the input, then divide this value into 3	2. 4
C. Subtract 3 from the input, then divide this value into 1	3. -4
	4. 0
	5. 1

## Solution

- A: 4
- B: 3
- C: 1

(From Unit 5, Lesson 2.)

## Problem 4

### Statement

Find a value of  $x$  that makes the equation true. Explain your reasoning, and check that your answer is correct.

$$-(-2x + 1) = 9 - 14x$$

## Solution

$x = \frac{5}{8}$ . This is the same as  $2x - 1 = 9 - 14x$ . If 1 is added to each side, that results in  $2x = 10 - 14x$ . If  $14x$  is added to each side, then  $16x = 10$ . Both sides are then multiplied by  $\frac{1}{16}$  to find  $x = \frac{10}{16}$  or  $\frac{5}{8}$ . This is correct because  $-(-2(\frac{5}{8}) + 1) = \frac{5}{4} - 1 = \frac{1}{4}$  and  $9 - 14(\frac{5}{8}) = \frac{36}{4} - \frac{35}{4} = \frac{1}{4}$ .

(From Unit 4, Lesson 4.)

# Section: Linear Functions and Rates of Change

## Lesson 8: Linear Functions

### Goals

- Comprehend that any linear function can be represented by an equation in the form  $y = mx + b$ , where  $m$  and  $b$  are rate of change and initial value of the function, respectively.
- Coordinate (orally and in writing) the graph of a linear function and its rate of change and initial value.

### Learning Targets

- I can determine whether a function is increasing or decreasing based on whether its rate of change is positive or negative.
- I can explain in my own words how the graph of a linear function relates to its rate of change and initial value.

### Lesson Narrative

This is the first of three lessons about linear functions. Students are already familiar with linear equations and their graphs from previous units.

In the first activity, students see that a proportional relationship between two quantities can be viewed as a function. They see that either quantity can be chosen as the independent variable and that the only difference in the equation and the graph is the constant of proportionality, which is visible on the graph as the slope of the line through the origin.

In the next activities, students investigate and make connections between linear functions as represented by graphs, descriptions, and by the equation  $y = mx + b$ . They interpret the slope of the line as the rate of change  $m$  of the dependent variable with respect to the independent variable and the vertical intercept of the line as the initial value  $b$  of the function. Students also compare properties of linear functions represented in different ways to determine, for example, which function has the greater rate of change. Consider using the optional activity if students need more practice comparing linear functions represented in different ways.

### Alignments

#### Addressing

- 8.F.A.2: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.
- 8.F.A.3: Interpret the equation  $y = mx + b$  as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function  $A = s^2$

giving the area of a square as a function of its side length is not linear because its graph contains the points  $(1, 1)$ ,  $(2, 4)$  and  $(3, 9)$ , which are not on a straight line.

- 8.F.B.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR5: Co-Craft Questions
- MLR8: Discussion Supports

### Student Learning Goals

Let's investigate linear functions.

## 8.1 Bigger and Smaller

### Warm Up: 5 minutes

The purpose of this warm-up is for students to reason about the values we can assign graphs based on which feature of the graph, such as slope and  $y$ -intercept, the viewer focuses on. Since there are no numbers on the graph, it is important for students to explain how they know the sign of the slope and  $y$ -intercept based on the position of the graph.

### Addressing

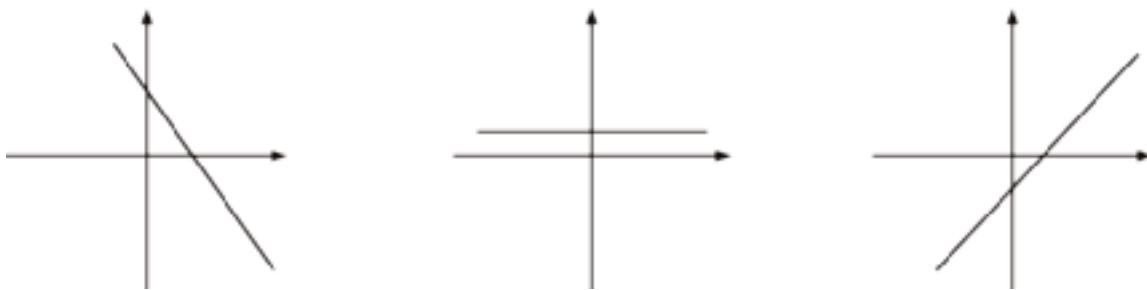
- 8.F.B.4

### Launch

Display the three graphs for all to see. Tell students that all three graphs have the same scale. Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

### Student Task Statement

Diego said that these graphs are ordered from smallest to largest. Mai said they are ordered from largest to smallest. But these are graphs, not numbers! What do you think Diego and Mai are thinking?



### Student Response

In the first graph, the line decreases when we read left to right. That means it has a negative slope. The second line stays horizontal the entire time, so it must have a slope of zero. The third graph increases as we read left to right, so it has a positive slope. That means the slopes are ordered from least to greatest.

On the other hand, the  $y$ -intercept of the graph on the left is positive and higher than the second graph. The  $y$ -intercept of the last graph is negative, so the  $y$ -intercepts are ordered from greatest to least.

Alternatively, Mai may just be looking at the left side of the graphs where they “start” while Diego is looking at the right side of the graph where they “end up.”

### Activity Synthesis

Display the three graphs for all to see. Invite students to share what they think Diego and Mai are thinking. Encourage students to reference the graphs in their explanation. Record and display their responses for all to see. Emphasize that even though there are no numbers shown, we can tell the sign of the slope and the sign of the  $y$ -intercept based on the position of the line.

## 8.2 Proportional Relationships Define Linear Functions

15 minutes

This activity begins connecting proportional relationships, which students learned in previous grades, to functions. Students use function language with proportional relationships and make connections between what they know about functions and what they know about proportional relationships. Students use similar contexts from previous grades to learn that proportional relationships are linear functions. They are also asked to determine the independent and dependent variables and the equation of the function.

Monitor for students who made opposite decisions assigning the independent and dependent variables. For the first problem, some may have written  $M = 7t$  and some may have written

$t = \frac{1}{7}M$ . For the second problem, some may have written  $f = 3y$ , and some may have written  $y = \frac{1}{3}f$ .

## Addressing

- 8.F.A.3
- 8.F.B.4

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

## Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share responses with their partner. Follow with a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example ask students to use the two different colors to represent the independent and dependent variables. Encourage students to use each respective color to label the graph, plot points, and represent variables in equations.

*Supports accessibility for: Visual-spatial processing*

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Display sentence frames to support students as they justify the independent and dependent variables they selected.

For example, “\_\_\_\_\_ is the independent variable, because \_\_\_\_\_. ” or “\_\_\_\_\_ depends on \_\_\_\_\_, because \_\_\_\_\_.”

*Design Principle(s): Support sense-making, Optimize output (for justification)*

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## Anticipated Misconceptions

Students may use different scales on the axes and then try to compare rates. Point out that in order to compare the constant rate of change visually, the scales and labels on the axes must be the same. For example, students who use the representation  $M = 7t$  versus the representation  $t = \frac{M}{7}$  may have graphs that look either very similar or very different depending on how they scaled the axes, and students who write the same equation but use different scales may have graphs that look different from each other.

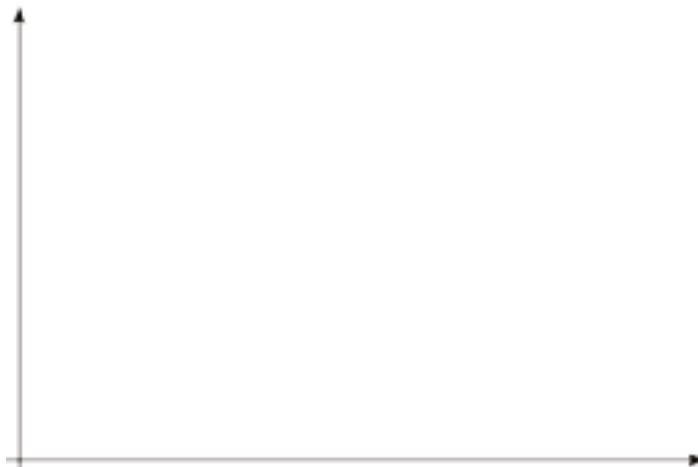
If students appear to be stuck on this misconception, consider selecting two graphs of the same function that use different scales and adding a point to the discussion about using scale to compare representations of the graph. Consider asking, "What could we do to the scale on the axes to see the constant rate of change on each graph and accurately compare them?" (Use the same scale on each axis, or graph both axes using the same length to represent 1 unit.)

### Student Task Statement

1. Jada earns \$7 per hour mowing her neighbors' lawns.
  - a. Name the two quantities in this situation that are in a functional relationship. Which did you choose to be the independent variable? What is the variable that depends on it?
  - b. Write an equation that represents the function.
  - c. Here is a graph of the function. Label the axes. Label at least two points with input-output pairs.

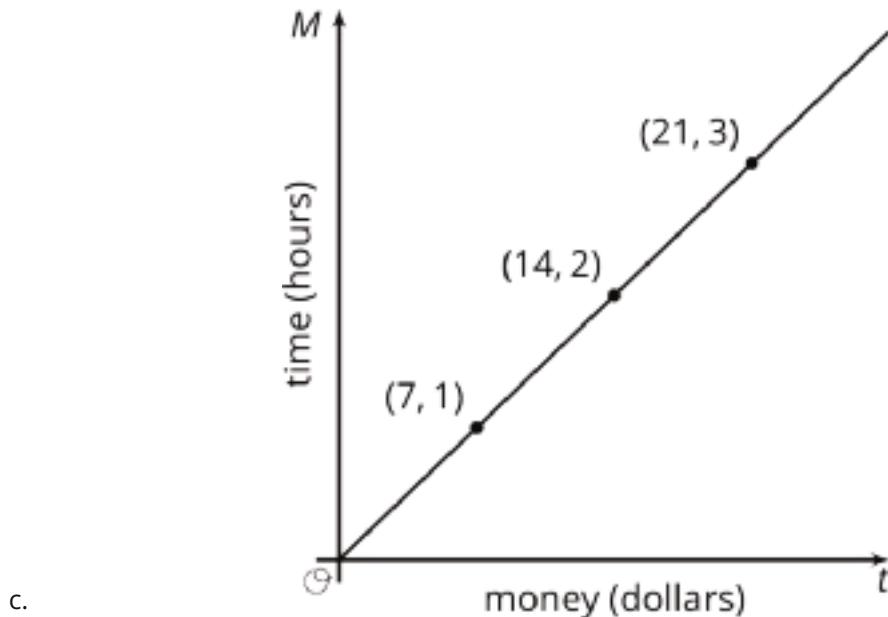


2. To convert feet to yards, you multiply the number of feet by  $\frac{1}{3}$ .
  - a. Name the two quantities in this situation that are in a functional relationship. Which did you choose to be the independent variable? What is the variable that depends on it?
  - b. Write an equation that represents the function.
  - c. Draw the graph of the function. Label at least two points with input-output pairs.



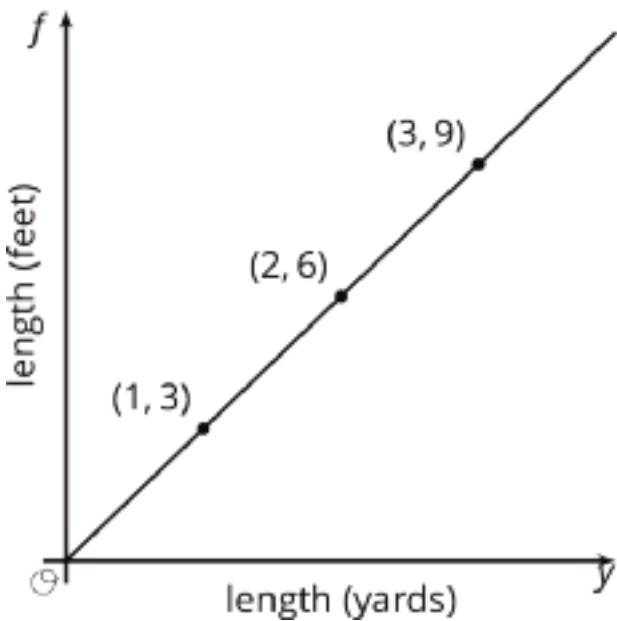
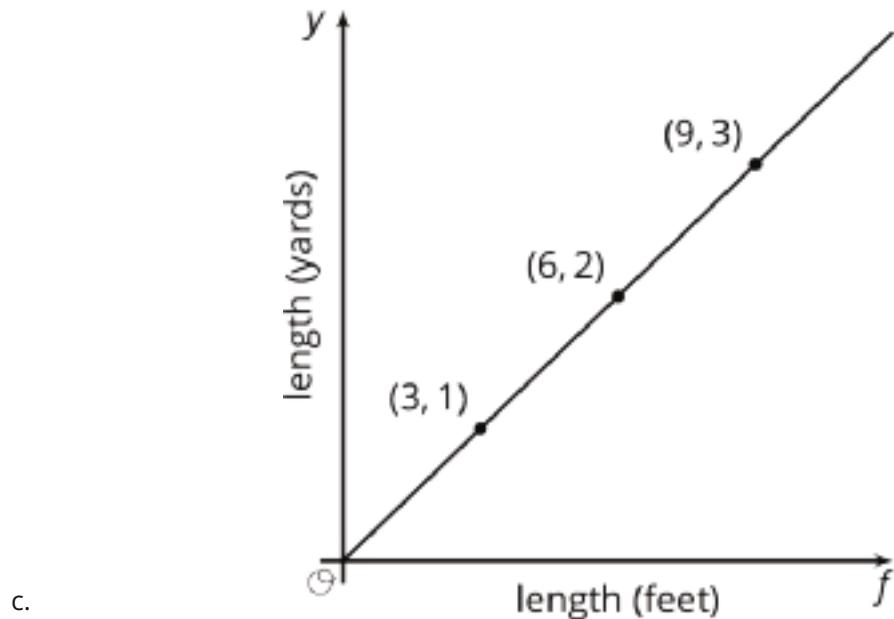
### Student Response

- The time,  $t$ , that Jada spent mowing lawns, is in a functional relationship with the amount of money,  $M$ , that Jada has earned. We can choose to think of  $t$  as a function of  $M$ , or vice versa.
  - We write  $M = 7t$  if we chose  $M$  as the dependent variable and  $t$  as the independent variable, or  $t = \frac{M}{7}$  if we chose  $t$  as the dependent variable and  $M$  as the independent variable.



- The value of a measurement in yards,  $y$ , is in a functional relationship with the value  $f$  of that same measurement in feet. We can choose to think of  $y$  as a function of  $f$ , or vice versa.

b. We write  $f = 3y$  if we choose  $f$  as the dependent variable and  $y$  as the independent variable, or  $y = \frac{f}{3}$  if we choose  $y$  as the dependent variable and  $f$  as the independent variable.



### Activity Synthesis

The purpose of this discussion is for students to understand that proportional relationships are functions and to connect the parts of functions to what they know about proportional relationships. Select previously identified students to share their equations and graphs. Sequence student work so that at least one example of each representation is shown for each of the problems, starting with the most common representation. Display their responses for all to see. Consider asking some of the following questions to help students make connections between the different representations:

- “For the first problem, if we wanted to know how many hours Jada needs to work to make a certain amount of money, which equation would make more sense to use? Why?” ( $t = \frac{1}{7}M$ , because in that equation, time worked,  $t$ , is expressed as a function of money earned,  $M$ .)
- “For the second problem, when would we want to use the equation  $f = 3y$ ?” (When we know the number of yards and need to calculate the number of feet.)
- “How do we know that each of these situations are represented by functions?” (For each valid input, there is only one output. For example, no matter which equation I use for the relationship between feet and yards, a specific number of feet will always equal the same number of yards.)

## 8.3 Is it Filling Up or Draining Out?

**10 minutes**

The purpose of this activity is to connect features of an equation representing a function to what that means in a context.

Students start with two functions that represent a tank being filled up and another being drained out and are asked to determine which equations represent which situation. This gives students the opportunity to connect initial value and slope, which they learned about in a previous unit, to the general form of the linear equation and to the fact that linear relationships are functions.

### Addressing

- 8.F.A.2
- 8.F.A.3
- 8.F.B.4

### Instructional Routines

- MLR5: Co-Craft Questions

### Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share their responses with their partner and reach agreement on their answers. Encourage partners to talk about specific parts of the graph and equation that indicate whether the tank is filling up or draining out. Follow with a whole-class discussion.

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### Access for Students with Disabilities

*Action and Expression: Internalize Executive Functions.* Chunk this task into more manageable parts to support students who benefit from support with organization and problem solving. For example present one question at a time.

*Supports accessibility for: Organization; Attention*

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### Access for English Language Learners

*Writing, Conversing: MLR5 Co-Craft Questions.* Display the task statement without the questions that follow. Ask pairs of students to write possible mathematical questions about the situation. Then, invite pairs to share their questions with the class. Look for questions that ask students to determine which tank is increasing or decreasing in volume. This helps students produce the language of mathematical questions and talk about the relationship between slope and the situation of water flowing in or out of a tank prior to solving the questions in this activity.

*Design Principle(s): Maximize meta-awareness; Support sense-making*

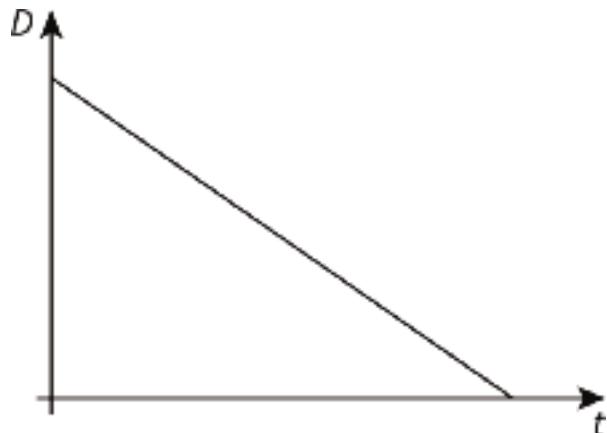
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### Student Task Statement

There are four tanks of water.

- The amount of water in gallons,  $A$ , in Tank A is given by the function  $A = 200 + 8t$ , where  $t$  is in minutes.
- The amount of water in gallons,  $B$ , in Tank B starts at 400 gallons and is decreasing at 5 gallons per minute. These functions work when  $t \geq 0$  and  $t \leq 80$ .
  1. Which tank started out with more water?
  2. Write an equation representing the relationship between  $B$  and  $t$ .
  3. One tank is filling up. The other is draining out. Which is which? How can you tell?
  4. The amount of water in gallons,  $C$ , in Tank C is given by the function  $C = 800 - 7t$ . Is it filling up or draining out? Can you tell just by looking at the equation?

5. The graph of the function for the amount of water in gallons,  $D$ , in Tank D at time  $t$  is shown. Is it filling up or draining out? How do you know?



### Student Response

1. Tank B. The two equations tell us that when  $t = 0$ , the volume of water in Tank A is 200 gallons, and the volume of water in Tank B is 400 gallons.
2.  $B = 400 - 5t$
3. Tank A is filling up, and Tank B is draining out. As time goes on, corresponding to larger values of  $t$ , the value of  $A$  gets bigger, but the value of  $B$  gets smaller.
4. Draining out. As  $t$  increases, the value of  $C$  decreases, since we are subtracting larger values from 800. In short, it is because we are subtracting multiples of  $t$  instead of adding them that we can quickly see that Tank C is draining.
5. Draining out. As time increases, the value of  $D$  goes down.

### Are You Ready for More?

- Pick a tank that was draining out. How long did it take for that tank to drain? What percent full was the tank when 30% of that time had elapsed? When 70% of the time had elapsed?
- What point in the plane is 30% of the way from  $(0, 15)$  to  $(5, 0)$ ? 70% of the way?
- What point in the plane is 30% of the way from  $(3, 5)$  to  $(8, 6)$ ? 70% of the way?

### Student Response

- Answers vary. Sample response: It takes 80 minutes for Tank B to drain, because  $400 \div 5 = 80$ . The tank is 70% full after 30% of that time has elapsed. The tank is 30% full after 70% of that time has elapsed.
- The point  $(1.5, 10.5)$  is 30% of the way from  $(0, 15)$  to  $(5, 0)$ , because  $0.3 \cdot 5 = 1.5$  and  $0.7 \cdot 15 = 10.5$ . The point  $(3.5, 4.5)$  is 70% of the way from  $(0, 15)$  to  $(5, 0)$ , because  $0.7 \cdot 5 = 3.5$  and  $0.3 \cdot 15 = 4.5$ .

- The point  $(4.5, 5.3)$  is 30% of the way from  $(3, 5)$  to  $(8, 6)$ . The  $x$ -coordinates are 5 units apart, because  $8 - 3 = 5$ . 30% of 5 is 1.5 and  $3 + 1.5 = 4.5$ . The  $y$ -coordinates are 1 unit apart, because  $6 - 5 = 1$ . 30% of 1 is 0.3 and  $5 + 0.3 = 5.3$ .  
The point  $(6.5, 5.7)$  is 70% of the way from  $(3, 5)$  to  $(8, 6)$ . 70% of 5 is 3.5 and  $3 + 3.5 = 6.5$ . 70% of 1 is 0.7 and  $5 + 0.7 = 5.7$ .

## Activity Synthesis

Consider asking some of the following questions to begin the discussion:

- “For the second problem, what in the equation tells you that the slope is decreasing? Increasing?” (Decreasing: the slope in the equation is negative; Increasing: the slope in the equation is positive.)
- “For the third problem, what is similar between the equation in this problem and the decreasing equation in the previous problem?”  
(Both slopes are negative.)
- “For the last problem, what in the graph tells you that Tank D is draining out? What would a graph that has a tank filling up look like? What would be different?” (I know it is draining out because the graph is going down from left to right. If the tank were filling up, the graph would be going up from left to right.)

Tell students that a linear function can always be represented with an equation of the form  $y = mx + b$ . The slope of the line,  $m$ , is the rate of the change of the function and the initial value of the function is  $b$ .

If time allows, give students the following scenario to come up with a possible equation for Tank D:

“Tank D started out with more water than Tank B but less water than Tank C. The water is draining from Tank D faster than from Tank B but slower than Tank C. What is a possible equation for the graph of the function for the amount of water  $D$  in Tank D over time  $t$ ?“ (Students should choose an initial value between 400 and 800, and a constant rate of change between -7 and -5. One possible such equation might be  $D = 600 - 6t$ .)

## 8.4 Which is Growing Faster?

### Optional: 10 minutes

The purpose of this activity is for students to connect their work with linear equations to functions. The two linear functions in this activity are represented differently and students are asked to compare various features of each representation.

Identify students who use different methods to answer the questions.

For example, students may write an equation to represent Noah’s account, and others may make a table to show the value in each account at different numbers of weeks by reasoning about the rate of change and the amount in each account when they were opened.

## Addressing

- 8.F.A.2
- 8.F.A.3
- 8.F.B.4

## Instructional Routines

- MLR1: Stronger and Clearer Each Time

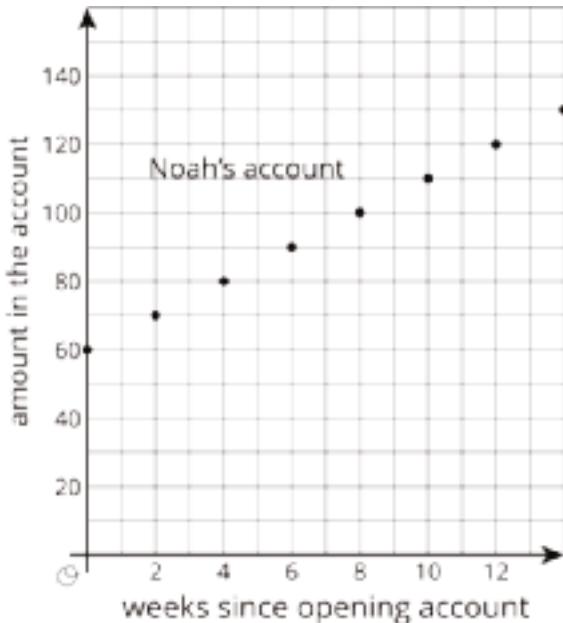
## Launch

Give students 1–2 minutes of quiet work time. Follow with a whole-class discussion.

### Student Task Statement

Noah is depositing money in his account every week to save money. The graph shows the amount he has saved as a function of time since he opened his account.

Elena opened an account the same day as Noah. The amount of money  $E$  in her account is given by the function  $E = 8w + 60$ , where  $w$  is the number of weeks since the account was opened.



1. Who started out with more money in their account? Explain how you know.
2. Who is saving money at a faster rate? Explain how you know.
3. How much will Noah save over the course of a year if he does not make any withdrawals? How long will it take Elena to save that much?

## Student Response

1. They are the same. At the left edge of the graph, representing the time when they opened the accounts, Noah had \$60. When  $t$  is 0, the money in Elena's account when it was opened is found by  $E = 8 \cdot 0 + 60$ , so she also had \$60.

2. Elena is saving money at a faster rate. Every 2 weeks, Noah's account increases by \$10 while Elena's account goes up by \$8 each week, so she makes \$16 in two weeks.
3. Noah will save \$260 over a year in addition to the \$60 he opened the account with, since he saves \$10 every 2 weeks and there are 52 weeks in the year. It will take Elena just 33 weeks to save the same amount since she also started with \$60 ( $260 \div 8 = 32.5$ , so rounding up, it will take 33 weeks).

## Activity Synthesis

Display the graph of Noah's savings over time and the equation for the amount of money in Elena's account for all to see. Select students previously identified to share their responses.

Consider asking the following questions to help student make connections between the different representations:

- "How did you determine the amount Noah saved in a year?" (I used the graph to figure out that Noah saves \$5 each week and multiplied that by 52 weeks.)
- "What equations could you use to solve the last question?" (I could use the equation  $N = 60 + 5w$  for the amount of money in Noah's account after  $w$  weeks. When  $w = 52$ , Noah has \$320. If I solve the equation  $320 = 8w + 60$  for  $w$ , I would know how many weeks it would take Elena to have \$320 in her account.)
- "How could you solve the last question without using an equation?" (I could extend the graph out to 52 weeks and plot the value of each account over the year.)

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## Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.

*Supports accessibility for: Attention; Social-emotional skills*

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## Access for English Language Learners

*Writing, Speaking: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to refine their explanation of how they determined who started out with more money in their account. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "What did you do first?", and "How did you determine how much money they each had at the start?", etc.). Students can borrow ideas and language from each partner to strengthen the final product. They can return to the first partner and revise their initial response.

*Design Principle(s): Optimize output (for explanation)*

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## Lesson Synthesis

To help students make connections between work they have done previously with linear equations and functions, consider asking some of the following questions:

- “How can we tell if a linear function is increasing from an equation? From a graph?” (In a linear equation  $y = mx + b$ , if  $m$  is positive, the linear function is increasing. In a graph, if the line is going up from left to right, then the function is increasing.)
- “How can we tell from the graph if the initial value of the function is positive? How can I tell from the equation?” (If the graph of the function crosses the vertical axis above 0, then the initial value is positive. In the equation,  $y = mx + b$  of a linear function, the  $b$  is positive when the initial value is positive.)

## 8.5 Beginning to See Daylight

Cool Down: 5 minutes

### Addressing

- 8.F.A.3
- 8.F.B.4

### Launch

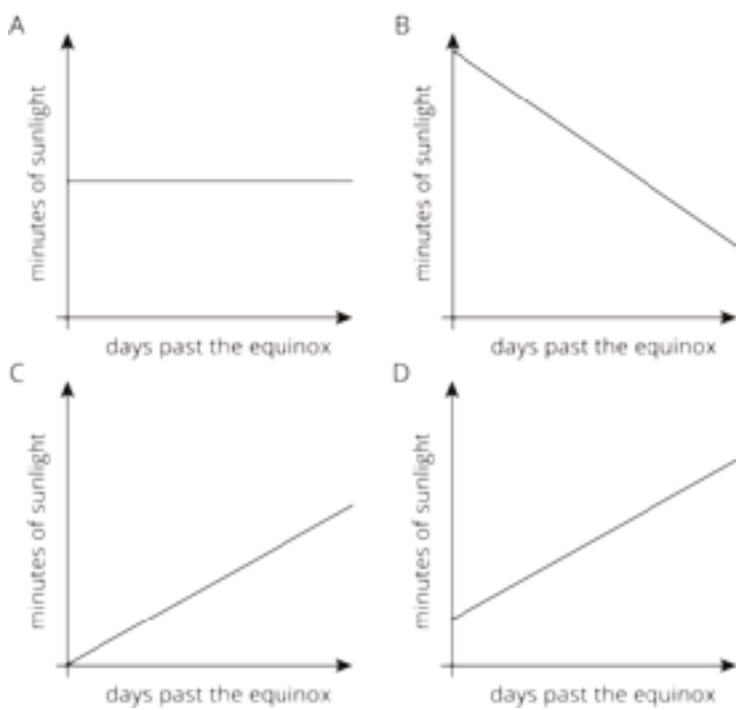
Explain that an equinox is the day when there is an approximately equal amount of daylight and darkness.

### Anticipated Misconceptions

Some students may think graph C is an acceptable choice, since some places do get zero hours of sunlight sometimes, but this never happens near the equinox or for any cities in France.

### Student Task Statement

In a certain city in France, they gain 2 minutes of daylight each day after the spring equinox (usually in March), but after the autumnal equinox (usually in September) they lose 2 minutes of daylight each day.



1. Which of the graphs is most likely to represent the graph of daylight for the month after the spring equinox?
2. Which of the graphs is most likely to represent the graph of daylight for the month after the autumnal equinox?
3. Why are the other graphs not likely to represent either month?

### Student Response

1. D
2. B
3. Graph A does not make sense because there is a constant amount of daylight. Graph C does not make sense because it goes through the origin, meaning it started with 0 minutes of daylight.

### Student Lesson Summary

Suppose a car is traveling at 30 miles per hour. The relationship between the time in hours and the distance in miles is a proportional relationship. We can represent this relationship with an equation of the form  $d = 30t$ , where distance is a function of time (since each input of time has exactly one output of distance). Or we could write the equation  $t = \frac{1}{30}d$  instead, where time is a function of distance (since each input of distance has exactly one output of time).

More generally, if we represent a linear function with an equation like  $y = mx + b$ , then  $b$  is the initial value (which is 0 for proportional relationships), and  $m$  is the rate of change of the

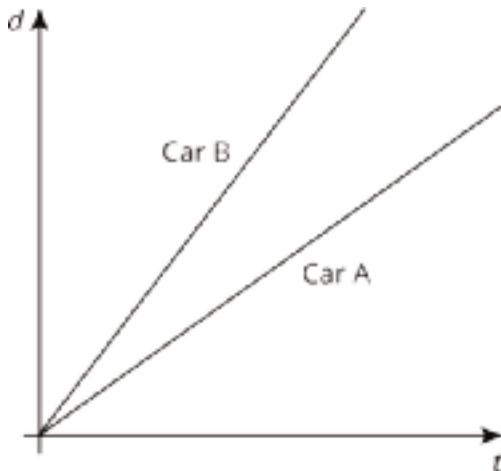
function. If  $m$  is positive, the function is increasing. If  $m$  is negative, the function is decreasing. If we represent a linear function in a different way, say with a graph, we can use what we know about graphs of lines to find the  $m$  and  $b$  values and, if needed, write an equation.

## Lesson 8 Practice Problems

### Problem 1

#### Statement

Two cars drive on the same highway in the same direction. The graphs show the distance,  $d$ , of each one as a function of time,  $t$ . Which car drives faster? Explain how you know.



#### Solution

Car B drives faster. The two cars began at the same place, but after any amount of time, Car B has traveled farther than Car A. Graphically, the slope of the line corresponding to Car B is greater than the slope of the line corresponding to Car A, so the rate of change of distance per time (speed) is higher for Car B.

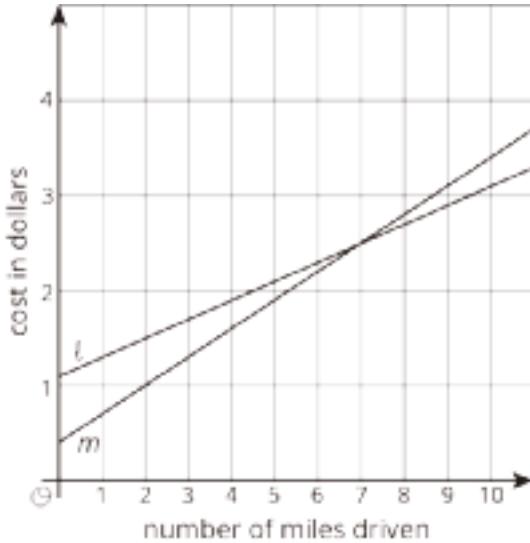
### Problem 2

#### Statement

Two car services offer to pick you up and take you to your destination. Service A charges 40 cents to pick you up and 30 cents for each mile of your trip. Service B charges \$1.10 to pick you up and charges  $c$  cents for each mile of your trip.

a. Match the services to the Lines  $\ell$  and  $m$ .

b. For Service B, is the additional charge per mile greater or less than 30 cents per mile of the trip? Explain your reasoning.



## Solution

a. Service A is represented by Line  $m$ . Service B is represented by Line  $\ell$ .

b. Less than 30 cents per mile since Line  $\ell$  is not increasing as quickly as Line  $m$ .

## Problem 3

### Statement

Kiran and Clare like to race each other home from school. They run at the same speed, but Kiran's house is slightly closer to school than Clare's house. On a graph, their distance from their homes in meters is a function of the time from when they begin the race in seconds.

a. As you read the graphs left to right, would the lines go up or down?

b. What is different about the lines representing Kiran's run and Clare's run?

c. What is the same about the lines representing Kiran's run and Clare's run?

## Solution

a. Down

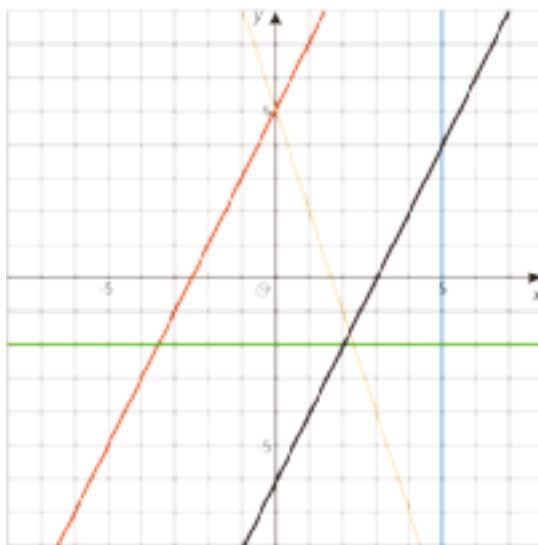
b. Answers vary. Sample response: Clare's line would be higher up since she started farther away from her house.

c. Answers vary. Sample response: The lines would have the same slope since they run at the same speed.

## Problem 4

### Statement

Write an equation for each line.



## Solution

Green line:  $y = -2$ , blue line:  $x = 5$ , black line:  $y = 2x - 6$ , yellow line:  $y = -3x + 5$ , red line:  $y = 2x + 5$

(From Unit 3, Lesson 11.)

# Lesson 9: Linear Models

## Goals

- Compare and contrast (orally and in writing) different linear models of the same data, and determine (in writing) the range of values for which a given model is a good fit for the data.
- Create a model of a non-linear data using a linear function, and justify (orally and in writing) whether the model is a good fit for the data.

## Learning Targets

- I can decide when a linear function is a good model for data and when it is not.
- I can use data points to model a linear function.

## Lesson Narrative

In this lesson, students use linear functions to model real-world situations (MP4). In the candle activity, they are given data for an almost linear relationship and develop a linear model. They use their model to make predictions and discuss the reasonableness of the model. In the shadow activity, it is difficult to tell from the information given if a linear model is appropriate, but when they are given more information, it becomes clear that the relationship is *not* linear. In the garbage recycling activity, different linear models apply to different time periods.

None of the given data are perfectly fit by a linear function, and students have to determine whether a linear approximation is reasonable and for which values it would be reasonable. Students should start to see both the value of linear models and their limitations. The garbage recycling activity leads into the next lesson, which is about modeling with piece-wise linear functions.

## Alignments

### Addressing

- 8.F.B.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

## Instructional Routines

- MLR6: Three Reads
- MLR8: Discussion Supports
- Poll the Class

## Required Materials

### Straightedges

A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a

straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

## Required Preparation

Provide access to straightedges to each student.

### Student Learning Goals

Let's model situations with linear functions.

## 9.1 Candlelight

### Warm Up: 10 minutes (there is a digital version of this activity)

In this warm-up, students work with data to determine if the situation represented by the data could be modeled by a linear function (MP4). Students are given 3 different data points and use what they know about linear functions and proportional relationships to estimate when the candle will burn out. Students are then asked to determine if this situation could be modeled by a linear function. The focus of the discussion should be around the last question and how students justify their reasoning.

### Addressing

- 8.F.B.4

### Instructional Routines

- Poll the Class

### Launch

Arrange students in groups of 2. Give students 1–2 minutes of quiet work time and ask them to pause after the first question. Poll the class for their response to the first question, and display the range of responses for all to see. Then, ask them to continue and discuss their response to the second question with their partner. If they don't agree, partners should work to understand each other's thinking. Follow with a whole-class discussion.

If using the digital activity, follow the structure above, as the prompts are the same. However, the digital activity allows students to plot points quickly without having to set up the axes from scratch. This means students may conclude the graph is not quite linear purely from a visual. Make sure these students can explain and understand their peers' rationale in answering the questions using more than just the plotted points. If any students attempt to guess a linear equation that fits the data, ask them to share during the discussion.

### Student Task Statement

A candle is burning. It starts out 12 inches long. After 1 hour, it is 10 inches long. After 3 hours, it is 5.5 inches long.

1. When do you think the candle will burn out completely?
2. Is the height of the candle a function of time? If yes, is it a linear function? Explain your thinking.

### Student Response

1. Answers vary. Sample response: Since it burns about 2 inches every hour, it will burn out between 5 and 6 hours after it was lit.
2. The height of a candle is a function of time, because at any given time, the candle will have one and only one height. It is not exactly linear, although it looks reasonable to approximate it as a linear function since the rate of burning is almost constant (2 inches per hour).

### Activity Synthesis

The purpose of this discussion is for students to justify how this situation can be modeled by a linear equation. Select students who answered yes to the last question and ask:

- "Was the data exactly linear? If not, what made you decide that you could treat it as such?"
- "Which data points did you use to predict when the candle would burn out?"
- "What was the slope between the first two data points? What was the slope between the last two data points? What does it mean that their slopes are different?"

Tell students that although the data is not precisely linear, it does make sense to model the data with a linear function because the points resemble a line when graphed. We can then use different data points to help predict when the candle would burn out. Answers might vary slightly, but it results in a close approximation.

Conclude the discussion by asking students to reconsider the range of values posted earlier for the first question and ask if they think that range is acceptable or if it needs to change (for example, students may now think the range should be smaller after considering the different slopes).

## 9.2 Shadows

### 10 minutes (there is a digital version of this activity)

The purpose of this activity is for students to determine if a given set of data can be modeled by a linear function. Students first view a set of pictures and data for the length of a shadow at 0, 20, and 60 minutes. Then, they make a prediction about how long a shadow will be after 95 minutes.

Students then compare their estimate with the actual length of the shadow and make conclusions about the model they used to make their estimate.

Monitor for students using different strategies to make their prediction. For example, students may use different pairs of points to make their prediction or they may try to use all three. The discussion for this activity focuses on how, if we are given two input-output pairs, we can always find a linear function with these inputs and outputs, but that doesn't mean a linear function is actually

appropriate for the situation. In this case, we can see when we get more data that a linear function is not appropriate.

### **Addressing**

- 8.F.B.4

### **Instructional Routines**

- MLR6: Three Reads

### **Launch**

Arrange students in groups of 2. Tell students to close their books or devices, and display the image and given data for all to see. Give students 1–2 minutes of quiet think time to estimate the length of the shadow after 95 minutes and discuss their responses with their partner. Encourage partners to discuss their estimation strategy and why their estimate makes sense. Invite groups to share their estimate and reasoning with the whole class.

Tell students to open their books or devices and give work time for the remaining questions. Follow with a whole-class discussion.

If using the digital activity, follow the directions above. In this lesson, the digital activity allows students to plot their points and test their thinking with a dynamic applet, however, the mathematics is truly the same.

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### **Access for Students with Disabilities**

*Engagement: Develop Effort and Persistence.* Encourage and support opportunities for peer interactions. Display sentence frames to support students when they explain their estimation with their partner. For example, “I predict \_\_\_\_ because . . .” and “How did you get . . .?”

*Supports accessibility for: Language; Social-emotional skills*

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## Access for English Language Learners

*Reading, Speaking, Listening: MLR6 Three Reads.* Use this routine to support reading comprehension without solving for students. In the first read, students read the problem and review the images with the goal of comprehending the situation (e.g., a photo of a stick's shadow was taken at different times; the length of the shadow was different at different times). In the second read, identify the important quantities by asking students what can be counted or measured (e.g., at 20 minutes, the shadow was 10.5 cm long; at 60 minutes, the shadow was 26 cm long). After the third read, ask students to brainstorm possible strategies to complete the task. This will help students connect the language in the problem and the reasoning needed to solve the problem, while keeping the intended level of cognitive demand in the task.

*Design Principle(s): Support sense-making*

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### Student Task Statement

When the Sun was directly overhead, the stick had no shadow. After 20 minutes, the shadow was 10.5 cm long. After 60 minutes, it was 26 cm long.



1. Based on this information, estimate how long it will be after 95 minutes.
2. After 95 minutes, the shadow measured 38.5 cm. How does this compare to your estimate?
3. Is the length of the shadow a function of time? If so, is it linear? Explain your reasoning.

### Student Response

1. Answers vary. Sample response: If we model this with the linear function that goes through  $(0, 0)$  and  $(20, 10.5)$ , we would predict that the length would be growing at a rate of  $\frac{10.5}{20}$  centimeters per minute. After 95 minutes, this would give a prediction of about 50 centimeters since  $95 \cdot \frac{10.5}{20} \approx 49.88$ .
2. Answers vary. Sample response: The prediction we made overestimated the length by about 11.5 centimeters.
3. The length of the shadow is a function of time, since every time determines and only one length. It is not a linear function of the time, since the points  $(0, 0)$ ,  $(20, 10.5)$ ,  $(60, 26)$  and  $(95, 38.5)$  do not lie on any one line. Based on the answer to the last part, it is not even very well approximated by a linear function.

## Activity Synthesis

Select groups that had different strategies for making their original prediction to share their reasoning about whether or not a linear model is a good fit for predicting the length of the shadow. In particular, make sure it is pointed out how much using the rate of change determined by the first two points over-predicts the length of the shadow after 95 minutes.

Tell students that if we only use two data points, it is always possible to model a situation with a linear function. We need additional data to help us determine if a linear model is appropriate. In this case, mathematicians have also used the geometry of Earth traveling around the Sun to provide a better model for the length of the shadow as a function of time that is not a linear function.

## 9.3 Recycling

**10 minutes**

The purpose of this activity is for students to approximate different parts of a graph with an appropriate line segment. This graph is from a previous activity, but students interact with it differently by sketching a linear function that models a certain part of the data. They take this model and consider its ability to predict input and output for other parts of the graph. This helps students think about subsets of data that might have different models from other parts of the data.

Identify students who draw in different lines for the first question to share during the Activity Synthesis.

### Addressing

- 8.F.B.4

### Instructional Routines

- MLR8: Discussion Supports

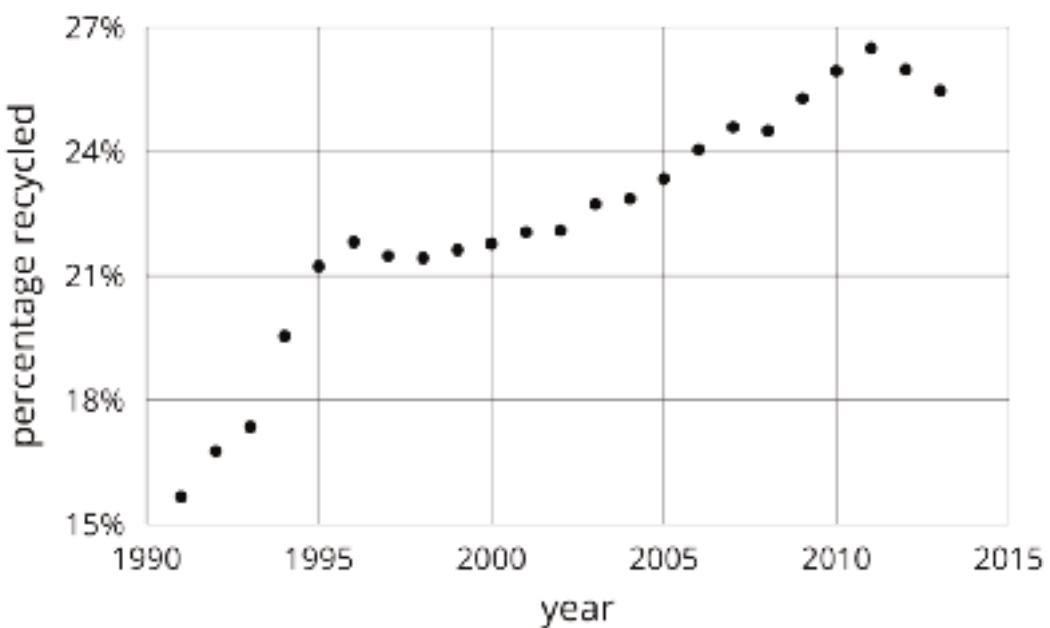
### Launch

Arrange students in groups of 2. Provide students with access to straightedges. Give students 3–5 minutes of quiet work time and then time to share their responses with their partner. Follow with a whole-class discussion.

#### Student Task Statement

In an earlier lesson, we saw this graph that shows the percentage of all garbage in the U.S. that was recycled between 1991 and 2013.

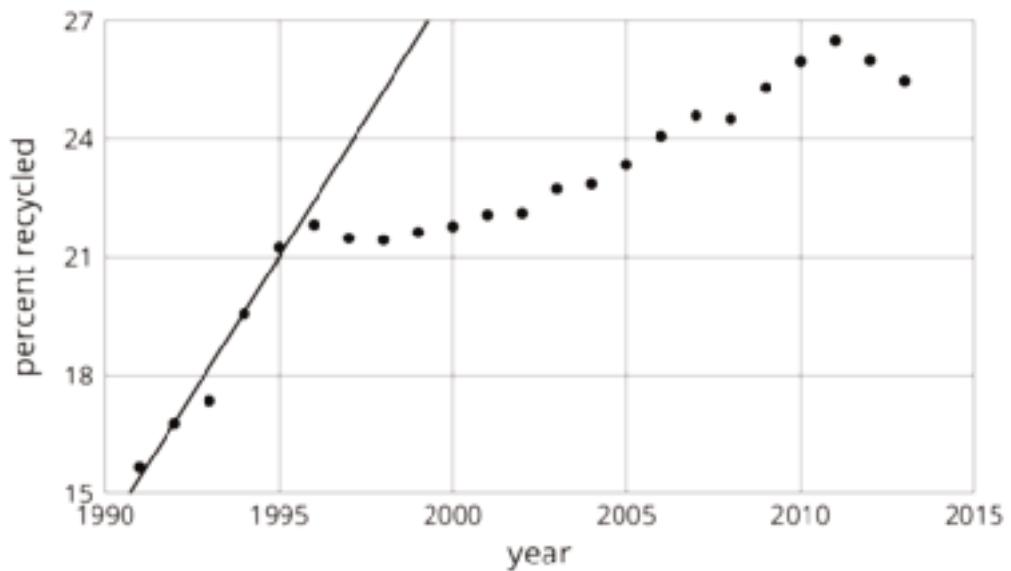




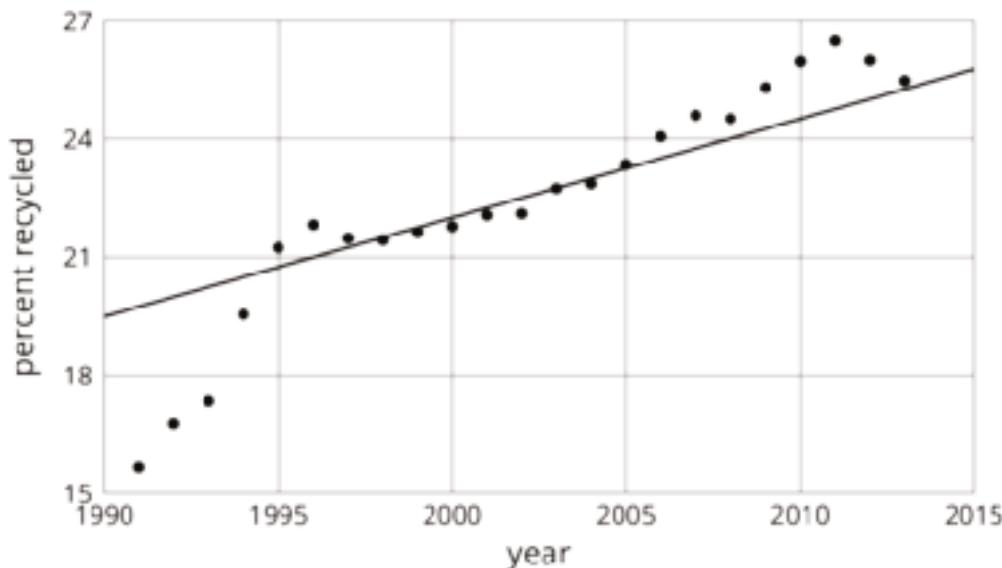
1. Sketch a linear function that models the change in the percentage of garbage that was recycled between 1991 and 1995. For which years is the model good at predicting the percentage of garbage that is produced? For which years is it not as good?
2. Pick another time period to model with a sketch of a linear function. For which years is the model good at making predictions? For which years is it not very good?

### Student Response

1. Answers vary. The one displayed is a reasonable approximation of the data between 1991 and 1996, and a bad approximation from then on.



2. Answers vary. The one displayed is a reasonable approximation of the data between 1994 and 2013, but a bad approximation from 1991 to 1993.



### Activity Synthesis

The purpose of this discussion is for students to understand that although you might find a good model for one part of a graph, that does not mean that model will work for other parts.

Select students previously identified to share their models. Display these for all to see throughout the entire discussion. Questions for discussion:

- “How much does your model for 1991 to 1995 overestimate 1996? 1997?”
- “If we drew in a single line to model 1997 to 2013, what would that line predict well? What would that line predict poorly?” (A single line modeling those years would reasonably predict the percent recycled from 1997 to 2010, but it wouldn’t be able to show how the percent recycled from 2011 to 2013 is decreasing.)

Conclude the discussion by telling students that there is a trade-off in number of years to include in the interval and accuracy. We could “connect the dots” and be accurate about everything, but then our model has limited use and is complicated with so many parts. (Just imaging writing an equation for each piece!)

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support whole-class discussion. For each model that is shared, ask students to summarize what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to speak, and better understand each model.

*Design Principle(s): Support sense-making*

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## Lesson Synthesis

Tell students that a mathematical *model* is a mathematical object like an equation, a function, or a geometric figure that we use to represent a real-life situation. Sometimes a situation can be modeled by a linear function. We have to use judgment about whether this is a reasonable thing to do based on the information we are given. We must also be aware that the model makes imprecise predictions, or may only be appropriate for certain ranges of values.

Give students 1–2 minutes to think of a situation that may seem linear but actually is not. Invite them to share their situations. For example, the height of humans may look linear for short periods of time, but eventually growth stops, so we wouldn't want to use a linear model for height over a large period of time.

## 9.4 Board Game Sales

Cool Down: 5 minutes

### Addressing

- 8.F.B.4

#### Student Task Statement

A small company is selling a new board game, and they need to know how many to produce in the future.

After 12 months, they sold 4 thousand games; after 18 months, they sold 7 thousand games; and after 36 months, they sold 15 thousand games.

1. Could this information be reasonably estimated using a single linear model?
2. If so, use the model to estimate the number of games sold after 48 months. If not, explain your reasoning.

## Student Response

1. Yes.
2. Answers vary. After 48 months there should be between 16 and 22 thousand sales depending on the data points used for the model.

## Student Lesson Summary

Water has different boiling points at different elevations. At 0 m above sea level, the boiling point is 100° C. At 2,500 m above sea level, the boiling point is 91.3° C. If we assume the boiling point of water is a linear function of elevation, we can use these two data points to calculate the slope of the line:

$$m = \frac{91.3 - 100}{2,500 - 0} = \frac{-8.7}{2,500}$$

This slope means that for each increase of 2,500 m, the boiling point of water decreases by 8.7° C. Next, we already know the  $y$ -intercept is 100° C from the first point, so a linear equation representing the data is

$$y = \frac{-8.7}{2,500}x + 100$$

This equation is an example of a mathematical *model*. A mathematical model is a mathematical object like an equation, a function, or a geometric figure that we use to represent a real-life situation. Sometimes a situation can be modeled by a linear function. We have to use judgment about whether this is a reasonable thing to do based on the information we are given. We must also be aware that the model may make imprecise predictions, or may only be appropriate for certain ranges of values.

Testing our model for the boiling point of water, it accurately predicts that at an elevation of 1,000 m above sea level (when  $x = 1,000$ ), water will boil at 96.5° C since  $y = \frac{-8.7}{2,500} \cdot 1000 + 100 = 96.5$ . For higher elevations, the model is not as accurate, but it is still close. At 5,000 m above sea level, it predicts 82.6° C, which is 0.6° C off the actual value of 83.2° C. At 9,000 m above sea level, it predicts 68.7° C, which is about 3° C less than the actual value of 71.5° C. The model continues to be less accurate at even higher elevations since the relationship between the boiling point of water and elevation isn't linear, but for the elevations in which most people live, it's pretty good.

# Lesson 9 Practice Problems

## Problem 1

### Statement



On the first day after the new moon, 2% of the Moon's surface is illuminated. On the second day, 6% is illuminated.

- Based on this information, predict the day on which the Moon's surface is 50% illuminated and 100% illuminated.
- The Moon's surface is 100% illuminated on day 14. Does this agree with the prediction you made?
- Is the percentage illumination of the Moon's surface a linear function of the day?

### Solution

- Answers vary. Sample response: A simple approach is to attempt a linear model starting at Day 1. If the illumination is increased by 4% every day, then after 11 more days (after Day 2) it reaches 50%. In 13 more days, illumination reaches 100%. This gives a prediction of Day 13 for 50% and Day 26 for 100%.
- No
- No (The linear model did a very bad job of approximating the data.)

## Problem 2

### Statement

In science class, Jada uses a graduated cylinder with water in it to measure the volume of some marbles. After dropping in 4 marbles so they are all under water, the water in the cylinder is at a height of 10 milliliters. After dropping in 6 marbles so they are all under water, the water in the cylinder is at a height of 11 milliliters.

- a. What is the volume of 1 marble?
- b. How much water was in the cylinder before any marbles were dropped in?
- c. What should be the height of the water after 13 marbles are dropped in?
- d. Is the relationship between volume of water and number of marbles a linear relationship? If so, what does the slope of a line representing this relationship mean? If not, explain your reasoning.

## Solution

- a. 0.5 ml
- b. 8 ml
- c. 14.5 ml
- d. Yes. The slope of the line represents the volume of 1 marble.

## Problem 3

### Statement

Solve each of these equations. Explain or show your reasoning.

$$2(3x + 2) = 2x + 28$$

$$5y + 13 = -43 - 3y$$

$$4(2a + 2) = 8(2 - 3a)$$

## Solution

- a.  $x = 6$ . Responses vary. Sample response: Distribute 2 on the left side, add  $-4$  to each side, add  $-2x$  to each side, then divide each side by 4.
- b.  $y = -7$ . Responses vary. Sample response: Add  $3y$  to each side, subtract 13 from each side, then divide each side by 8.
- c.  $a = \frac{1}{4}$ . Responses vary. Sample response: Divide each side by 4, distribute 2 on the right side, subtract 2 from each side, add  $6a$  to each side, then divide each side by 8.

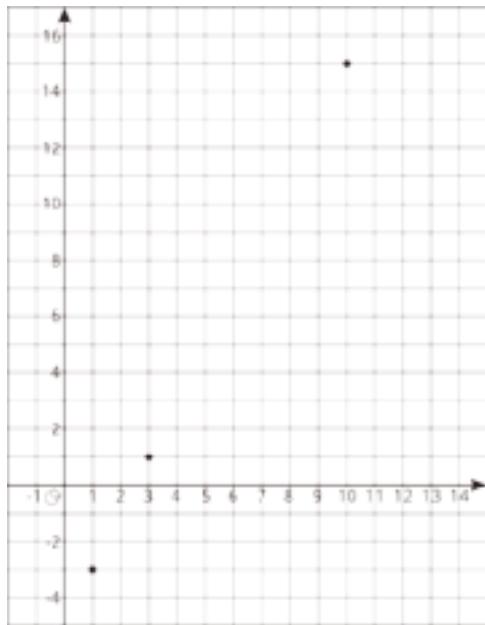
(From Unit 4, Lesson 5.)

## Problem 4

### Statement

For a certain city, the high temperatures (in degrees Celsius) are plotted against the number of days after the new year.

Based on this information, is the high temperature in this city a linear function of the number of days after the new year?



### Solution

Answers vary. Sample response: Although this data does fit a linear model, it does not make sense to use a linear model for this situation. For example, after only 2 months, the high temperature would be more than the boiling point of water, which is unlikely.

## Problem 5

### Statement

The school designed their vegetable garden to have a perimeter of 32 feet with the length measuring two feet more than twice the width.

- Using  $\ell$  to represent the length of the garden and  $w$  to represent its width, write and solve a system of equations that describes this situation.
- What are the dimensions of the garden?

### Solution

a.  $2\ell + 2w = 32$ ,  $\ell = 2w + 2$

b.  $\ell = 11\frac{1}{3}$ ,  $w = 4\frac{2}{3}$

(From Unit 4, Lesson 15.)

# Lesson 10: Piecewise Linear Functions

## Goals

- Calculate the different rates of change of a piecewise linear function using a graph, and interpret (orally and in writing) the rates of change in context.
- Create a model of a non-linear function using a piecewise linear function, and describe (orally) the benefits of having more or less segments in the model.

## Learning Targets

- I can create graphs of non-linear functions with pieces of linear functions.

## Lesson Narrative

This lesson picks up on the idea planted in the previous lesson about creating linear models for data. Specifically, in some situations where a quantity changes at different constant rates over different time intervals, we can model the situation with a piecewise linear function (MP4). Students look at temperature data, which changes at different, almost-constant rates during different parts of the day. They also use piecewise linear graphs to find information about the real-life situation they represent. The focus of this lesson is not necessarily to find equations for the piecewise linear functions (though students may choose to do so in some instances), but rather to study the graphs qualitatively and to compute and compare the different rates of change.

## Alignments

### Addressing

- 8.F.B: Use functions to model relationships between quantities.
- 8.F.B.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
- 8.F.B.5: Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR5: Co-Craft Questions
- MLR8: Discussion Supports

- Notice and Wonder

### Student Learning Goals

Let's explore functions built out of linear pieces.

## 10.1 Notice and Wonder: Lines on Dots

### Warm Up: 5 minutes

This warm-up connects to the previous lesson and is meant to elicit ideas that will be helpful for students in the next activity in this lesson. Students should notice that the points in the graph are not connected and wonder how well the lines model the sections of data they span, which is explored further in the next activity. For that reason, the discussion should focus on collecting all the things students notice and wonder but not explaining the things they wonder.

### Addressing

- 8.F.B

### Instructional Routines

- Notice and Wonder

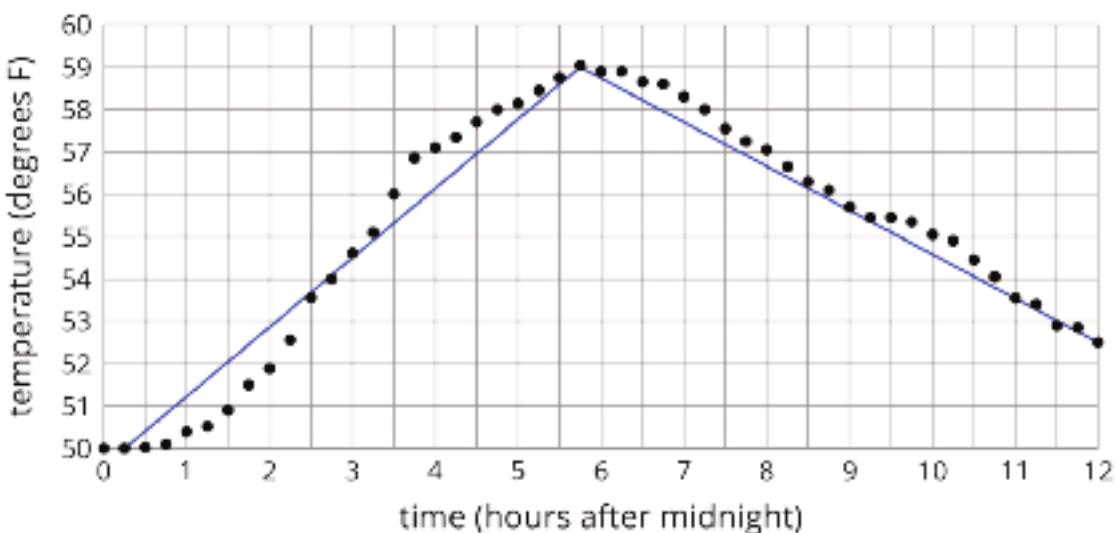
### Launch

Tell students they will look at a graph and their job is to think of at least one thing they notice and at least one thing they wonder about the graph. Display the graph for all to see and give 1 minute of quiet think time.

Ask students to give a signal when they have noticed or wondered about something.

### Student Task Statement

What do you notice? What do you wonder?



## Student Response

Things students may notice:

- Not many of the points are on the blue line.
- The temperature gets warmer and then cooler as time goes on.
- At about 5:45 it is the warmest at about 59 degrees.
- The second blue line connects the highest point and the point furthest to the right.

Things students may wonder:

- What location does this data represent?
- Why is it warmer at 6:00 am then it is at noon?
- Why aren't the points connected?
- Why is the second line lower than almost all the points?

## Activity Synthesis

Ask students to share their ideas. Record and display the responses for all to see. If no one mentions that the dots are not connected or what they think the blue lines mean, bring these ideas to their attention and tell them they will be working more with these ideas in the next activity.

# 10.2 Modeling Recycling

## 10 minutes

In this activity, students work with a graph that clearly cannot be modeled by a single linear function, but pieces of the graph could be reasonably modeled using different linear functions, leading to the introduction of piecewise linear functions (MP4). Students find the slopes of their piecewise linear model and interpret them in the context.

Monitor for students who:

- Choose different numbers of line segments to represent the function (e.g., 3, 4, and 5 segments).
- Choose different endpoints for the segments (e.g., two students have chosen 4 segments but different placement of those segments).

## Addressing

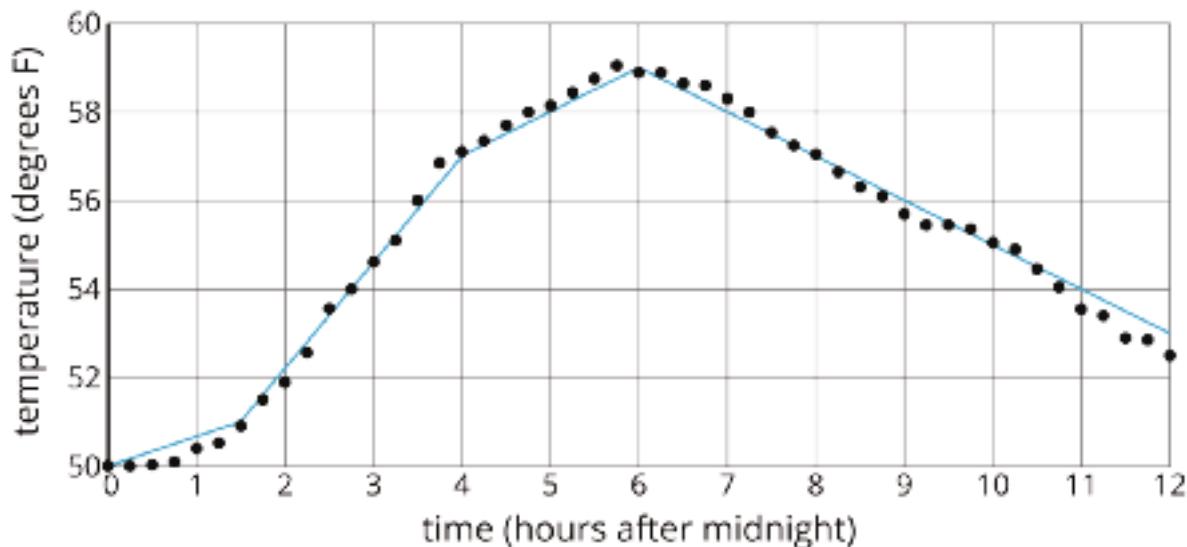
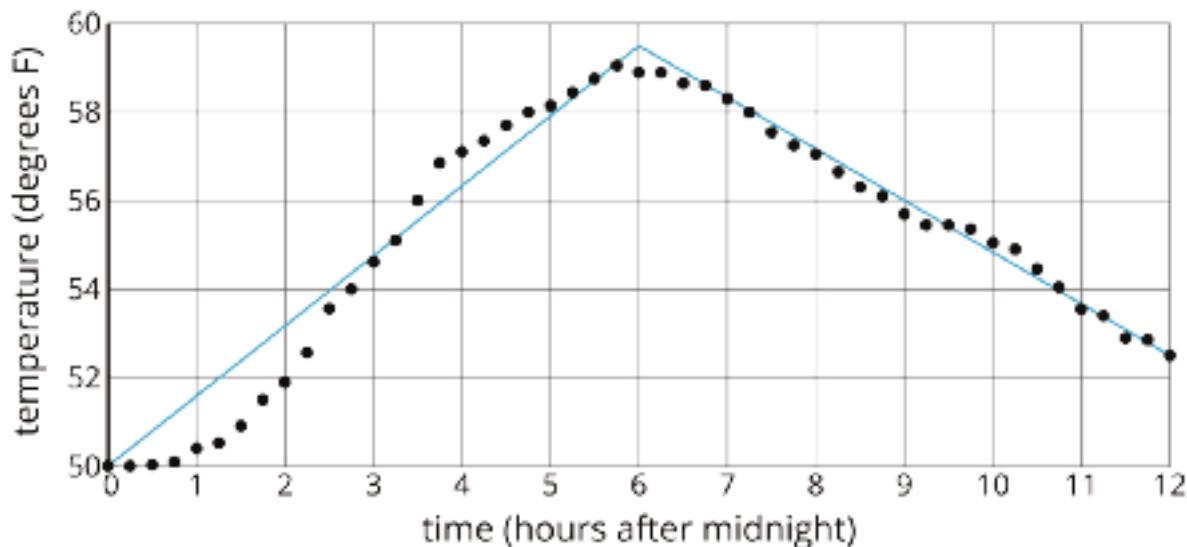
- 8.F.B.4

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

## Launch

Arrange students in groups of 2. Display images for all to see. Tell students that sometimes we model functions with multiple line segments in different places. These models are called piecewise linear functions. For example, here are two different piecewise linear models of the same temperature data (note that the first image is not the same as the image in the warm-up):



Give students 3–5 minutes of quiet work time and then time to share their responses with their partners. Follow with a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Develop Language and Symbols.* Add to the display created in an earlier lesson of important terms and vocabulary. Add the following term and continue to maintain the display for reference throughout the remainder of the unit: piecewise linear functions.

*Supports accessibility for: Memory; Language*

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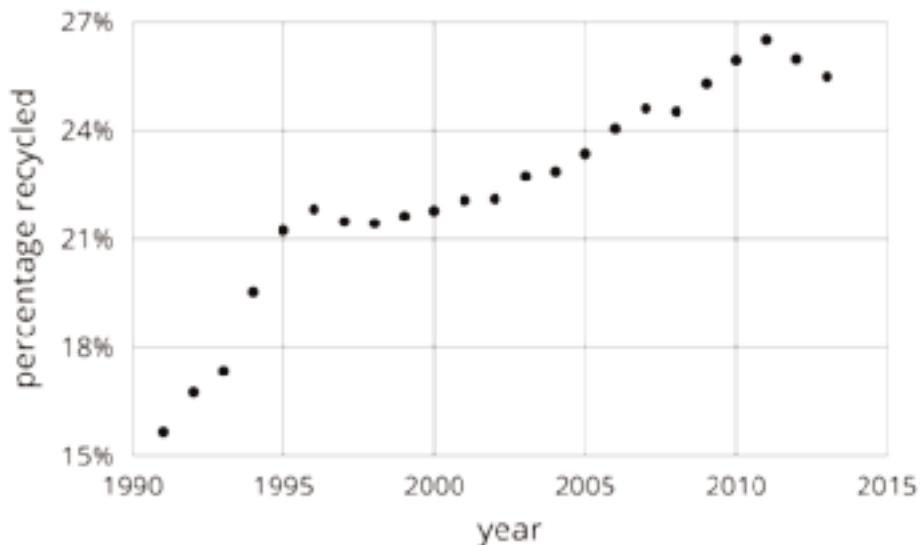
### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Amplify mathematical language students use to communicate about piecewise linear functions. As students share what they notice between the two graphs, revoice their statements using the term “piecewise linear functions” while gesturing. Invite students to use the term “piecewise linear function” when describing what they noticed. Some students may benefit from chorally repeating the phrases that include the word “piecewise linear function” in context.

*Design Principle(s): Support sense-making; Optimize output (for explanation)*

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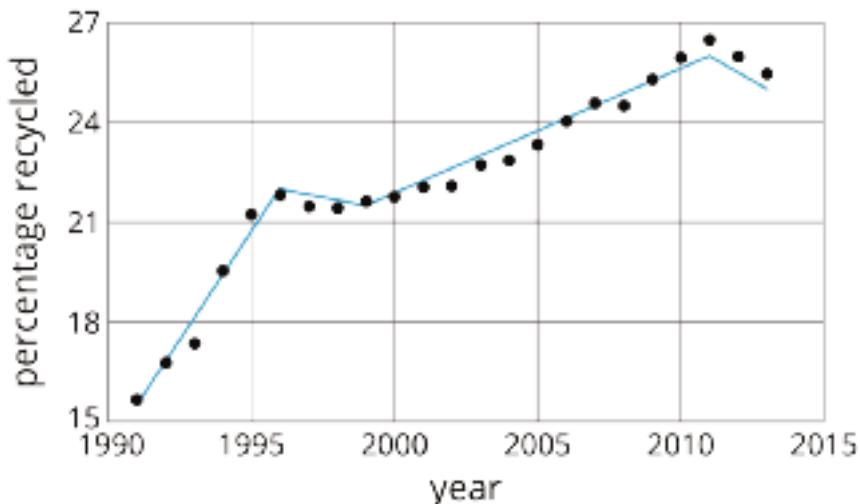
### Student Task Statement



1. Approximate the percentage recycled each year with a piecewise linear function by drawing between three and five line segments to approximate the graph.
2. Find the slope for each piece. What do these slopes tell you?

### Student Response

1. Answers vary. Here is a piecewise linear model of the graph using four line segments.



2. Answers vary. The endpoints of the four segments given in the answer to the previous part are: (1991, 16), (1996, 22), (1999, 21.5), (2011, 26), and (2013, 25), so we can find that the four line segments have respective slopes  $\frac{6}{5}$ ,  $-\frac{1}{6}$ ,  $\frac{3}{8}$ , and  $-\frac{1}{2}$ . These slopes describe the approximate rate at which the percent recycled increased or decreased over those times. For example, from 2011 to 2013, the percent recycled decreased by approximately 0.5 percent per year.

### Activity Synthesis

Select previously identified students to share their responses and display each student's graph for all to see. Sequence the student responses in order from fewest to greatest number of segments.

To highlight the use and interpretation of piecewise linear models, ask:

- “What do the slopes of the different lines mean?” (The slopes of the lines tell us the rate of change of the different linear pieces for the specific intervals of time.)
- “Can we use this information to predict information for recycling in the future?” (If the data continues to decrease as it does from 2011 to 2013, yes. If the data starts to increase again, our model may not make very good predictions.)
- “What are the benefits of having fewer segments in the piecewise linear function? What are the benefits of having more segments?” (Fewer segments are easier to write equations for and help show long-term trends. More segments give a more accurate model of the data.)

## 10.3 Dog Bath

15 minutes

The purpose of this activity is to give students more exposure to working with a situation that can be modeled with a piecewise linear function. Here, the situation has already been modeled and students must calculate the rate of change for the different pieces of the model and interpret it in

the context. A main discussion point should be around what the different rates of change mean in the situation and connecting features of the graph to the events in the context.

### **Addressing**

- 8.F.B.4
- 8.F.B.5

### **Instructional Routines**

- MLR5: Co-Craft Questions

### **Launch**

Students in groups of 2. Give 3–5 minutes of quiet work time and then ask them to share their responses with a partner and reach an agreement. Follow with a whole-class discussion.

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### **Access for English Language Learners**

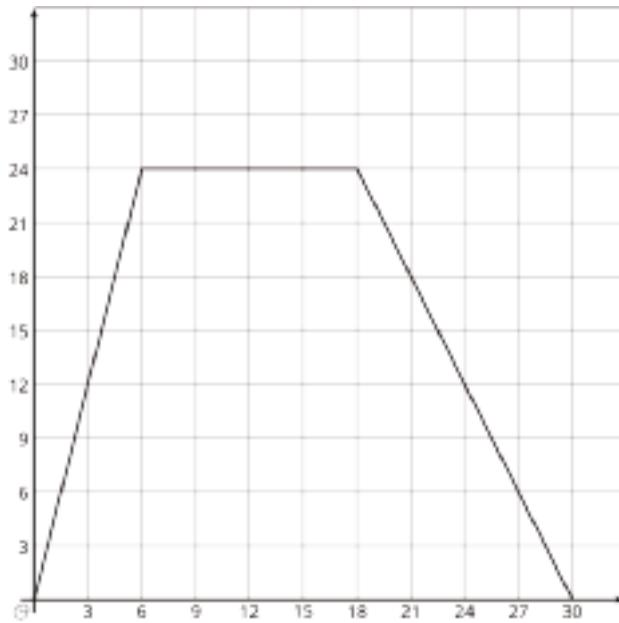
*Conversing, Writing: MLR5 Co-Craft Questions.* Display the situation and the graph without revealing the questions that follow. Invite students to write mathematical questions that could be answered by the graph. Invite students to share their questions with a partner before selecting 2–3 to share with the whole class. Highlight questions that ask students to make sense of the rate of change in context. Next, reveal the questions of the activity. This routine allows students to produce the language of mathematical questions and talk about the relationship between quantities and rates of change that are represented graphically.

*Design Principle(s): Maximize meta-awareness; Support sense-making*

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### **Student Task Statement**

Elena filled up the tub and gave her dog a bath. Then she let the water out of the tub.



1. The graph shows the amount of water in the tub, in gallons, as a function of time, in minutes. Add labels to the graph to show this.
2. When did she turn off the water faucet?
3. How much water was in the tub when she bathed her dog?
4. How long did it take for the tub to drain completely?
5. At what rate did the faucet fill the tub?
6. At what rate did the water drain from the tub?

### Student Response

1. Horizontal axis: "time (minutes)" (or equivalent). Vertical axis: "volume of water (gallons)" (or equivalent).
2. After 6 minutes. The volume of water is increasing until  $t = 6$ , so this must be when Elena turned off the faucet.
3. 24 gallons. She bathed her dog after she turned the water off, so we look to the graph to see the volume of water at  $t = 6$ .
4. 12 minutes. The volume of water starts decreasing at  $t = 18$  minutes, and takes until  $t = 30$  to completely drain.
5. 4 gallons per minute. It took 6 minutes to fill 24 gallons, giving a rate of 4 gallons per minute ( $\frac{24}{6}$ ).
6. 2 gallons per minute. It took 12 minutes to drain 24 gallons, giving a rate of 2 gallons per minute ( $\frac{24}{12}$ ).

## Activity Synthesis

The purpose of this discussion is for students to make sense of what the rate of change and other features of the model mean in the context of this situation. Consider asking the following questions:

- “Did the tub fill faster or drain faster? How can you tell?” (The tub filled faster. The slope of the line representing the interval the water was filling the tub is steeper than the slope of the line representing the interval the water was draining from the tub.)
- “If you were going to write a linear equation for the first piece of the graph, what would you use for  $m$ ? For  $b$ ?” (I would use  $m = 4$  and  $b = 0$ , because the tub filled at a rate of 4 gallons per minute and the initial amount of water was 0.)
- “Which part of the graph represents the 2 gallons per minute you calculated?” (The last part of the piecewise function has a slope of -2, which is when the tub was draining at 2 gallons per minute.)

## 10.4 Distance and Speed

**Optional: 5 minutes**

This activity is similar to the previous activity in that students are interpreting a graph and making sense of what situation the graph is representing. The difference here is that the specificity with numbers has been removed, so students need to think a bit more abstractly about what the changes in the graph represent and how they connect to the situation.

Monitor for students describing the graph with different levels of detail, particularly for any students who state that the car got up to speed faster than the car slowed down to 0.

### Addressing

- 8.F.B.4
- 8.F.B.5

### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time

### Launch

Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

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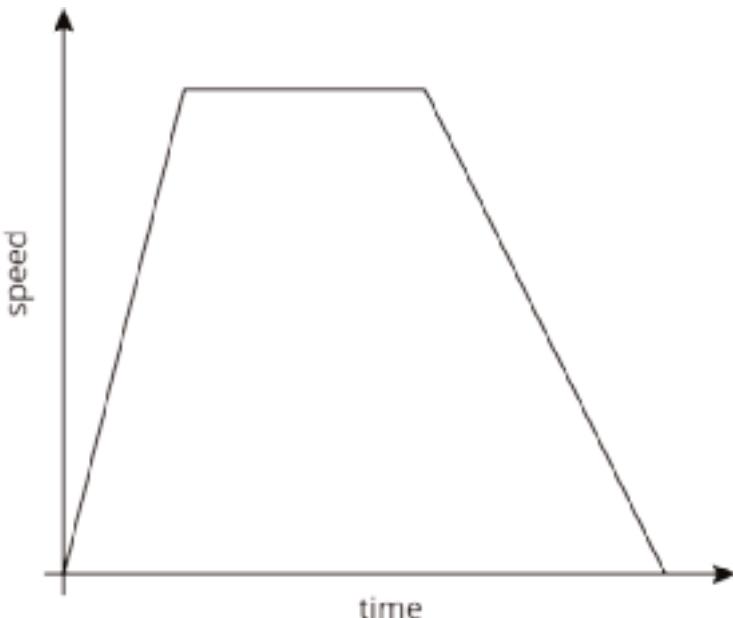
## Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.

*Supports accessibility for: Attention; Social-emotional skills*

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### Student Task Statement



The graph shows the speed of a car as a function of time. Describe what a person watching the car would see.

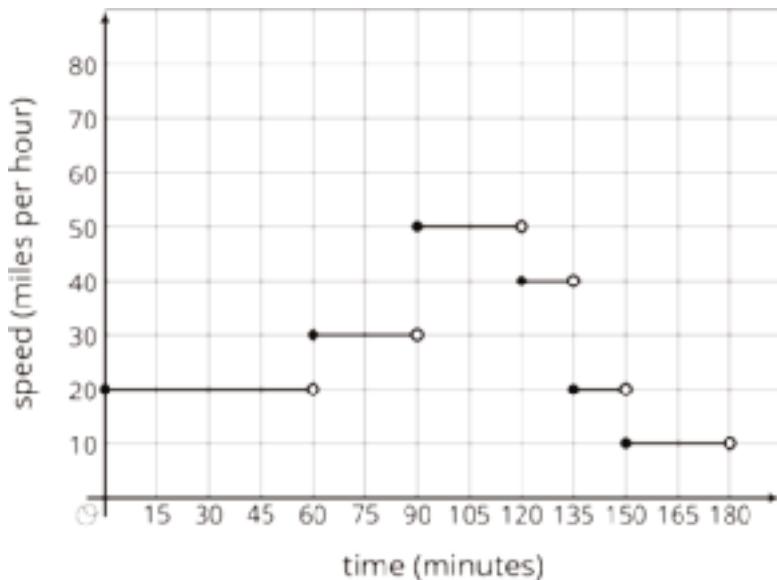
### Student Response

The car begins at rest and then quickly picks up speed. After some time, it reaches its maximum speed, stays at that speed for a while, and then gradually slows back down until it comes to a stop.

### Are You Ready for More?

The graph models the speed of a car over a function of time during a 3-hour trip. How far did the car go over the course of the trip?

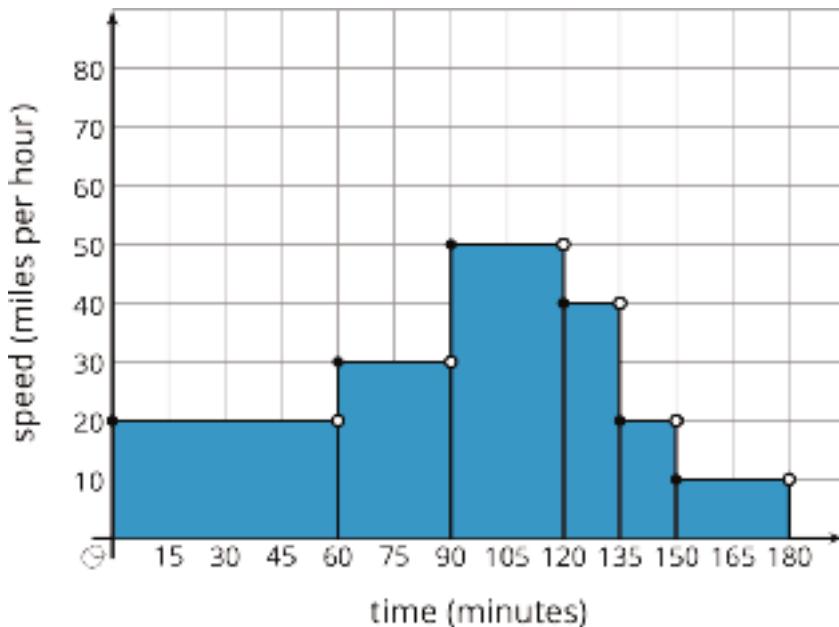
There is a nice way to visualize this quantity in terms of the graph. Can you find it?



### Student Response

We can divide the trip into segments over which the speed is constant, and use “distance =  $speed \times time$ .” Over the first hour, the car traveled 20 miles per hour, for a total of 20 miles. Then for half an hour, the car traveled 30 miles per hour, for a total of 15 miles. Similarly, the remaining four segments correspond to distances traveled of 25 miles, 10 miles, 5 miles, and 5 miles. Summing these gives a total of 80 miles.

The quantity “ $speed \times time$ ” is represented graphically by a length measured along the  $x$ -axis times a height measured along the  $y$ -axis, giving the area of a rectangle. The distance traveled over each segment is the area of the rectangle under that segment, and the total distance is the total area of the shaded region.



## Activity Synthesis

The purpose of this activity is for students to connect what is happening in a graph to a situation. Display the graph for all to see. Select previously identified students to share the situation they came up with. Sequence students from least descriptive to most descriptive. Have students point out the parts on the graph as they share their story about the situation.

Consider asking the following questions:

- “Did the car speed up faster or slow down faster? How do you know?” (The car sped up faster because the first part of the model is steeper than the third part of the model.)
- “How did you know that the car stayed that speed for a period of time?” (The graph stays at the same height for a while, so the speed was not changing during that time.)

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## Access for English Language Learners

*Reading, Writing, Conversing: MLR1 Stronger and Clearer Each Time.* Use this routine to provide students with a structured opportunity to revise and refine the explanation of their thinking about the shape and patterns in the graph. After students have written a response to the task statement, ask students to meet with 2–3 partners in a row for feedback. Each time, partners should explain their thinking without reading from their written work. Provide students with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “How do you know that from the graph?”, “What in the graph makes you think that?”, “Can you give an example?”, etc.). Students can borrow ideas and language from each partner to strengthen their final version.

*Design Principle(s): Optimize output (for explanation)*

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## Lesson Synthesis

Ask students, “How would you describe a piecewise linear function to someone who has never seen one?” and give 1 minute of quiet think time and then time to share their response with a partner. Invite partners to share their responses with the class while recording them for all to see. (A piecewise linear function is a function whose graph is pieced together out of line segments. For different ranges of input, the output is changing at different approximately constant rates so a different line is used for each range.) If students don’t include the different constant rates over different intervals of the independent variable, make sure that is made clear.

If time allows, ask students, “Can you think of another situation that changes at different constant rates over time?” and give partners 1 minute of think time before selecting groups to share their situations.

## 10.5 Lin’s Phone Charge

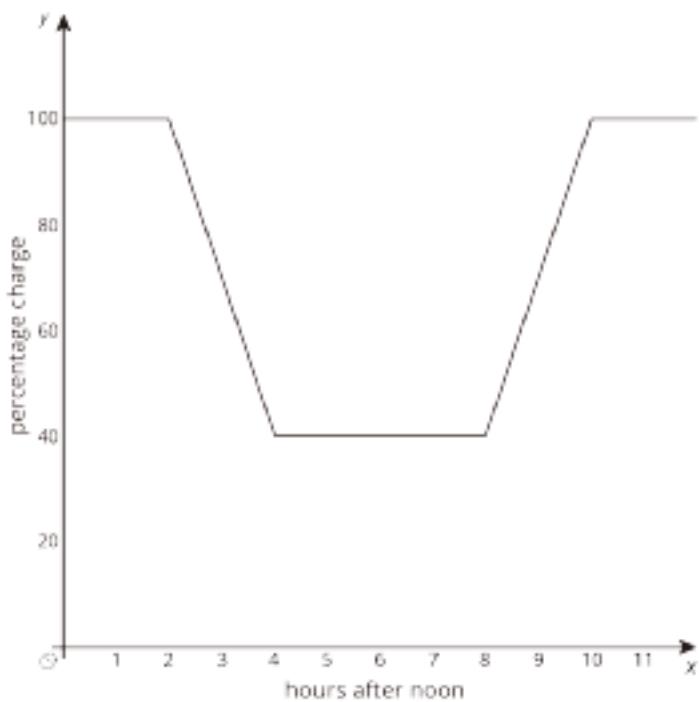
**Cool Down: 5 minutes**

## Addressing

- 8.F.B.5

### Student Task Statement

Lin uses an app to graph the charge on her phone.



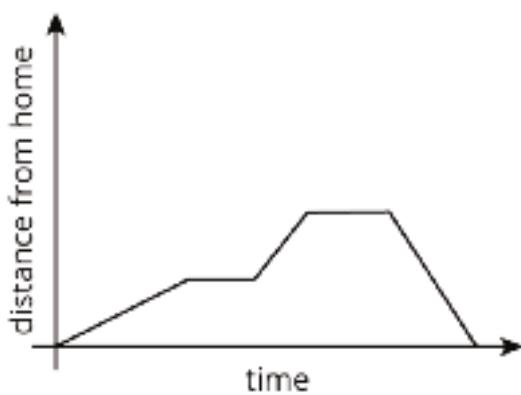
- When did she start using her phone?
- When did she start charging her phone?
- While she was using her phone, at what rate was Lin's phone battery dying?

### Student Response

- Lin started using her phone 2 hours after noon or at 2:00 p.m., since that is where the negative slope begins.
- Lin started charging her phone 8 hours after noon or at 8:00 p.m., since that is where the positive slope begins.
- The battery was dying at 30% per hour since it decreased 60% over 2 hours.

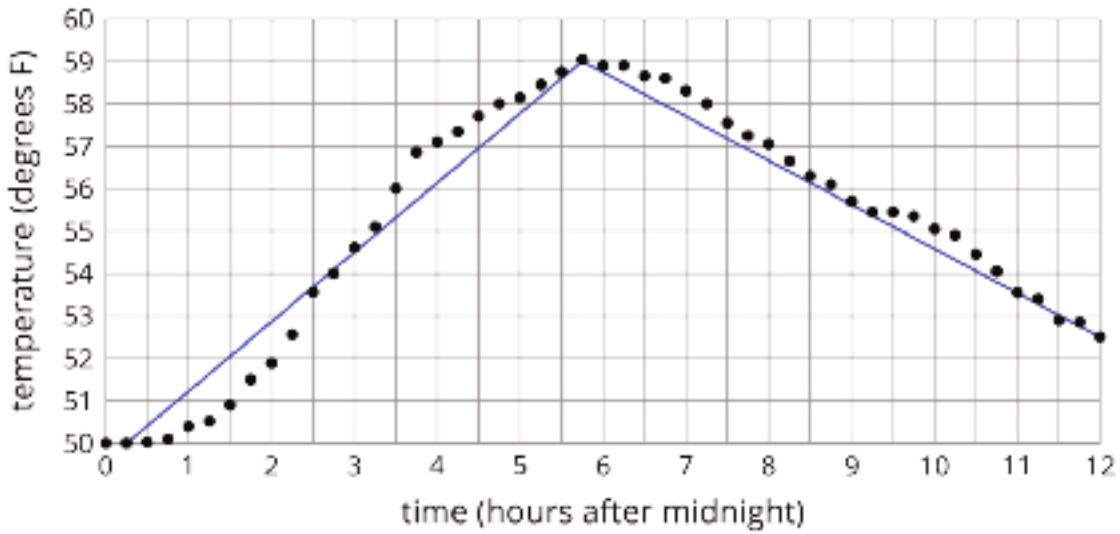
### Student Lesson Summary

This graph shows Andre biking to his friend's house where he hangs out for a while. Then they bike together to the store to buy some groceries before racing back to Andre's house for a movie night. Each line segment in the graph represents a different part of Andre's travels.



This is an example of a piecewise linear function, which is a function whose graph is pieced together out of line segments. It can be used to model situations in which a quantity changes at a constant rate for a while, then switches to a different constant rate.

We can use piecewise functions to represent stories, or we can use them to model actual data. In the second example, temperature recordings at several times throughout a day are modeled with a piecewise function made up of two line segments. Which line segment do you think does the best job of modeling the data?

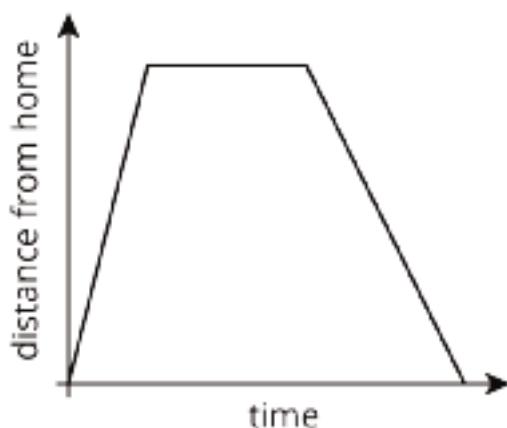


# Lesson 10 Practice Problems

## Problem 1

### Statement

The graph shows the distance of a car from home as a function of time.



Describe what a person watching the car may be seeing.

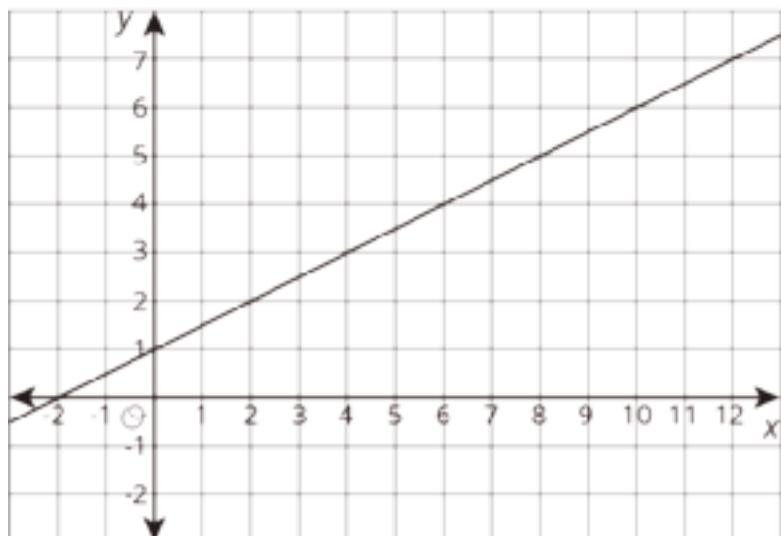
### Solution

Answers vary. Sample response: The car is driven away from home, then waits. The car is then driven back home at a slower speed than it was when driven away from home.

## Problem 2

### Statement

The equation and the graph represent two functions. Use the equation  $y = 4$  and the graph to answer the questions.



a. When  $x$  is 4, is the output of the equation or the graph greater?

b. What value for  $x$  produces the same output in both the graph and the equation?

## Solution

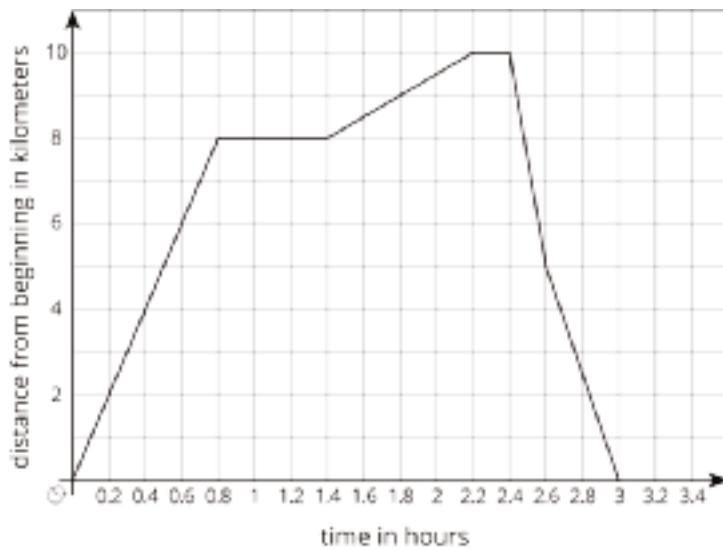
- a. Equation
- b. 6

(From Unit 5, Lesson 7.)

## Problem 3

### Statement

This graph shows a trip on a bike trail. The trail has markers every 0.5 km showing the distance from the beginning of the trail.



- a. When was the bike rider going the fastest?
- b. When was the bike rider going the slowest?
- c. During what times was the rider going away from the beginning of the trail?
- d. During what times was the rider going back towards the beginning of the trail?
- e. During what times did the rider stop?

## Solution

- a. Between 2.4 and 2.6 hours
- b. Between 1.4 and 2.2 hours, except the times the rider stopped
- c. Between 0 and 0.8 hours and between 1.4 and 2.2 hours because the biker was stopped between 0.8 and 1.4 hours

- d. Between 2.4 and 3 hours
- e. Between 0.8 and 1.4 hours and between 2.2 and 2.4 hours

## Problem 4

### Statement

The expression  $-25t + 1250$  represents the volume of liquid of a container after  $t$  seconds. The expression  $50t + 250$  represents the volume of liquid of another container after  $t$  seconds. What does the equation  $-25t + 1250 = 50t + 250$  mean in this situation?

### Solution

Responses vary. Sample response: The equation says that the volume in one container is equal to the volume in the other container. This equation can be solved for  $t$  to find the time at which both containers have the same volume.

(From Unit 4, Lesson 9.)

# Section: Cylinders and Cones

## Lesson 11: Filling Containers

### Goals

- Create a graph of a function from collected data, and interpret (in writing) a point on the graph.
- Draw a container for which the height of water as a function of volume would be represented as a piecewise linear function, and explain (orally) the reasoning.
- Interpret (orally and in writing) a graph of heights of certain cylinders as a function of volume, and compare the rates of change of the functions.

### Learning Targets

- I can collect data about a function and represent it as a graph.
- I can describe the graph of a function in words.

### Lesson Narrative

This lesson is the beginning of a sequence of lessons that interweaves the development of the function concept with the development of formulas for volumes of cylinders and cones. Because students have not yet learned these formulas, the context of filling a cylindrical container with water is useful for developing the abstract concept of function. It makes physical sense that the height of the water is a function of its volume even if we cannot write down an equation for the function. At the same time, considering how changing the diameter of the cylinder changes the graph of the function helps students develop a geometric understanding of how the volume is related to the height and the diameter. In later lessons they will learn a formula for that relation.

In this lesson, students fill a graduated cylinder with different amounts of water and draw the graph of the height as a function of the volume. They next consider how their data and graph would change if their cylinder had a different diameter. The following activity turns the situation around: when given a graph showing the height of water in a container as a function of the volume of water in the container, can students create a sketch of what the container must look like?

### Alignments

#### Addressing

- 8.F.B: Use functions to model relationships between quantities.
- 8.F.B.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two  $(x, y)$  values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

## **Building Towards**

- 8.G.C: Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

## **Instructional Routines**

- MLR1: Stronger and Clearer Each Time
- MLR8: Discussion Supports
- Which One Doesn't Belong?

## **Required Materials**

**Graduated cylinders**

## **Required Preparation**

Students work in groups of 3–4 for the activity Height and Volume. Each group needs 1 graduated cylinder and water.

### **Student Learning Goals**

Let's fill containers with water.

## **11.1 Which One Doesn't Belong: Solids**

### **Warm Up: 5 minutes**

The purpose of this warm-up is for students to compare different objects that may not be familiar and think about how they are similar and different from objects they have encountered in previous activities and grade levels. To allow all students to access the activity, each object has one obvious reason it does not belong. Encourage students to move past the obvious reasons (e.g., Figure A has a point on top) and find reasons based on geometrical properties (e.g., Figure D when looked at from every side is a rectangle). During the discussion, listen for important ideas and terminology that will be helpful in upcoming work of the unit.

## **Building Towards**

- 8.G.C

## **Instructional Routines**

- Which One Doesn't Belong?

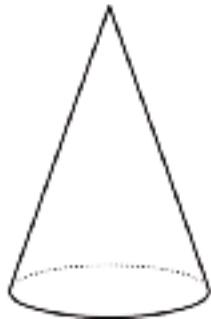
## **Launch**

Arrange students in groups of 2. Display the image for all to see. Ask students to indicate when they have noticed one object that does not belong and can explain why. Give students 2 minutes of quiet think time and then time to share their thinking with their partner.

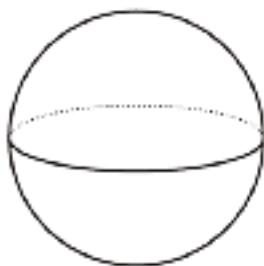
## Student Task Statement

These are drawings of three-dimensional objects. Which one doesn't belong? Explain your reasoning.

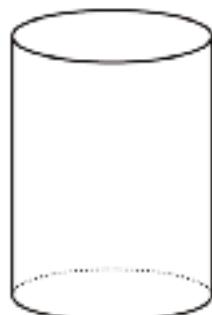
A



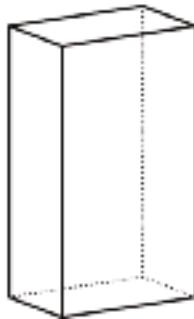
B



C



D



## Student Response

Answers vary. Sample responses:

Figure A doesn't belong because:

- It's the only object with exactly two surfaces.
- It's the only object with exactly one base.

Figure B doesn't belong because:

- It's the only object that has no edges or planar faces.
- It's the only object that isn't stable if you set it down (i.e., it would roll around).

Figure C doesn't belong because:

- It's the only object in which a side is a rectangle in two dimensions but curved in three dimensions.
- It's one object with exactly two bases.

Figure D doesn't belong because:

- It's the only object in which all the faces are flat planes.
- It's the only object with rectangular faces.

## Activity Synthesis

Ask students to share one reason why a particular object might not belong. Record and display the responses for all to see. After each response, ask the rest of the class if they agree or disagree.

Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given are correct. During the discussion, prompt students to explain the meaning of any terminology they use, such as diameter, radius, vertex, edge, face, or specific names of the figures: sphere, cylinder, cone, rectangular prism.

## 11.2 Height and Volume

**20 minutes (there is a digital version of this activity)**

In this activity, students investigate how the height of water in a graduated cylinder is a function of the volume of water in the graduated cylinder. Students make predictions about how the graph will look and then test their prediction by filling the graduated cylinder with different amounts of water, gathering and graphing the data (MP4).

### Addressing

- 8.F.B.4

### Building Towards

- 8.G.C

### Instructional Routines

- MLR8: Discussion Supports

### Launch

Arrange students in groups of 3–4. Be sure students know how to measure using a graduated cylinder. If needed, display a graduated cylinder filled to a specific measurement for all to see and demonstrate to students how to read the measurement. Give each group access to a graduated cylinder and water.

Give groups 8–10 minutes to work on the task, follow with a whole-class discussion.

For classrooms with access to the digital materials or those with no access to graduated cylinders, an applet is included here. Physical measurement tools and an active lab experience are preferred.

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Display sentence frames to support small-group discussion.

For example, "I think \_\_\_\_ , because \_\_\_\_ ." or "I (agree/disagree) because \_\_\_\_ ."

*Design Principle(s): Support sense-making; Optimize output for (explanation)*

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### Student Task Statement

Your teacher will give you a graduated cylinder, water, and some other supplies. Your group will use these supplies to investigate the height of water in the cylinder as a function of the water volume.

1. Before you get started, make a prediction about the shape of the graph.

2. Fill the cylinder with different amounts of water and record the data in the table.

volume (ml)					
height (cm)					

3. Create a graph that shows the height of the water in the cylinder as a function of the water volume.



4. Choose a point on the graph and explain its meaning in the context of the situation.

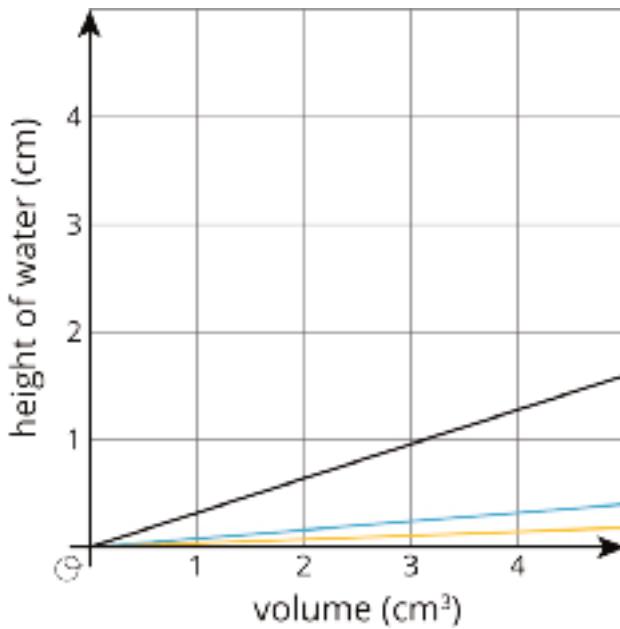
### Student Response

Answers vary according to the shape of the cylinder and the specific measurements taken. Sample response: For interpreting points, the point (150, 3) on the graph would signify that when 150 milliliters of water is poured into the cylinder, the height of the water would be 3 centimeters.

### Activity Synthesis

Select groups to share the graph for the third question and display it for all to see. Consider asking students the following questions:

- “What do you notice about the shape of your graph?”
- “What is the independent variable of your graph? Dependent variable?”
- “How does this graph differ from what you predicted the shape would be?”
- “For the last question, what point did you choose, and what does that point mean in the context of this activity?”
- “What would the endpoint of the graph be?” (There is a maximum possible volume for the cylinder. Once it’s filled, any extra water will spill out and not raise the water height.)



Ask students to predict how the graph would change if their cylinder had double the diameter. After a few responses, display this graph for all to see: Explain that each line represents the graph of a cylinder with a different radius. One cylinder has a radius of 1 cm, another has a radius of 2 cm, and another has a radius of 3 cm. Have students consider which line must represent which cylinder. Ask, “how did the slope of each graph change as the radius increased?” (As the radius is larger, the slope is less steep. This is because for a cylinder with a larger base, the same volume of water will not fill as high up the side of the cylinder.)

## 11.3 What Is the Shape?

10 minutes

In the previous activity, students were given a container and asked to draw the graph of the height as a function of the volume. In this activity, students are given the graph and asked to draw a sketch of the container that could have generated that height function. Since students have worked on the two previous activities, they have an idea of what the data for a graduated cylinder and a graduated cylinder with twice the diameter looks like and can use that information to compare to while working on this task.

### Addressing

- 8.F.B

### Building Towards

- 8.G.C

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

## Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share their drawings with their partner. Follow with a whole-class discussion.

If time is short, consider having half of the class work on the first question and the other half work on the second question and then complete the last question as part of the Activity Synthesis.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Begin by providing students with a range of different sized containers or students to test to determine if their volume and height could be represented by the given graphs.

*Supports accessibility for: Conceptual processing*

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### Access for English Language Learners

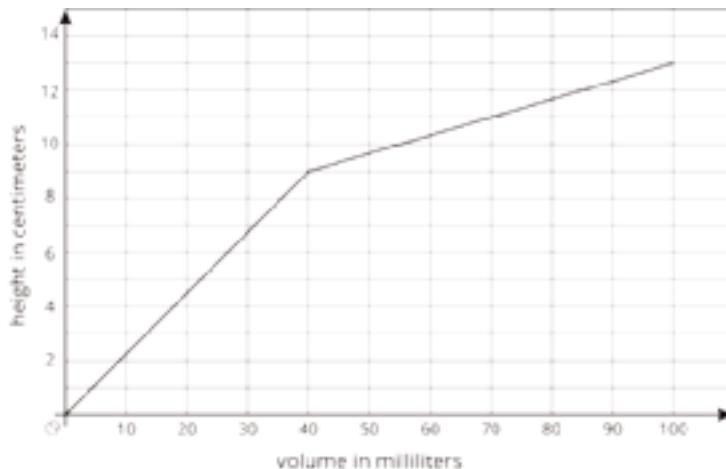
*Reading, Writing, Conversing: MLR1 Stronger and Clearer Each Time.* Use this routine to provide students with a structured opportunity to revise and refine their response to the last question. Ask students to meet with 2–3 partners for feedback. Provide students with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “Why do you think...?”, “What in the graph makes you think that?”, “Can you give an example?”, etc.). Students can borrow ideas and language from each partner to strengthen their final version.

*Design Principle(s): Optimize output (for explanation)*

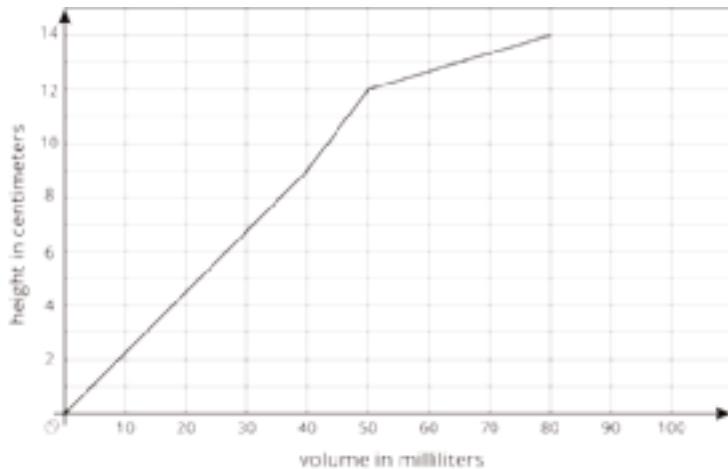
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### Student Task Statement

1. The graph shows the height vs. volume function of an unknown container. What shape could this container have? Explain how you know and draw a possible container.



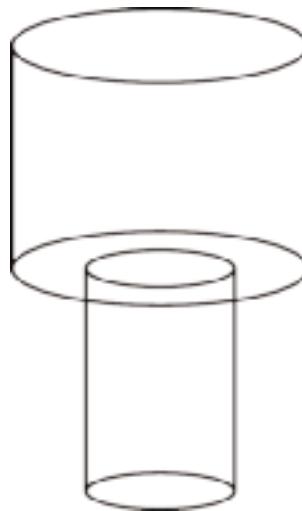
2. The graph shows the height vs. volume function of a different unknown container. What shape could this container have? Explain how you know and draw a possible container.



3. How are the two containers similar? How are they different?

### Student Response

1. Answers vary. Sample response: A shape in the form of two cylinders stacked on top of each other, with the upper cylinder having a greater radius. The height grows linearly with the volume in each cylinder, but as the water level rises into the second container, the height will begin to grow less quickly (since it takes more volume to achieve the same increase in height).

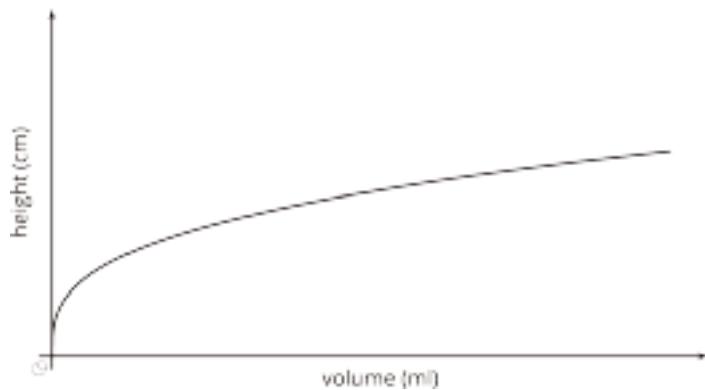


2. Answers vary. Sample response: 3 cylinders stacked on top of each other. The bottom cylinder should be the tallest. The middle cylinder should be shorter and have a smaller radius than the bottom. The top cylinder should be the shortest but have the largest radius.

3. Answers vary. Sample response: Both containers are made up of cylinders stacked on top of each other. The containers are different because the first container is made up of two parts, while the second is made up of three parts.

### Are You Ready for More?

The graph shows the height vs. volume function of an unknown container. What shape could this container have? Explain how you know and draw a possible container.



### Student Response

This graph in particular was made using the shape of a cone with its vertex at the bottom. Any shape that was very thin at the bottom and gradually got wider as you go up would be a reasonable answer.

### Activity Synthesis

Select students to share the different containers they drew. Display their drawings and the graph for all to see. Ask students to explain how they came up with their drawing and refer to parts in the graph that determined the shape of their container.

### Lesson Synthesis

Have students make their own graph showing the height and volume of a container. Tell students to use 2–5 lines for their container. Once the graphs are made, have students swap with a partner and try to draw the shape of their partner’s container. Ask a few groups to share their graphs and container drawings.

## 11.4 Which Cylinder?

Cool Down: 5 minutes

### Addressing

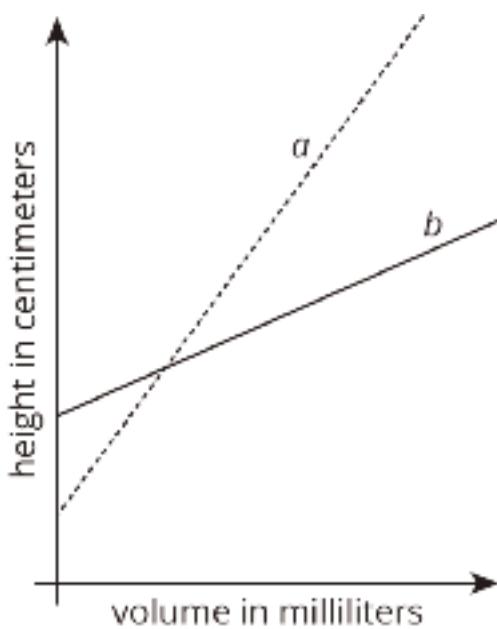
- 8.F.B

### Building Towards

- 8.G.C

### Student Task Statement

Two cylinders,  $a$  and  $b$ , each started with different amounts of water. The graph shows how the height of the water changed as the volume of water increased in each cylinder. Which cylinder has the larger radius? Explain how you know.

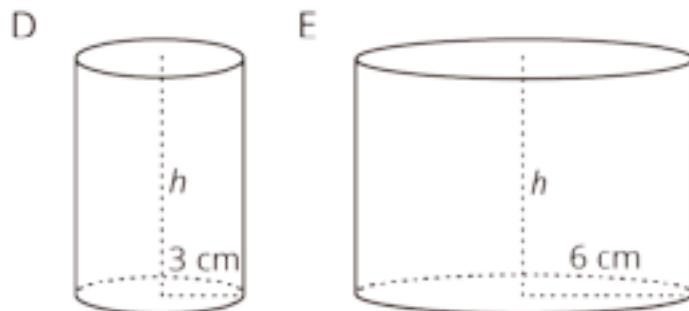


### Student Response

Cylinder  $b$ . Sample reasoning: a cylinder with a large radius would have a smaller change in height (slope) for the same volume of water added when compared to a cylinder with a smaller radius. Since the line for  $b$  has the smaller slope, it must be the cylinder with the larger radius.

### Student Lesson Summary

When filling a shape like a cylinder with water, we can see how the dimensions of the cylinder affect things like the changing height of the water. For example, let's say we have two cylinders,  $D$  and  $E$ , with the same height, but  $D$  has a radius of 3 cm and  $E$  has a radius of 6 cm.



If we pour water into both cylinders at the same rate, the height of water in  $D$  will increase faster than the height of water in  $E$  due to its smaller radius. This means that if we made graphs of the height of water as a function of the volume of water for each cylinder, we would have two lines and the slope of the line for cylinder  $D$  would be greater than the slope of the line for cylinder  $E$ .

## Glossary

- cylinder

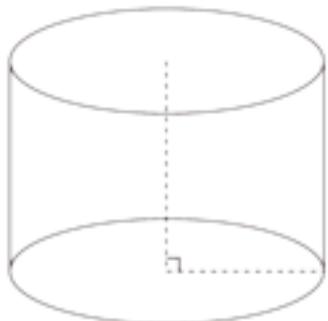
# Lesson 11 Practice Problems

## Problem 1

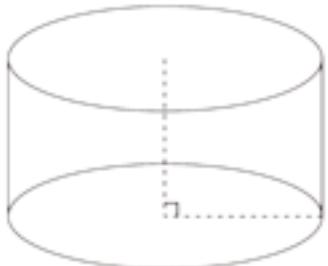
### Statement

Cylinder A, B, and C have the same radius but different heights. Put the cylinders in order of their volume from least to greatest.

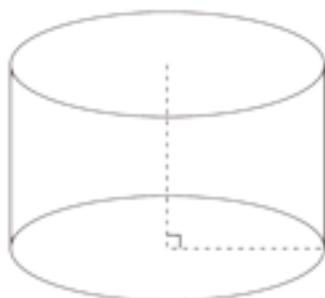
A



B



C



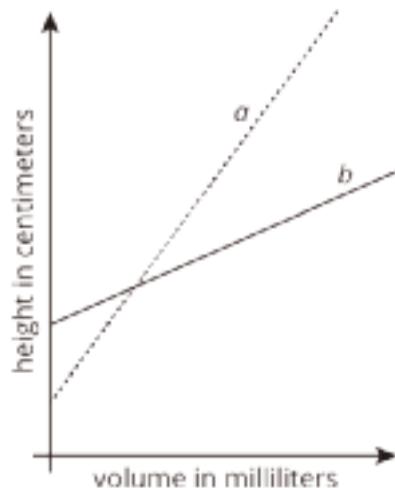
### Solution

Cylinder B, Cylinder C, Cylinder A

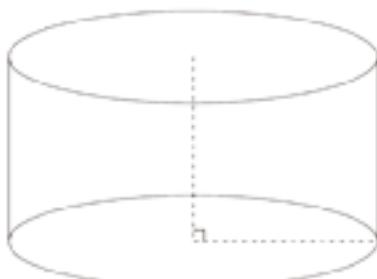
## Problem 2

### Statement

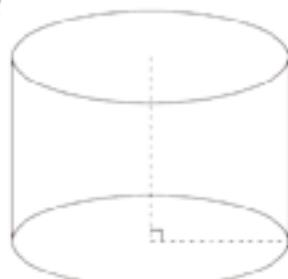
Two cylinders,  $a$  and  $b$ , each started with different amounts of water. The graph shows how the height of the water changed as the volume of water increased in each cylinder. Match the graphs of  $a$  and  $b$  to Cylinders P and Q. Explain your reasoning.



P



Q



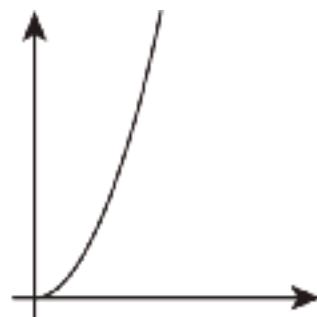
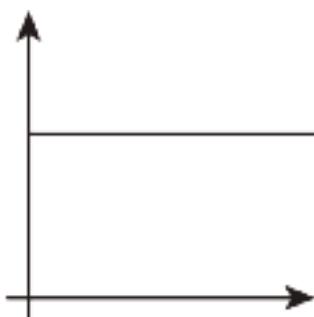
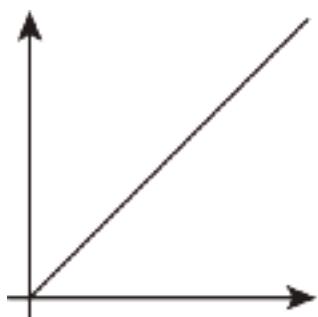
## Solution

Line  $a$  matches Cylinder Q and line  $b$  matches Cylinder P. Sample reasoning: A cylinder with a large radius would have a smaller rate of change (slope) for the same volume of water added when compared to a cylinder with a smaller radius. Since line  $b$  has the smaller slope, it must be the cylinder with the larger radius.

## Problem 3

### Statement

Which of the following graphs could represent the volume of water in a cylinder as a function of its height? Explain your reasoning.



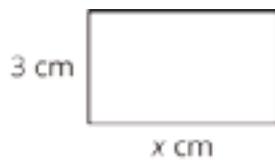
## Solution

The linear, increasing graph. Sample reasoning: As the height of water in a cylinder increases, the volume increases by the same scale factor.

## Problem 4

### Statement

Together, the areas of the rectangles sum to 30 square centimeters.



a. Write an equation showing the relationship between  $x$  and  $y$ .

b. Fill in the table with the missing values.

$x$	3		8		12
$y$		5		10	

## Solution

a.  $3x + 2y = 30$

b.

$x$	3	$6\frac{2}{3}$	8	$3\frac{1}{3}$	12
$y$	10.5	5	3	10	not possible

(From Unit 5, Lesson 3.)

# Lesson 12: How Much Will Fit?

## Goals

- Draw a cylinder and label its height and radius, describe (in writing) the shape of the “base” of the figure.
- Estimate the volumes of various containers using different units of measure, and explain (orally) the reasoning.

## Learning Targets

- I know that volume is the amount of space contained inside a three-dimensional figure.
- I recognize the 3D shapes cylinder, cone, rectangular prism, and sphere.

## Lesson Narrative

The purpose of this lesson is to remind students of the tangible meaning of volume: that it's the amount of space contained in a three-dimensional figure. Students estimate the amount of stuff different containers hold, recalling units of measurement commonly used for volume, like fluid ounces, cups, liters, gallons, cubic feet, and cubic centimeters (also known as milliliters). They revisit the names of figures learned prior to this unit: cylinders, cones, rectangular prisms, and spheres, and see some physical containers that can be modeled with these. It is important for students to make these connections between physical and mathematical objects so that, later on, real-world objects can be modeled with idealized figures.

Students also learn a method for quickly drawing a cylinder. Later in the unit, they also learn quick methods for sketching a cone and a sphere. This skill was included both because it is a handy thinking tool to have access to in problem solving and also because it helps students better understand the meaning of terms like radius and height as they apply to these mathematical objects.

## Alignments

### Addressing

- 8.G.C: Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

## Instructional Routines

- MLR7: Compare and Connect
- MLR8: Discussion Supports
- Poll the Class

## Required Preparation

Consider bringing in containers, dried rice, and measuring tools for the What's Your Estimate activity. It is also recommended that you have various-sized solid objects for students to pass around during the Do You Know These Figures activity.

### Student Learning Goals

Let's reason about the volume of different shapes.

## 12.1 Two Containers

### Warm Up: 5 minutes

In the previous lesson, students studied the relationship between volume of liquid and the height of the liquid when poured into a cylindrical container. The purpose of this warm-up is to shift students' attention toward other types of containers and to consider how the volume of two containers differs. This warm-up is direct preparation for the first activity of the lesson in which students reason about volumes of several container types and re-familiarize themselves with the language of three-dimensional objects.

### Addressing

- 8.G.C

### Instructional Routines

- Poll the Class

### Launch

Tell students to close their books or devices. Arrange students in groups of 2. Display the image of the two containers filled with beans for all to see.



Give partners 1 minute to estimate how many beans are in each container. Poll the class for their estimates and display these values for all to see, in particular the range of values expressed.

Tell students that the smaller container holds 200 beans. Ask students to open their books or devices and reconsider their estimate for the large container now that they have more information. Give 1–2 minutes for students to write down a new estimate. Follow with a whole-class discussion.

### **Anticipated Misconceptions**

Some students may not be sure how to start estimating the amount of beans in the larger jar once the number of beans in the smaller jar is known. Encourage them to start by estimating how the dimensions (e.g., height, width) of the two containers compare. For example, since the larger jar is more than twice the height of the smaller and has a greater width, then it must have at least twice as many beans.

### **Student Task Statement**

Your teacher will show you some containers. The small container holds 200 beans. Estimate how many beans the large jar holds.

### **Student Response**

Sample response: approximately 1,000 beans.

### **Activity Synthesis**

Poll the class for their new estimates for the number of beans in the larger container and display these next to the original estimates for all to see. Tell the class that the large container actually holds about 1,000 beans.

Discuss:

- “How did you and your partner calculate your estimate for the large jar?” (We estimated the large jar holds 900 beans since the large jar is about 3 times taller than the smaller jar, and it’s about 1.5 times wider and  $200 \cdot 3 \cdot 1.5 = 900$ .)
- “Is there a more accurate way to measure the difference in volume between the two containers than ‘number of beans.’” (Yes, we could use something smaller than beans so there is less air, such as rice or water.)
- “What are some examples of units used to measure volume? Where have you seen them used in your life?” (Cups, tablespoons, gallons, liters, cubic centimeters, etc. Drinks often have fluid ounces, gallons, or liters written on them. Recipes may use cups or tablespoons.)

## **12.2 What’s Your Estimate?**

**15 minutes**

The purpose of this activity is for students to practice using precise language to describe how they estimated volumes of objects. Starting from an object of known volume, students must consider the

difference in dimensions between the two objects. The focus here is on strategies to estimate the volume and units of measure used, not on exact answers or calculating volume using a formula (which will be the focus of later lessons). Notice students who:

- have clear strategies to estimate volume of contents inside container
- have an estimate that is very close to the actual volume

### **Addressing**

- 8.G.C

### **Instructional Routines**

- MLR8: Discussion Supports

### **Launch**

Arrange students in groups of 2.

Option 1: Bring in real containers and have students estimate how much rice each would hold, one at a time, preferably with one container whose volume is stated so students have a visual reference for their estimates. Also bring plenty of dried rice and measuring tools, such as tablespoons or cups. After collecting students' estimates, you can demonstrate how much rice each container holds using whichever units of measure the class deems reasonable. Note that 1 tablespoon is 0.5 ounces or around 15 milliliters. 1 cup is 8 ounces or around 240 milliliters. 1 milliliter is the same as 1 cubic centimeter.

Option 2: Display images one at a time for all to see. Give students 1–2 minutes to work with their partner and write down an estimate for the objects of unknown volumes in the picture. Follow with a whole-class discussion.

1.





2.



3.

4.



#### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Provide access to 3-D models of the objects depicted in the images for students to manipulate. Ask students to use the 3-D models to estimate the quantities in the other figures. Encourage students to place objects with known volumes inside or next to others to ensure an accurate estimate.

*Supports accessibility for: Visual-spatial processing; Conceptual processing*

#### Anticipated Misconceptions

Students may think there is a single right answer. Measurements are always approximate. Some of the measurements given by the authors of this activity were calculated using estimates from the photos, and may not be very precise. Measurements listed on the sides of packages are more accurate, but actual contents may vary slightly.

#### Student Task Statement

Your teacher will show you some containers.

1. If the pasta box holds 8 cups of rice, how much rice would you need for the other rectangular prisms?
2. If the pumpkin can holds 15 fluid ounces of rice, how much do the other cylinders hold?
3. If the small cone holds 2 fluid ounces of rice, how much does the large cone hold?
4. If the golf ball were hollow, it would hold about 0.2 cups of water. If the baseball were hollow, how much would the sphere hold?

## Student Response

Some answers are more exact than others.

1. pudding: 1.5 cups; pasta: 8 cups; chicken stock: 32 fluid ounces (or 4 cups); answers are from the information on the container.
2. tuna: 5.5 fluid ounces; oatmeal:  $5\frac{2}{3}$  cups (or about 45 fluid ounces); pumpkin: 15 fluid ounces; answers are from the information on the container.
3. cone with uneven top: approximately 3 fluid ounces (holds more if tilted); cone with smooth top: approximately 2 fluid ounces.
4. baseball: approximately 0.9 cup; golf ball: approximately 0.2 cup.

## Activity Synthesis

For each set of containers, display the image and select previously identified students to share their strategies for estimating the volume. Once strategies for each set of containers are shared, discuss:

- “How do the estimates differ if we measure using water versus rice?” (Measuring with rice leaves a bit of empty space between the grains, while water, being liquid, leaves no empty space, so it’s more accurate.)
- “If the containers we used were much larger (like a water tank), would our units of measure change? Why?” (If we were measuring larger volumes, we might want to use a larger unit, like gallons. 4000 ml sounds big, but it’s only a bit more than 1 gallon, which isn’t that much water.)

Conclude the discussion by asking students to compare some other units of measure for volume that they know of. Have students recall what they know about unit conversion between some units of measure. Example:

- Fluid ounces, quarts, cups, liters, milliliters
- Cubic feet, cubic meters, cubic yards
- Note that cubic centimeters are special, because  $1 \text{ cc} = 1 \text{ ml}$

If it comes up, here is the scoop on *ounces*: units called “ounces” are used to measure both volume and weight. It is important to be clear about what quantity you are measuring! To differentiate between them, people refer to the units of measure for volume as “fluid ounces.” For water, 1 fluid ounce is very close to 1 ounce by weight. This is not true for other substances! For example, mercury is much denser than water. 1 fluid ounce of mercury weighs about 13.6 ounces! Motor oil is less dense than water (that’s why it floats), so 1 fluid ounce of oil weighs only about 0.8 ounces. The metric system is not so confusing for quantities that would be measured in ounces, since it’s common to measure mass instead of weight and measure it in grams, whereas volume is measured in milliliters.

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support whole-class discussion. For each strategy that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to speak as they make sense of the reasoning of others.

*Design Principle(s): Support sense-making*

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## 12.3 Do You Know These Figures?

**10 minutes**

The purpose of this activity is for students to learn or remember the names of the figures they worked with in the previous activity and learn a quick method for sketching a cylinder. Students start by determining the shapes that are the faces of the four shapes. They also determine which shape would be considered the base of each figure shown. This allows students to connect previously learned two-dimensional figures to new three-dimensional figures introduced here. The last question introduces students to a way to sketch a cylinder. This is a skill they will continue to use throughout the unit when working on problems that do not provide a visual example of a situation. Identify students who sketch cylinders that are different sizes or drawn sideways.

### Addressing

- 8.G.C

### Instructional Routines

- MLR7: Compare and Connect

### Launch

It is strongly recommended that you provide physical, solid objects for students to hold and look at. If using physical objects, pass around the objects for students to see and feel before starting the activity.

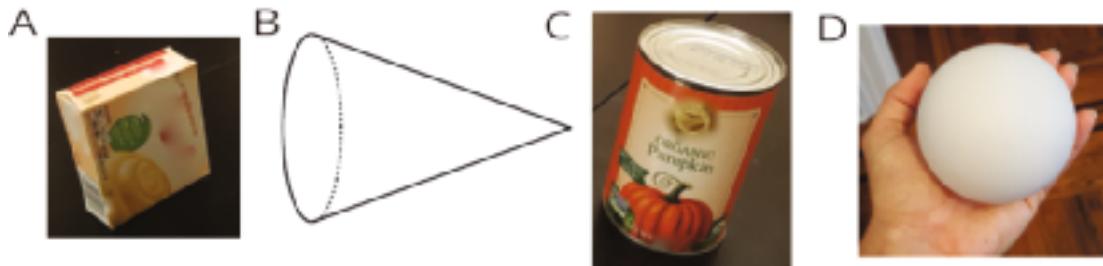
Give students 3–5 minutes of quiet work time, followed by a whole-class discussion.

### Anticipated Misconceptions

If students struggle to visualize the shapes of the faces, position the object so that students can only see two dimensions. Ask students what two-dimensional shape they see.

## Student Task Statement

- What shapes are the faces of each type of object shown here? For example, all six faces of a cube are squares.



1. Which faces could be referred to as a “base” of the object?

2. Here is a method for quickly sketching a cylinder:

- Draw two ovals.
- Connect the edges.
- Which parts of your drawing would be hidden behind the cylinder? Make these parts dashed lines.
- Practice sketching some cylinders. Sketch a few different sizes, including short, tall, narrow, wide, and sideways. Label the radius  $r$  and height  $h$  on each cylinder.



## Student Response

1. rectangular prism: rectangles; cone: circle and a curved surface; cylinder: 2 circles and a curved surface; sphere: one curved surface.
2. rectangular prism: any rectangular face can be called the base; cone: the circular face is the base; cylinder: either circular face can be called the base; sphere: has no base.
3. Answers vary.

## Are You Ready for More?



A soccer ball is a polyhedron with 12 black pentagonal faces and 20 white hexagonal faces. How many edges in total are on this polyhedron?

### Student Response

Since each pentagonal face has 5 edges and there are 12 pentagonal faces, there are 60 edges on pentagonal faces. Similarly, there are  $20 \cdot 6 = 120$  edges on the 20 hexagonal faces. Combined, this would make 180 edges, except that every edge is counted twice by this process. Dividing by 2, we conclude that there are 90 edges on a soccer ball.

### Activity Synthesis

If using physical objects, display each object one at a time for all to see. If using images, display the images for all to see, and refer to each object one at a time. Ask students to identify:

- which figure the object is an example of
- the different shapes that make up the faces of the figure
- the shape that is the base of the figure

Select previously identified students to share their sketches of cylinders. The goal is to ensure that students see a variety of cylinders: short, tall, sideways, narrow, etc. If no student drew a “sideways” cylinder, sketch one for all to see and make sure students understand that even though it is sideways, the height is still the length perpendicular to the base.

Tell students that we will be working with these different three-dimensional figures for the rest of this unit. Consider posting a display in the classroom that shows a diagram of each object labeled with its name, and where appropriate, with one side labeled “base.” As volume formulas are developed, the formulas can be added to the display.

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### Access for English Language Learners

*Representing: MLR7 Compare and Connect.* Invite students to do a gallery walk, to observe each other's sketches. As students circulate, ask students to describe what is similar and what is different about the cylinders to a partner. Draw students' attention to the different characteristics of the cylinders (e.g., laying sideways, short, narrow, long, etc.). In this discussion, demonstrate precise ways to describe shapes, such as using the terms cylinders, cones, spheres, and rectangular prisms. These exchanges strengthen students' mathematical language use and reasoning about cylinders.

*Design Principle(s): Optimize output (for comparison); Cultivate conversation*

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## Lesson Synthesis

The volume of a three-dimensional figure is the amount of space it encloses. Ask students:

- "What are some shapes you worked with in today's lesson?" (Cylinders, cones, spheres, rectangular prisms)
- "What are some different units of measure we use to calculate volume of these figures?" (Cubic feet, fluid ounces, gallons, cubic nanometers, rice)
- "What are some examples of objects we see in our world that are very similar to these figures that you didn't see in the pictures earlier?" (A basketball is like a sphere. A cell phone is like a rectangular prism.)

## 12.4 Rectangle to Round

Cool Down: 5 minutes

### Addressing

- 8.G.C

#### Student Task Statement



Here is a box of pasta and a cylindrical container. The two objects are the same height, and the cylinder is just wide enough for the box to fit inside with all 4 vertical edges of the box touching the inside of the cylinder. If the box of pasta fits 8 cups of rice, estimate how many cups of rice will fit inside the cylinder. Explain or show your reasoning.

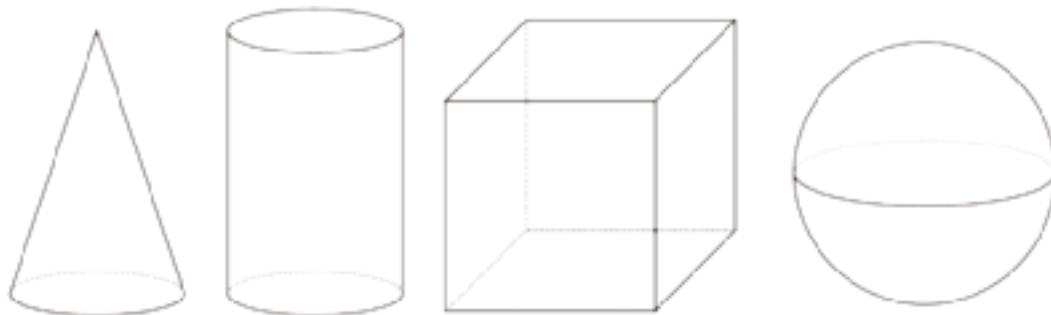
#### Student Response

Answers vary. Sample response: About 11 cups of rice since it should be a little more than the box, but not a lot.

## Student Lesson Summary

The volume of a three-dimensional figure, like a jar or a room, is the amount of space the shape encloses. We can measure volume by finding the number of equal-sized volume units that fill the figure without gaps or overlaps. For example, we might say that a room has a volume of 1,000 cubic feet, or that a pitcher can carry 5 gallons of water. We could even measure volume of a jar by the number of beans it could hold, though a bean count is not really a measure of the volume in the same way that a cubic centimeter is because there is space between the beans. (The number of beans that fit in the jar do depend on the volume of the jar, so it is an okay estimate when judging the relative sizes of containers.)

In earlier grades, we studied three-dimensional figures with flat faces that are polygons. We learned how to calculate the volumes of rectangular prisms. Now we will study three-dimensional figures with circular faces and curved surfaces: cones, cylinders, and spheres.



To help us see the shapes better, we can use dotted lines to represent parts that we wouldn't be able to see if a solid physical object were in front of us. For example, if we think of the cylinder in this picture as representing a tin can, the dotted arc in the bottom half of that cylinder represents the back half of the circular base of the can. What objects could the other figures in the picture represent?

## Glossary

- cone
- sphere

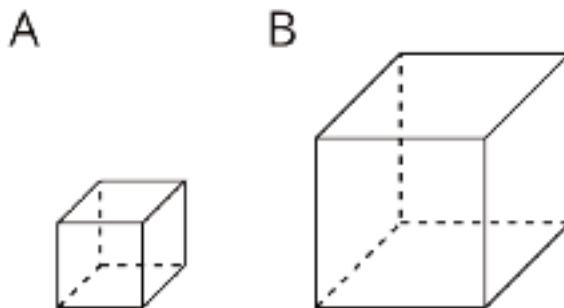
## Lesson 12 Practice Problems

### Problem 1

#### Statement

- Sketch a cube and label its side length as 4 cm (this will be Cube A).
- Sketch a cube with sides that are twice as long as Cube A and label its side length (this will be Cube B).
- Find the volumes of Cube A and Cube B.

#### Solution



- Cube A, with a labeled side length of 4 cm
- Cube B, with a labeled side length of 8 cm
- Cube A:  $64 \text{ cm}^3$ , Cube B:  $512 \text{ cm}^3$

### Problem 2

#### Statement

Two paper drink cups are shaped like cones. The small cone can hold 6 oz of water. The large cone is  $\frac{4}{3}$  the height and  $\frac{4}{3}$  the diameter of the small cone. Which of these could be the amount of water the large cone holds?

- 8 cm
- 14 oz
- 4.5 oz
- 14 cm

#### Solution

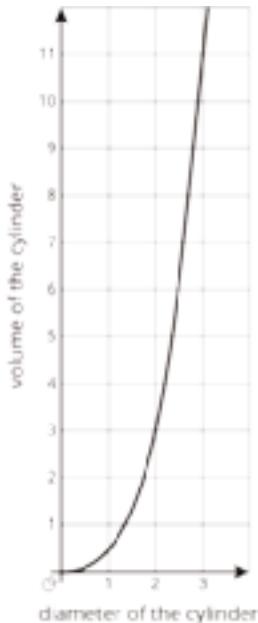
B

## Problem 3

### Statement

The graph represents the volume of a cylinder with a height equal to its radius.

- When the diameter is 2 cm, what is the radius of the cylinder?
- Express the volume of a cube of side length  $s$  as an equation.
- Make a table for volume of the cube at  $s = 0$  cm,  $s = 1$  cm,  $s = 2$  cm, and  $s = 3$  cm.
- Which volume is greater: the volume of the cube when  $s = 3$  cm, or the volume of the cylinder when its diameter is 3 cm?



### Solution

- The radius is 1 cm.
- $s^3$

c.

$s$	volume of cube
0 cm	0 cm <sup>3</sup>
1 cm	1 cm <sup>3</sup>
2 cm	8 cm <sup>3</sup>
3 cm	27 cm <sup>3</sup>

- The volume of the cube at  $s = 3$  cm. (Its volume is  $27 \text{ cm}^3$ , while the volume of the cylinder when its diameter is 3 cm is  $\frac{27\pi}{8} \text{ cm}^3$ , or about  $10.6 \text{ cm}^3$ .)

(From Unit 5, Lesson 7.)

## Problem 4

### Statement

Select all the points that are on a line with slope 2 that also contains the point  $(2, -1)$ .

- A.  $(3, 1)$
- B.  $(1, 1)$
- C.  $(1, -3)$
- D.  $(4, 0)$
- E.  $(6, 7)$

### Solution

`["A", "C", "E"]`

(From Unit 3, Lesson 10.)

## Problem 5

### Statement

Several glass aquariums of various sizes are for sale at a pet shop. They are all shaped like rectangular prisms. A 15-gallon tank is 24 inches long, 12 inches wide, and 12 inches tall. Match the dimensions of the other tanks with the volume of water they can each hold.

A. Tank 1: 36 inches long, 18 inches wide, and 12 inches tall	1. 5 gallons
	2. 10 gallons
B. Tank 2: 16 inches long, 8 inches wide, and 10 inches tall	3. 20 gallons
	4. 30 gallons
C. Tank 3: 30 inches long, 12 inches wide, and 12 inches tall	
D. Tank 4: 20 inches long, 10 inches wide, and 12 inches tall	

### Solution

- A: 4
- B: 1
- C: 3
- D: 2

## Problem 6

### Statement

Solve: 
$$\begin{cases} y = -2x - 20 \\ y = x + 4 \end{cases}$$

### Solution

(-8, -4)

(From Unit 4, Lesson 14.)

# Lesson 13: The Volume of a Cylinder

## Goals

- Calculate the volume of a cylinder, and compare and contrast (orally) the formula for volume of a cylinder with the formula for volume of a prism.
- Explain (orally) how to find the volume of a cylinder using the area of the base and height of the cylinder.

## Learning Targets

- I can find the volume of a cylinder in mathematical and real-world situations.
- I know the formula for volume of a cylinder.

## Lesson Narrative

In this lesson students learn that the volume of a cylinder is the area of the base times the height, just like a prism. This is accomplished by considering 1-unit-tall layers of a rectangular prism side by side with 1-unit-tall layers of a cylinder. After thinking about how to compute the volume of specific cylinders, students learn the general formulas  $V = Bh$  and  $V = \pi r^2 h$ .

In the warm-up, students recall that a circle's area can be determined given its radius or diameter. Students also become familiar with what is meant by *radius* and *height* as those terms apply to cylinders. Finally, students compute the volume of a cylinder by multiplying the area of its base by its height. A volume expressed using the exact number  $\pi$  versus the same volume computed using 3.14 as an approximation for  $\pi$  is discussed. The following lesson provides opportunities to practice these skills and solve related problems.

## Alignments

### Building On

- 6.G.A.2: Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas  $V = lwh$  and  $V = bh$  to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
- 7.G.B.4: Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

### Addressing

- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR3: Clarify, Critique, Correct
- MLR8: Discussion Supports
- Think Pair Share

## Required Materials

### Colored pencils

### Required Preparation

A good way to manage the various formulas in this unit is to create a display for each formula as each one is introduced. Frequently draw students' attention to the displays and use them as a reference.

For the Circular Volumes activity, consider building a rectangular prism from 48 snap cubes to match the diagram in the print statement.

Provide access to colored pencils.

### Student Learning Goals

Let's explore cylinders and their volumes.

## 13.1 A Circle's Dimensions

### Warm Up: 10 minutes

The purpose of this warm-up is for students to review how to compute the area of a circle. This idea should have been carefully developed in grade 7. This warm-up gives students an opportunity to revisit this idea in preparation for finding the volume of a cylinder later in the lesson.

Students begin the activity with a whole-class discussion in which they identify important features of a circle including its radius and diameter. They use this information and the formula for the area of the circle to choose expressions from a list that are equivalent to the area of the circle. In the final question, students are given the area of the circle and find the corresponding radius.

As students are working, monitor for students who can explain why  $16\pi$ ,  $\pi 4^2$ , and "approximately 50" square units represent the area of the circle.

### Building On

- 7.G.B.4

### Launch

Display the diagram from the task statement for all to see and ask students:

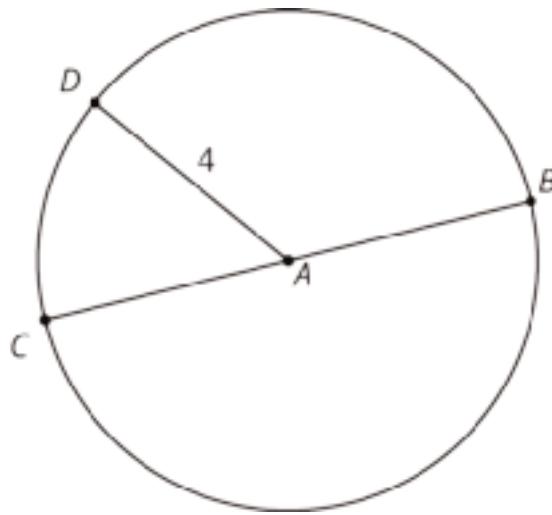
- “Name a segment that is a radius of circle A.” (AC, AD, and AB or those with the letters reversed are all radii.) Review the meaning of the radius of a circle.
- “What do we call a segment like BC, with endpoints on the circle that contains the center of the circle?” (A diameter.) Review the meaning of a diameter of a circle.
- “What is the length of segment  $AB$ ?” (4 units.) Review the fact that all radii of a circle have the same length.

Give students 3 minutes of quiet work time and follow with a whole-class discussion.

### Anticipated Misconceptions

If students struggle to recall how to find the area of a circle, encourage them to look up a formula using any available resources.

#### Student Task Statement



Here is a circle. Points  $A$ ,  $B$ ,  $C$ , and  $D$  are drawn, as well as Segments  $AD$  and  $BC$ .

1. What is the area of the circle, in square units? Select all that apply.
  - $4\pi$
  - $\pi 8$
  - $16\pi$
  - $\pi 4^2$
  - approximately 25
  - approximately 50
2. If the area of a circle is  $49\pi$  square units, what is its radius? Explain your reasoning.

### Student Response

1. c, d, and f. Since the radius is 4, the area of the circle is  $\pi \cdot 4^2 = 16\pi$ . This is approximately 50.3 square units.
2. 7 units. The square of the radius is 49 since the area is  $\pi$  times the square of the radius, and the area is  $49\pi$  square units. The radius is 7 units, because  $49 = 7^2$ .

### Activity Synthesis

The purpose of this discussion is to make sure students remember that the area of a circle can be found by squaring its radius and multiplying by  $\pi$ .

Select previously identified students to share answers to the first question and explain why each of the solutions represents the area of the circle. If not brought up during the discussion, tell students that sometimes it is better to express an area measurement in terms of  $\pi$ . Other times it may be better to use an approximation of  $\pi$ , like 3.14, to represent the area measurement in decimal form. In this unit, we will often express our answers in terms of  $\pi$ .

Display in a prominent place for all to see for the next several lessons: Let  $A$  be the area of a circle of radius  $r$ , then  $A = \pi r^2$ .

## 13.2 Circular Volumes

15 minutes

The purpose of this activity is for students to connect their previous knowledge of the volume of rectangular prisms to their understanding of the volume of cylinders. From previous work, students should know that the volume of rectangular prisms is found by multiplying the area of the base by the height. Here we expand upon that to compute the volume of a cylinder.

Students start by calculating the volume of a rectangular prism. Then they extrapolate from that to calculate the volume of a cylinder given the area of its base and its height. If some students don't know they should multiply the area of the base by its height, then they are prompted to connect prisms and cylinders to make a reasonable guess.

We want students to conjecture that the volume of a cylinder is the area of its base multiplied by its height.

### Building On

- 6.G.A.2

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

- Think Pair Share

## Launch

Arrange students in groups of 2. Remind students that a rectangular prism has a base that is a rectangle, and a cylinder has a base that is a circle. It may have been some time since students have thought about the meaning of a result of a volume computation. Consider showing students a rectangular prism built from 48 snap cubes with the same dimensions as Figure A. It may help them to see that one layer is made of 16 cubes. Give students 3–5 minutes of quiet work time followed by a partner discussion. During their discussion, partners compare the volumes they found for the cylinders. If they guessed the volumes, partners explain their reasoning to one another. Follow with a whole-class discussion.

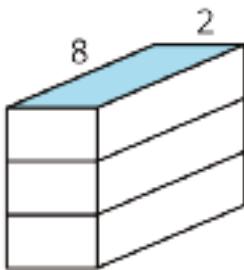
### Access for Students with Disabilities

*Representation: Develop Language and Symbols.* Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide connecting cubes for students to create rectangular prisms and calculate the volume of. Ask students to represent the volume of the rectangular prism in an equation and connect the base, width, and height to the 3-D model.  
*Supports accessibility for: Visual-spatial processing; Conceptual processing*

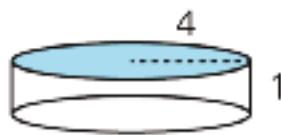
### Student Task Statement

What is the volume of each figure, in cubic units? Even if you aren't sure, make a reasonable guess.

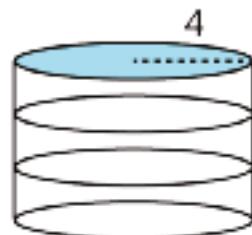
A



B



C



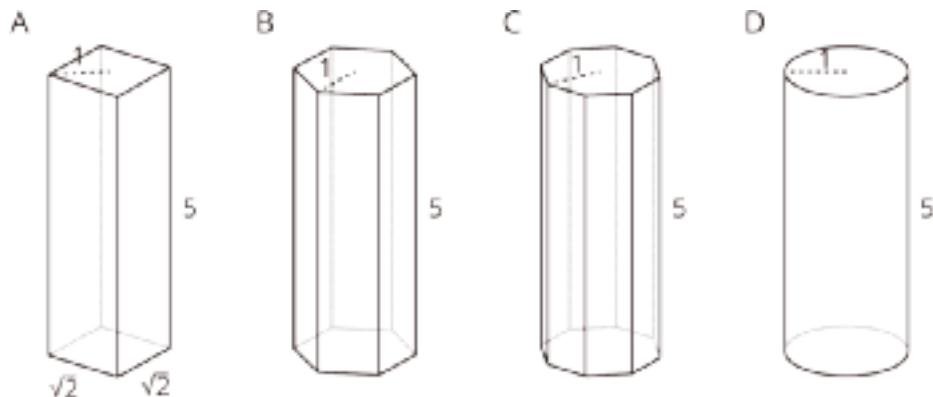
1. Figure A: A rectangular prism whose base has an area of 16 square units and whose height is 3 units.
2. Figure B: A cylinder whose base has an area of  $16\pi$  square units and whose height is 1 unit.
3. Figure C: A cylinder whose base has an area of  $16\pi$  square units and whose height is 3 units.

### Student Response

1. 48 cubic units. The area of the rectangular base is 16, which is multiplied by the height 3 to find the volume.
2.  $16\pi$  cubic units. The area of the circular base is  $\pi \cdot 4^2$ , which is multiplied by the height 1 to find the volume.
3.  $48\pi$  cubic units. The area of the circular base is  $\pi \cdot 4^2$ , which is multiplied by the height 3 to find the volume.

### Are You Ready for More?

prism	prism	prism	cylinder
base: square	base: hexagon	base: octagon	base: circle



Here are solids that are related by a common measurement. In each of these solids, the distance from the center of the base to the furthest edge of the base is 1 unit, and the height of the solid is 5 units. Use 3.14 as an approximation for  $\pi$  to solve these problems.

1. Find the area of the square base and the circular base.
2. Use these areas to compute the volumes of the rectangular prism and the cylinder. How do they compare?
3. Without doing any calculations, list the figures from smallest to largest by volume. Use the images and your knowledge of polygons to explain your reasoning.
4. The area of the hexagon is approximately 2.6 square units, and the area of the octagon is approximately 2.83 square units. Use these areas to compute the volumes of the prisms with the hexagon and octagon bases. How does this match your explanation to the previous question?

### Student Response

1. Area of the square is 2 square units. Area of the circle is approximately 3.14 square units.

2. Volume of the rectangular prism is 10 cubic units. Volume of the cylinder is 15.7 cubic units.
3. Answers vary. Sample response: the area of the polygons increase as the number of sides increase. This means that the volumes will also increase, since the height stays the same.
4. Volume of the hexagonal prism is 13 cubic units. Volume of the octagonal prism is 14.15 cubic units.

### Activity Synthesis

Highlight the important features of cylinders and their definitions: the radius of the cylinder is the radius of the circle that forms its base; the height of a cylinder is the length between its circular top and bottom; a cylinder of height 1 can be thought of as a “layer” in a cylinder with height  $h$ . To highlight the connection between finding the area of a rectangular prism and finding the area of a cylinder, ask:

- “How are prisms and cylinders different?” (A prism has a base that is a polygon, and a cylinder has a base that is a circle.)
- “How are prisms and cylinders the same?” (The volume of cylinders and prisms is found by multiplying the area of the base by the height.  $V = Bh$ )
- “How do you find the area of the base,  $B$ , of a cylinder?” ( $B = \pi r^2$ ).

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### Access for English Language Learners

*Writing, Conversing: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise a response to the question, “What is similar and different about cylinders and rectangular prisms?” Ask each student to meet with 2–3 other partners in a row for feedback. Provide student with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “Can you give an example?”, “Why do you think that?”, “Can you say more about ...?”, etc.). Students can borrow ideas and language from each partner to strengthen their final explanation.

*Design Principle(s): Optimize output (for explanation)*

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## 13.3 A Cylinder's Dimensions

**Optional: 10 minutes**

In this optional activity, students use colored pencils (or pens or highlighters) to label the radius and height on different pictures of cylinders. Then they sketch their own cylinders and label the radius and heights of those. The purpose of this activity is for students to practice identifying the radius and height of various cylinders, some of which are in context.

This activity can also be abbreviated if students demonstrate prior understanding of how to draw or label cylinders and only need a brief refresh.

## Addressing

- 8.G.C.9

## Instructional Routines

- MLR3: Clarify, Critique, Correct

## Launch

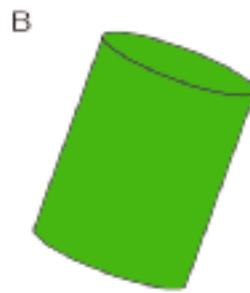
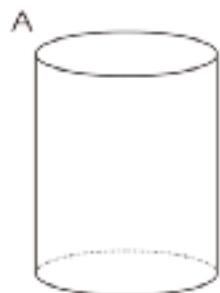
Distribute colored pencils. Give students 1–2 minutes of quiet work time followed by a whole-class discussion.

## Anticipated Misconceptions

In diagrams D and E, some students might mistake the diameter for the height of the cylinder. Some students may mark a height taller than the cylinder due to the tilt of the figure.

### Student Task Statement

1. For cylinders A–D, sketch a radius and the height. Label the radius with an  $r$  and the height with an  $h$ .



2. Earlier you learned how to sketch a cylinder. Sketch cylinders for E and F and label each one's radius and height.

### Student Response

1. Answers vary.
2. Answers vary.

## Activity Synthesis

Select students to share where they marked the radius and height and the cylinders sketched for images E and F. Discuss examples of other cylinders students see in real life.

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### Access for English Language Learners

*Speaking, Listening: MLR3 Clarify, Critique, Correct.* Present an incorrect drawing that reflects a possible misunderstanding from the class about the height of a cylinder. For example, draw a cylinder with the diameter incorrectly placed at its height and the height at its diameter.

Prompt students to identify the error, (e.g., ask, “Do you agree with the representation? Why or why not?”), correct the representation, and write feedback to the author explaining the error. This helps students develop an understanding and make sense of identifying the heights of differently-oriented cylinders.

*Design Principle(s): Support sense-making; Maximize meta-awareness*

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## 13.4 A Cylinder's Volume

10 minutes

The purpose of this activity is to give students opportunities to find the volumes of some cylinders. By finding the area of the base before finding the volume, students are encouraged to compute the volume by multiplying the area of its base by its height. This way of thinking about volume might be more intuitive for students than the formula  $V = \pi r^2 h$ . Notice students who plug the radius into a formula for the volume and students who find the area of the base and multiply that by the cylinder's height.

The second problem of this activity focuses on exploring cylinders in a context. Generally, the volume of a container is the amount of space inside, but in this context, that also signifies the amount of material that fits into the space. Students are not asked to find the area of the base of the silo. Notice students that find the area before computing the volume and those that use the volume formula and solve for it directly. Notice at what step in the computations students approximate  $\pi$ .

When working with problems in a given context, it is sometimes convenient or practical to use an approximation of  $\pi$ . An example of this is given in the question regarding the volume of a grain silo and interpretations of the answer.

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR8: Discussion Supports

## Launch

Provide access to colored pencils to shade the cylinder's base. Give students 5–6 minutes of quiet work time followed by a whole-class discussion.

### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems and other text-based content.

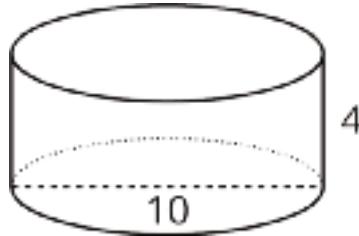
*Supports accessibility for: Language; Conceptual processing*

### Anticipated Misconceptions

Students might use the silo's diameter instead of the radius to find the volume. Remind them of the formula for area of a circle and the discussion about diameter and radius earlier in this lesson.

### Student Task Statement

1. Here is a cylinder with height 4 units and diameter 10 units.



- a. Shade the cylinder's base.
- b. What is the area of the cylinder's base? Express your answer in terms of  $\pi$ .
- c. What is the volume of this cylinder? Express your answer in terms of  $\pi$ .

2. A silo is a cylindrical container that is used on farms to hold large amounts of goods, such as grain. On a particular farm, a silo has a height of 18 feet and diameter of 6 feet. Make a sketch of this silo and label its height and radius. How many cubic feet of grain can this silo hold? Use 3.14 as an approximation for  $\pi$ .

### Student Response

1. a. The image of the cylinder should have either its top or bottom shaded in.  
b.  $25\pi$ . The radius of the base is half of 10 or 5 units. The area of the base is  $25\pi$  square units since  $\pi 5^2 = 25\pi$ .  
c.  $100\pi$ . The volume of the cylinder is the area of its base times its height, which is  $100\pi$  cubic units, since  $25\pi \cdot 4 = 100\pi$ .

2. Approximately 509 cubic feet. The diameter of the silo's base is 6 feet, which means the base has a radius of 3 feet. The area of the base of the silo is  $9\pi$  square feet.  $\pi 3^2 = 9\pi$ . The volume is  $162\pi$  cubic feet ( $18 \cdot 9\pi = 162\pi$ ). Using  $\pi \approx 3.14$ , the volume of the silo is approximately 509 cubic feet.

### Are You Ready for More?

One way to construct a cylinder is to take a rectangle (for example, a piece of paper), curl two opposite edges together, and glue them in place.

Which would give the cylinder with the greater volume: Gluing the two dashed edges together, or gluing the two solid edges together?



### Student Response

Gluing the two solid edges together will create a cylinder with the greater volume.

Whichever two lines are glued together become the height of the cylinder, and the other lines represent the circumference of the circular base. For the cylinder created by gluing the dashed lines together, the height is 3 units, and the circumference is 2 units. Since circumference of a circle is equal to  $\pi$  times the diameter, the radius of the circular base must be  $\frac{1}{2}(\frac{2}{\pi}) = \frac{1}{\pi}$ . Therefore, the volume can be determined by  $V = \pi(\frac{1}{\pi})^2(3)$ , which is about .95 units<sup>3</sup>.

For the cylinder created by gluing the solid lines together, the height is 2 units, and the radius is  $\frac{1}{2}(\frac{3}{\pi}) = \frac{3}{2\pi}$ . Therefore, the volume can be determined by  $V = \pi(\frac{3}{2\pi})^2(2)$ , which is about 1.43 units<sup>3</sup>.

### Activity Synthesis

The goal of this discussion is to ensure students understand how to use the area of the cylinder's base to calculate its volume. Consider asking the following questions:

- “How does knowing the area of a circular base help determine the volume of a cylinder?” (The volume is this area multiplied by the height of the cylinder.)
- “If the cylinder were on its side, how do you know which measurements to use for the volume?” (Since we need area of the base first, the radius or diameter of the circle will always be the measurement used for  $r$ , and the height is always the distance between the bases, or the measurement perpendicular to the bases. It doesn't matter which direction the cylinder is turned.)

- “When is it better to use approximations of pi instead of leaving it exact?” (Approximating pi helps us interpret an answer that has pi as a factor, like the area of a circular region or the volume of a cylindrical container.)

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support whole-class discussion. For each response that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will help students produce and make sense of the language needed to communicate about different strategies for calculating volume of cylinders.

*Design Principle(s): Support sense-making; Optimize output (for explanation)*

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## Lesson Synthesis

Make a display that includes the formula for a cylinder’s volume,  $V = Bh$ , along with a labeled diagram of a cylinder. This display should be kept posted in the classroom for the remaining lessons within this unit.

Previously, students computed the volume of prisms by multiplying the area of the base by the prism’s height. To help students summarize the ideas in this lesson, ask “How is finding the volume of a cylinder like finding the volume of a prism?” and give students 2–3 minutes to write a response. Encourage students to use diagrams to show their thinking. Invite students to share their ideas, displaying any diagrams for all to see.

## 13.5 Liquid Volume

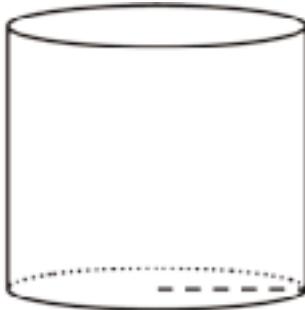
### Cool Down: 5 minutes

Students compute the volume of a cylinder given its height and radius. Students are given the dimensions of the cylinder and the units.

### Addressing

- 8.G.C.9

## Student Task Statement



The cylinder shown here has a height of 7 centimeters and a radius of 4 centimeters.

1. What is the area of the base of the cylinder? Express your answer in terms of  $\pi$ .
2. How many cubic centimeters of fluid can fill this cylinder? Express your answer in terms of  $\pi$ .
3. Give a decimal approximation of your answer to the second question using 3.14 to approximate  $\pi$ .

## Student Response

1.  $16\pi \text{ cm}^2$ . The square of the radius of the base is  $4^2 = 16$ , which is multiplied by  $\pi$  ( $\pi \cdot 4^2 = 16\pi$ ).
2.  $112\pi \text{ cm}^3$ . The height of the cylinder is 7, which is multiplied by the area of the base ( $16\pi \cdot 7 = 112\pi$ ).
3.  $351.68 \text{ cm}^3$ , because  $112 \cdot 3.14 \approx 351.68$

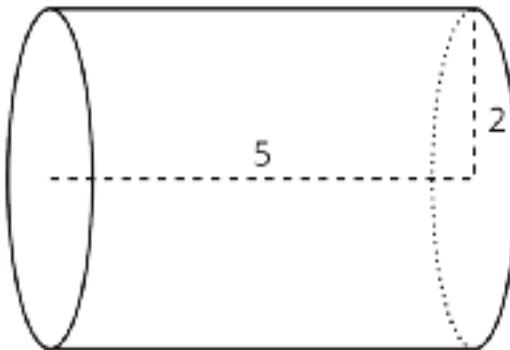
## Student Lesson Summary

We can find the volume of a cylinder with radius  $r$  and height  $h$  using two ideas we've seen before:

- The volume of a rectangular prism is a result of multiplying the area of its base by its height.
- The base of the cylinder is a circle with radius  $r$ , so the base area is  $\pi r^2$ .

Remember that  $\pi$  is the number we get when we divide the circumference of any circle by its diameter. The value of  $\pi$  is approximately 3.14.

Just like a rectangular prism, the volume of a cylinder is the area of the base times the height. For example, take a cylinder whose radius is 2 cm and whose height is 5 cm.



The base has an area of  $4\pi \text{ cm}^2$  (since  $\pi \cdot 2^2 = 4\pi$ ), so the volume is  $20\pi \text{ cm}^3$  (since  $4\pi \cdot 5 = 20\pi$ ). Using 3.14 as an approximation for  $\pi$ , we can say that the volume of the cylinder is approximately  $62.8 \text{ cm}^3$ .

In general, the base of a cylinder with radius  $r$  units has area  $\pi r^2$  square units. If the height is  $h$  units, then the volume  $V$  in cubic units is

$$V = \pi r^2 h$$

# Lesson 13 Practice Problems

## Problem 1

### Statement

- Sketch a cylinder.
- Label its radius 3 and its height 10.
- Shade in one of its bases.

### Solution

Answers vary.

## Problem 2

### Statement

At a farm, animals are fed bales of hay and buckets of grain. Each bale of hay is in the shape a rectangular prism. The base has side lengths 2 feet and 3 feet, and the height is 5 feet. Each bucket of grain is a cylinder with a diameter of 3 feet. The height of the bucket is 5 feet, the same as the height of the bale.

- a. Which is larger in area, the rectangular base of the bale or the circular base of the bucket? Explain how you know.
- b. Which is larger in volume, the bale or the bucket? Explain how you know.

### Solution

- a. The bucket's base. The area of the bale's base is 6 square feet. The area of the bucket's base is just over 7 square feet, because  $\pi(1.5)^2 \approx 7.07$ .
- b. The bucket. The bale and the bucket have the same height, and the bucket's base area is larger.

## Problem 3

### Statement

Three cylinders have a height of 8 cm. Cylinder 1 has a radius of 1 cm. Cylinder 2 has a radius of 2 cm. Cylinder 3 has a radius of 3 cm. Find the volume of each cylinder.

### Solution

- Cylinder 1 has a volume of  $8\pi \approx 25.12 \text{ cm}^3$ .
- Cylinder 2 has a volume of  $32\pi \approx 100.48 \text{ cm}^3$ .

- Cylinder 3 has a volume of  $72\pi \approx 226.08 \text{ cm}^3$ .

## Problem 4

### Statement

A one-quart container of tomato soup is shaped like a rectangular prism. A soup bowl shaped like a hemisphere can hold 8 oz of liquid. How many bowls will the soup container fill? Recall that 1 quart is equivalent to 32 fluid ounces (oz).

### Solution

4 bowls

(From Unit 5, Lesson 12.)

## Problem 5

### Statement

Match each set of information about a circle with the area of that circle.

A. Circle A has a radius of 4 units.	1. $4\pi$ square units
B. Circle B has a radius of 10 units.	2. approximately 314 square units
C. Circle C has a diameter of 16 units.	3. $64\pi$ square units
D. Circle D has a circumference of $4\pi$ units.	4. $16\pi$ square units

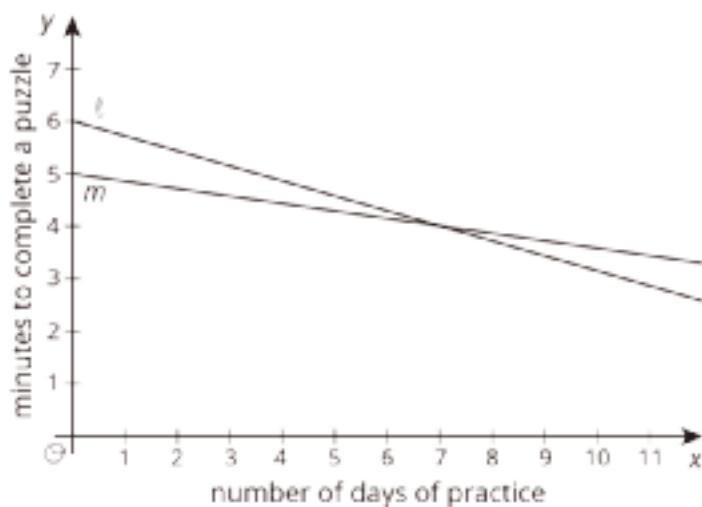
### Solution

- A: 4
- B: 2
- C: 3
- D: 1

## Problem 6

### Statement

Two students join a puzzle solving club and get faster at finishing the puzzles as they get more practice. Student A improves their times faster than Student B.



- Match the students to the Lines  $l$  and  $m$ .
- Which student was faster at puzzle solving before practice?

## Solution

- Student A is represented by Line  $l$ . Student B is represented by Line  $m$ .
- Student B

(From Unit 5, Lesson 8.)

# Lesson 14: Finding Cylinder Dimensions

## Goals

- Calculate the value of one dimension of a cylinder, and explain (orally and in writing) the reasoning.
- Create a table of dimensions of cylinders, and describe (orally) patterns that arise.

## Learning Targets

- I can find missing information about a cylinder if I know its volume and some other information.

## Lesson Narrative

In this lesson, students use the formula  $V = \pi r^2 h$  for the volume of a cylinder to solve a variety of problems. They compute volumes given radius and height, and find radius or height given a cylinder's volume and the other dimension by reasoning about the structure of the volume formula.

## Alignments

### Addressing

- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR8: Discussion Supports
- Think Pair Share

### Student Learning Goals

Let's figure out the dimensions of cylinders.

## 14.1 A Cylinder of Unknown Height

### Warm Up: 5 minutes

The purpose of this warm-up is to assess students' understanding of the volume of a cylinder. Students learned that the volume of either a cylinder or prism is found by multiplying the area of the base by its height. In this warm-up, students are given information to find the area of a cylinder's base, but they are not given the height. Students propose a volume for the cylinder and explain why it works. Since the diameter of the base is 8, the area of the base is  $16\pi$ .

If students have trouble getting started, ask them:

- "Do you have enough information to calculate the area of the base?"

- “What is the radius?”

Identify students who use these strategies:

- find the area of the base first then set up the equation  $V = 16\pi h$ .
- choose a specific value for  $h$  then solve for the volume.

## Addressing

- 8.G.C.9

## Instructional Routines

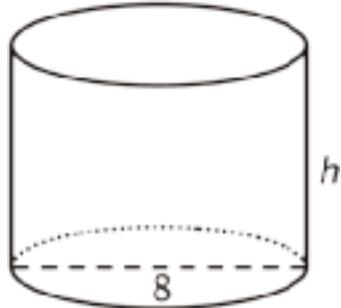
- Think Pair Share

## Launch

Arrange students in groups of 2. Tell students that in a previous lesson, they learned how to find the volume of a cylinder if they know the cylinder’s radius and height. Draw their attention to where volume formulas are displayed in the classroom as the unit progresses. Give students 1–2 minutes of quiet work time followed by time to explain their reasoning to their partner. Follow this with a whole-class discussion.

### Student Task Statement

What is a possible volume for this cylinder if the diameter is 8 cm? Explain your reasoning.



### Student Response

Answers vary. Sample response: The radius of the cylinder’s base is 4 cm, which means the area of the base is  $16\pi \text{ cm}^2$  since  $4^2 \cdot \pi = 16\pi$ . If the height is 1 cm, then the volume would be  $16\pi \text{ cm}^3$  since  $16\pi \cdot 1 = 16\pi$ .

### Activity Synthesis

The goal of this discussion is for students to communicate how the height of a cylinder is related to its volume. Invite students to share their solutions and their reasoning. Record and display the dimensions and volumes of cylinders that correspond to solutions given by students.

## 14.2 What's the Dimension?

15 minutes

In this activity, students find the missing dimensions of cylinders when given the volume and the other dimension. A volume equation representing the cylinder is given for each problem.

Identify students who use these strategies: guess and check, divide each side of the equation by the same value to solve for missing variable, or use the structure of the volume equation to reason about the missing variable

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

### Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by time to share their explanation for the first problem with their partners.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Activate or supply background knowledge of strategies students can use to solve for unknown variables in equations. Allow students to use calculators to ensure inclusive participation in the activity.

*Supports accessibility for: Memory; Conceptual processing*

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### Access for English Language Learners

*Writing, Conversing: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise a response to one or both of the questions. Ask each student to meet with 2–3 other partners in a row for feedback. Provide student with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “What did you do first?”, “How do you know  $h$  is...?”, and “How did you use the volume formula?”, etc.). Students can borrow ideas and language from each partner to strengthen their final explanation.

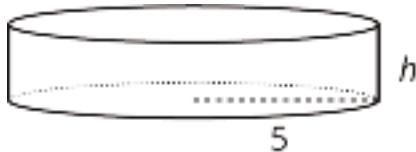
*Design Principle(s): Optimize output (for explanation)*

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### Student Task Statement

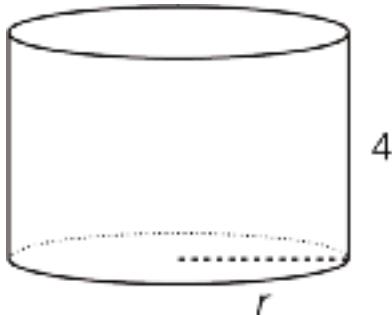
The volume  $V$  of a cylinder with radius  $r$  is given by the formula  $V = \pi r^2 h$ .

1. The volume of this cylinder with radius 5 units is  $50\pi$  cubic units. This statement is true:  
 $50\pi = 5^2\pi h$



What does the height of this cylinder have to be? Explain how you know.

2. The volume of this cylinder with height 4 units is  $36\pi$  cubic units. This statement is true:  
 $36\pi = r^2\pi 4$



What does the radius of this cylinder have to be? Explain how you know.

### Student Response

1. The statement  $50\pi = 5^2\pi h$  is equivalent to  $50 = 25h$ . Since 50 is 25 times 2,  $h = 2$  units.
2. The statement  $36\pi = r^2\pi 4$  is equivalent to  $36 = r^2 \cdot 4$ . Since 36 is 4 times 9,  $r^2 = 9$ . This implies  $r = 3$  units.

### Are You Ready for More?

Suppose a cylinder has a volume of  $36\pi$  cubic inches, but it is not the same cylinder as the one you found earlier in this activity.

1. What are some possibilities for the dimensions of the cylinder?
2. How many different cylinders can you find that have a volume of  $36\pi$  cubic inches?

### Student Response

1. The volume for the cylinder is  $36\pi = \pi \cdot r^2 \cdot h$ , which implies  $36 = h \cdot r^2$ . Answers vary. Sample responses: the cylinder could have  $r = 3$  and  $h = 4$  or  $r = 9$  and  $h = \frac{4}{9}$ .
2. There are an infinite number of cylinders with a volume of  $36\pi$  cubic inches. No matter what value for  $r$  is chosen, a value for  $h$  can be calculated using the formula  $36\pi = h \cdot r^2 \cdot \pi$ .

## Activity Synthesis

Select previously identified students to explain the strategies they used to find the missing dimension in each problem. If not brought up in students' explanations. Discuss the following strategies and explanations:

- Guess and check: plug in numbers for  $h$ , a value that make the statements true. Since the solutions for these problems are small whole numbers, this strategy works well. In other situations, this strategy may be less efficient.
- Divide each side of the equation by the same value to solve for the missing variable: for example, divide each side of  $36\pi = r^2\pi 4$  by the common factor,  $4\pi$ . It's important to remember  $\pi$  is a number that can be multiplied and divided like any other factor.
- Use the structure of the equation to reason about the missing variable: for example,  $50\pi$  is double  $25\pi$ , so the missing value must be 2.

## 14.3 Cylinders with Unknown Dimensions

15 minutes

The purpose of this activity is for students to use the structure of the volume formula for cylinders to find missing dimensions of a cylinder given other dimensions. Students are given the image of a generic cylinder with marked dimensions for the radius, diameter, and height to help their reasoning about the different rows in the table..

While completing the table, students work with approximations and exact values of  $\pi$  as well as statements that require reasoning about squared values. The final row of the table asks students to find missing dimensions given an expression representing volume that uses letters to represent the height and the radius. This requires students to manipulate expressions consisting only of variables representing dimensions.

Encourage students to make use of work done in some rows to help find missing information in other rows. Identify students who use this strategy and ask them to share during the discussion.

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR8: Discussion Supports

### Launch

Give students 6–8 minutes of work time followed by a whole-class discussion.

If short on time, consider assigning students only some of the rows to complete.

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### Access for Students with Disabilities

*Engagement: Internalize Self Regulation.* Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to choose and complete five out of eight rows in the table. Encourage students to select rows that contain varied information.

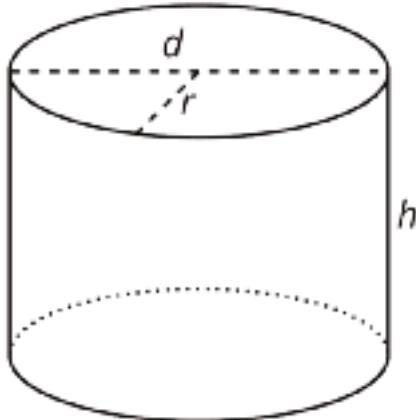
*Supports accessibility for: Organization; Attention*

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### Anticipated Misconceptions

Students might try to quickly fill in the missing dimensions without the proper calculations. Encourage students to use the volume of a cylinder equation and the given dimensions to figure out the unknown dimensions.

#### Student Task Statement



Each row of the table has information about a particular cylinder. Complete the table with the missing dimensions.

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume (cubic units)
	3		5	
12				$108\pi$
			11	$99\pi$
8				$16\pi$
			100	$16\pi$
	10			$20\pi$
20				314
			$b$	$\pi \cdot b \cdot a^2$

### Student Response

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume (cubic units)
6	3	$9\pi$	5	$45\pi$
12	6	$36\pi$	3	$108\pi$
6	3	$9\pi$	11	$99\pi$
8	4	$16\pi$	1	$16\pi$
0.8	0.4	$0.16\pi$	100	$16\pi$
20	10	$100\pi$	0.2 (or $\frac{1}{5}$ )	$20\pi$
20	10	$100\pi$	1	314
$2a$	$a$	$a^2\pi$	$b$	$\pi \cdot b \cdot a^2$

### Activity Synthesis

Select previously identified students to share their strategies. Ask students:

- “What patterns did you see as you filled out the table?” (Sample reasoning: Rows that had the same base area were easier to compare because their volume was the base area times height.)
- “Look at rows 1 and 3 in the table. How did having one row filled out help you fill out the other more efficiently?” (If the base areas were the same, then the radius and diameter must be the same also.)
- “How did you reason about the last row?”

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Provide sentence frames to support students as they share their strategies for completing the table. For example, “I noticed \_\_\_\_\_ in the (rows/columns).” or “I noticed \_\_\_\_\_, and it tells me that \_\_\_\_\_.”

*Design Principle(s): Support sense-making; Optimize output (for justification)*

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## Lesson Synthesis

Working in groups of 2, tell students to choose one partner to name a value for the radius and one partner to name a value for the volume of a cylinder. Together, partners determine the height and make a sketch of their cylinder, including labels on the dimensions of their sketch. Display sketches and invite students to share their strategies for determining height.

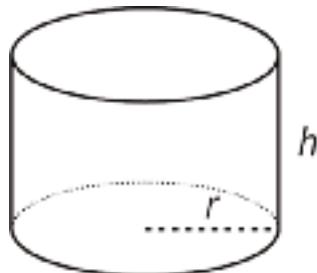
## 14.4 Find the Height

Cool Down: 5 minutes

### Addressing

- 8.G.C.9

#### Student Task Statement



This cylinder has a volume of  $12\pi$  cubic inches and a diameter of 4 inches. Find the cylinder's radius and height.

## Student Response

The radius is 2 inches, and the height is 3 inches. Since the diameter is 4 inches, the radius is half of 4 inches. The volume is  $12\pi = 2^2\pi h$ , which means  $12\pi = 4\pi h$  and  $h = 3$ .

## Student Lesson Summary

In an earlier lesson we learned that the volume,  $V$ , of a cylinder with radius  $r$  and height  $h$  is

$$V = \pi r^2 h$$

We say that the volume depends on the radius and height, and if we know the radius and height, we can find the volume. It is also true that if we know the volume and one dimension (either radius or height), we can find the other dimension.

For example, imagine a cylinder that has a volume of  $500\pi \text{ cm}^3$  and a radius of 5 cm, but the height is unknown. From the volume formula we know that

$$500\pi = \pi \cdot 25 \cdot h$$

must be true. Looking at the structure of the equation, we can see that  $500 = 25h$ . That means that the height has to be 20 cm, since  $500 \div 25 = 20$ .

Now imagine another cylinder that also has a volume of  $500\pi \text{ cm}^3$  with an unknown radius and a height of 5 cm. Then we know that

$$500\pi = \pi \cdot r^2 \cdot 5$$

must be true. Looking at the structure of this equation, we can see that  $r^2 = 100$ . So the radius must be 10 cm.

# Lesson 14 Practice Problems

## Problem 1

### Statement

Complete the table with all of the missing information about three different cylinders.

diameter of base (units)	area of base (square units)	height (units)	volume (cubic units)
4		10	
6			$63\pi$
	$25\pi$	6	

### Solution

Answers vary. Sample response:

diameter of base (units)	area of base (square units)	height (units)	volume (cubic units)
4	$4\pi$	10	$40\pi$
6	$9\pi$	7	$63\pi$
10	$25\pi$	6	$150\pi$

## Problem 2

### Statement

A cylinder has volume  $45\pi$  and radius 3. What is its height?

### Solution

5 units (Solve  $45\pi = \pi \cdot 3^2 \cdot h$ .)

## Problem 3

### Statement

Three cylinders have a volume of  $2826 \text{ cm}^3$ . Cylinder A has a height of 900 cm. Cylinder B has a height of 225 cm. Cylinder C has a height of 100 cm. Find the radius of each cylinder. Use 3.14 as an approximation for  $\pi$ .

## Solution

- Cylinder A has a radius of 1 cm.
- Cylinder B has a radius of 2 cm.
- Cylinder C has a radius of 3 cm.

## Problem 4

### Statement

A gas company's delivery truck has a cylindrical tank that is 14 feet in diameter and 40 feet long.

- Sketch the tank, and mark the radius and the height.
- How much gas can fit in the tank?

### Solution

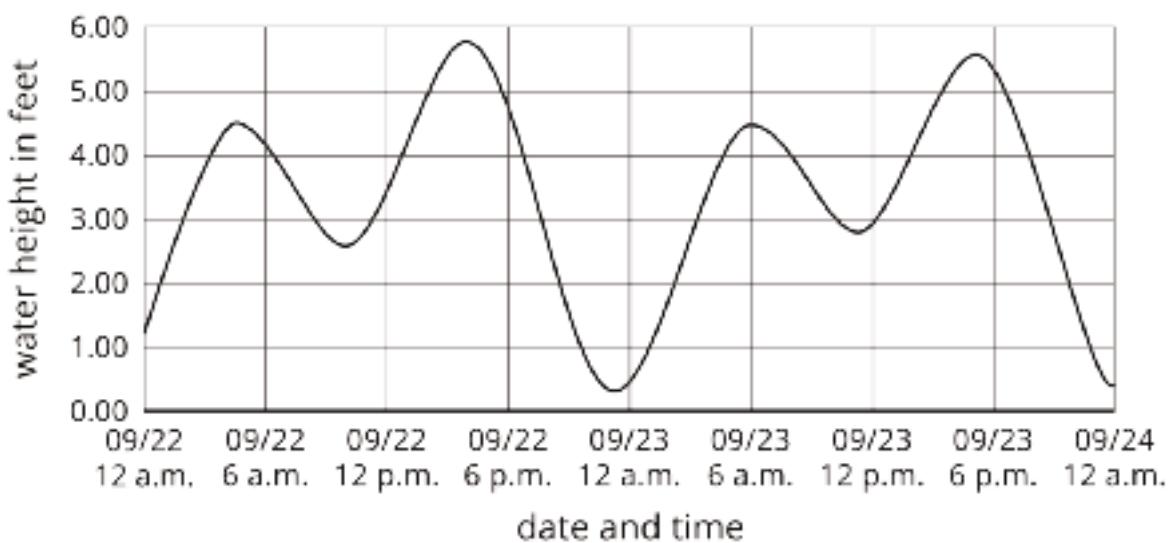
- Answers vary. The radius is 7 feet, and the height is 40 feet.
- About 6,158 cubic feet (The volume of the cylinder is given by  $V = \pi \cdot r^2 \cdot h$  where  $r = 7$  and  $h = 40$ . Using a close approximation of  $\pi$  gives an approximate volume of 6,158 cubic feet, but different answers may be found if a different approximation of  $\pi$  is used.)

(From Unit 5, Lesson 13.)

## Problem 5

### Statement

Here is a graph that shows the water height of the ocean between September 22 and September 24, 2016 in Bodega Bay, CA.



- Estimate the water height at 12 p.m. on September 22.
- How many times was the water height 5 feet? Find two times when this happens.
- What was the lowest the water got during this time period? When does this occur?
- Does the water ever reach a height of 6 feet?

## Solution

- 3.5 feet
- 4 times: approximately 2 p.m. and 6 p.m. on 9/22, 3 p.m. and 7 p.m. on 9/23
- The lowest height was at about 11 p.m. on 9/22. The water height then was about 0.3 feet.
- No, the highest it ever got was about 5.8 feet.

(From Unit 5, Lesson 5.)

# Lesson 15: The Volume of a Cone

## Goals

- Calculate the volume of a cone and cylinder given the height and radius, and explain (orally) the solution method.
- Compare the volumes of a cone and a cylinder with the same base and height, and explain (orally and in writing) the relationship between the volumes.

## Learning Targets

- I can find the volume of a cone in mathematical and real-world situations.
- I know the formula for the volume of a cone.

## Lesson Narrative

In this lesson students start working with cones, and learn that the volume of a cone is  $\frac{1}{3}$  the volume of a cylinder with a congruent base and the same height. First, students learn a method for quickly sketching a cone, and the meaning of the *radius* and *height* of a cone. Then they watch a video (or if possible, a live demonstration) showing that it takes three cones of water to fill a cylinder with the same radius and height. At this point, it is taken as a mysterious and beautiful fact that the volume of a cone is one third the volume of the associated cylinder. A proof of this fact requires mathematics beyond grade level.

Students write the volume of a cone given a specific volume of a cylinder with the same base and height, and vice versa. Then they use the formula for the volume of a cylinder learned in previous lessons to write the general formula  $V = \frac{1}{3}\pi r^2 h$  for the volume,  $V$ , of a cone in terms of its height,  $h$ , and radius,  $r$ . Finally, students practice computing the volumes of some cones. There are opportunities for further practice in the next lesson.

## Alignments

### Addressing

- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## Instructional Routines

- MLR3: Clarify, Critique, Correct
- MLR8: Discussion Supports
- Notice and Wonder
- Think Pair Share

## Required Preparation

For the Which Has a Larger Volume activity, it is suggested that students have access to geometric solids.

During the From Cylinders to Cones activity, students will need to view a video. Alternatively, do a demonstration with a cone that could be filled with water and poured into a cylinder.

### Student Learning Goals

Let's explore cones and their volumes.

## 15.1 Which Has a Larger Volume?

**Warm Up:** 5 minutes

The purpose of this activity is for students to think about how the volume of a cone might relate to the volume of a cylinder with the same base and height. Additionally, students learn one method for sketching a cone. In this activity, just elicit students' best guess about how many cone-contents would fit into the cylinder (or, what fraction of the cylinder's volume is the cone's volume). In the next activity, they will watch a demonstration that verifies the actual amount.

### Addressing

- 8.G.C.9

### Instructional Routines

- Think Pair Share

### Launch

If you have access to appropriate geometric solids that include a cylinder and a cone with congruent bases and equal heights, consider showing these to students, even passing them around for students to hold if time permits.

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time, followed by time to discuss fractional amount with partner. Follow with a whole-class discussion.

### Anticipated Misconceptions

If students think the two shapes will have the same volume, ask them to imagine dropping the cone into the cylinder and having extra space around the cone and still inside the cylinder.

### Student Task Statement

The cone and cylinder have the same height, and the radii of their bases are equal.

1. Which figure has a larger volume?
2. Do you think the volume of the smaller one is more or less than  $\frac{1}{2}$  the volume of the larger one? Explain your reasoning.
3. Sketch two different sized cones. The oval doesn't have to be on the bottom! For each drawing, label the cone's radius with  $r$  and height with  $h$ .

Here is a method for quickly sketching a cone:



- Draw an oval.
- Draw a point centered above the oval.
- Connect the edges of the oval to the point.
- Which parts of your drawing would be hidden behind the object? Make these parts dashed lines.

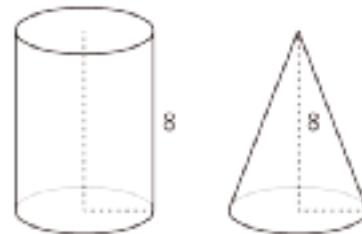
### Student Response

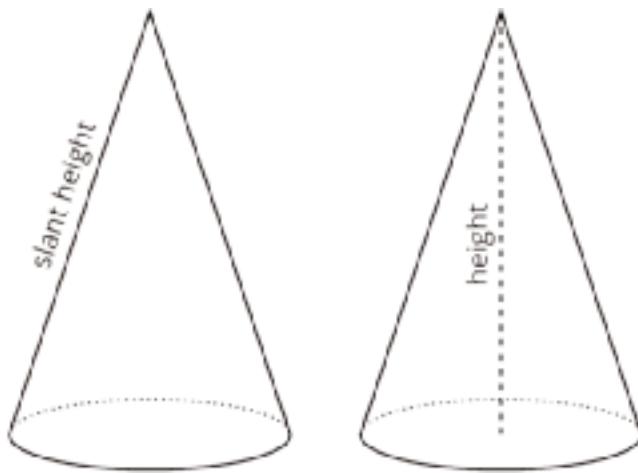
1. The cylinder has a larger volume.
2. Answers vary.
3. Answers vary.

### Activity Synthesis

Invite students to share their answers to the first two questions. The next activity includes a video that shows that it takes 3 cones to fill a cylinder that has the same base and height as the cone, so it is not necessary that students come to an agreement about the second question, just solicit student's best guesses, and tell them that we will find out the actual fractional amount in the next activity.

End the discussion by selecting 2–3 students to share their sketches. Display these for all to see and compare the different heights and radii. If no student draws a perpendicular height or slant height, display the image shown here for all to see and remind students that in earlier units we learned that height creates a right angle with something in the figure. In the case of the cones, the height is perpendicular to the circular base.





## 15.2 From Cylinders to Cones

20 minutes

In this activity, students use the relationship that the volume of a cone is  $\frac{1}{3}$  of the volume of a cylinder to calculate the volume of various cones. Students start by watching a video (or demonstration) that shows that it takes the contents of 3 cones to fill the cylinder when they have congruent bases and equal heights. Students use this information to calculate the volume of various cones and cylinders. For the last question, identify students who:

- write the equation as  $\frac{1}{3}V$  (or  $V \div 3$ ), where  $V$  represents the volume of a cylinder with the same base and height as the cone.
- write the equation in terms of  $r$  and  $h$  ( $V = \frac{1}{3}\pi r^2 h$ ).

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR8: Discussion Supports
- Notice and Wonder

### Launch

Video 'How Many Cones Does it Take to Fill a Cylinder with the Same Base and Height?' available here: <https://player.vimeo.com/video/309581286>.

Either conduct a demonstration or show the video and tell students to write down anything they notice or wonder while watching. Pause for a whole-class discussion. Record what students noticed and wondered for all to see. Ensure that students notice that it takes the contents of 3 cones to fill

the cylinder, or alternatively, that the volume of the cone is  $\frac{1}{3}$  the volume of the cylinder. Then, set students to work on the questions in the task, followed by a whole-class discussion.

### Access for English Language Learners

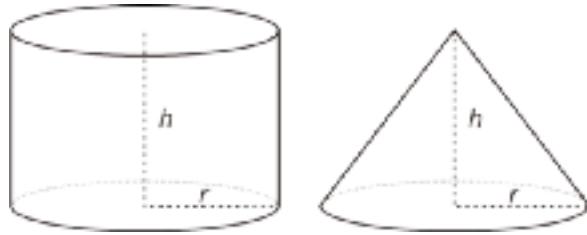
*Speaking: MLR8 Discussion Supports.* As students describe what they notice and wonder about from the video, revoice student ideas to demonstrate mathematical language use while incorporating gestures and referring to the context in the video. This will help students to produce and make sense of the language needed to communicate their own ideas about the relationship between the volume of cones and cylinders.

*Design Principle(s): Support sense-making; Optimize output (for explanation)*

### Student Task Statement

A cone and cylinder have the same height and their bases are congruent circles.

1. If the volume of the cylinder is  $90 \text{ cm}^3$ , what is the volume of the cone?
2. If the volume of the cone is  $120 \text{ cm}^3$ , what is the volume of the cylinder?
3. If the volume of the cylinder is  $V = \pi r^2 h$ , what is the volume of the cone? Either write an expression for the cone or explain the relationship in words.



### Student Response

1.  $30 \text{ cm}^3$ . The volume of a cone is  $\frac{1}{3}$  the volume of a cylinder and  $\frac{1}{3} \cdot 90$ .
2.  $360 \text{ cm}^3$ . The volume of the cylinder is 3 times larger than the volume of a cone and  $3 \cdot 120$ .
3. Answers vary. Sample responses:  $V = \frac{1}{3} \pi r^2 h$  (or equivalent) or "the volume of the cone is  $\frac{1}{3}$  that of the volume of the cylinder."

### Activity Synthesis

Select previously identified students to share the volume equation they wrote for the last question. Display examples for all to see and ask "Are these equations the same? How can you know for sure?" (The calculated volume is the same when you use both equations.)

If no student suggests it, connect  $\frac{1}{3}V$ , where  $V$  represents the volume of a cylinder with the same base and height as the cone, to the volume of the cone,  $\frac{1}{3}\pi r^2 h$ . Reinforce that these are equivalent expressions.

Add the formula  $V = \frac{1}{3}\pi r^2 h$  and a diagram of a cone to your classroom displays of the formulas being developed in this unit.

## 15.3 Calculate That Cone

**10 minutes**

The purpose of this activity is for students to calculate the volume of cones given their height and radius. Students are given a cylinder with the same height and radius and use the volume relationship they learned in the previous activity to calculate the volume of the cone. They then calculate the volume of a cone given a height and radius using the newly learned formula for volume of a cone. For the last problem, an image is not provided to give students the opportunity to sketch one if they need it.

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR3: Clarify, Critique, Correct

### Launch

Give students 3–5 minutes of quiet work time followed by a whole-class discussion.

### Anticipated Misconceptions

In the first problem, students might use 10 as the radius length. Ask students what the length of 10 in the picture is called. Ask students to recall the formula for the volume of a cylinder.

For students who are not sure where to begin the last problem since it does not have an image, encourage them to sketch and label their own.

### Student Task Statement

1. Here is a cylinder and cone that have the same height and the same base area.

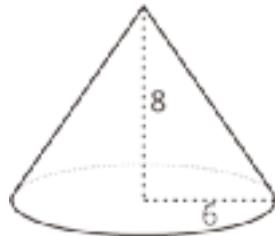
What is the volume of each figure? Express your answers in terms of  $\pi$ .



2. Here is a cone.

a. What is the area of the base? Express your answer in terms of  $\pi$ .

b. What is the volume of the cone? Express your answer in terms of  $\pi$ .



3. A cone-shaped popcorn cup has a radius of 5 centimeters and a height of 9 centimeters. How many cubic centimeters of popcorn can the cup hold? Use 3.14 as an approximation for  $\pi$ , and give a numerical answer.

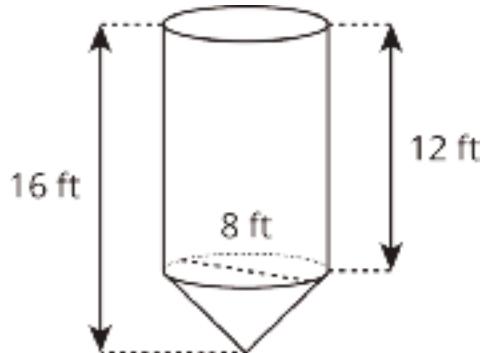
### Student Response

1. Cylinder:  $100\pi$ ; cone:  $\frac{100}{3}\pi$ . To calculate the volume of the cylinder, find the area of the base and multiply it by the height of the cylinder. The area of the base is  $\pi r^2$ , and the height is 4 ( $\pi 5^2 \cdot 4 = 100\pi$ ). The volume of the cone is  $\frac{1}{3}$  of the cylinder's volume, which is  $\frac{100}{3}\pi$ .
2. a.  $36\pi$  square units because  $A = \pi r^2$ .  
b.  $96\pi$  cubic units because the volume is  $\frac{1}{3}$  of the area of the base multiplied by the height of the cone ( $\frac{1}{3} \cdot 36\pi \cdot 8 = 96\pi$ ).
3.  $235.5 \text{ cm}^3$  because with a radius of 5 cm and a height of 9 cm, the volume is calculated with the equation  $V = \frac{1}{3} \cdot 3.14 \cdot 5^2 \cdot 9$ .

### Are You Ready for More?

A grain silo has a cone shaped spout on the bottom in order to regulate the flow of grain out of the silo. The diameter of the silo is 8 feet. The height of the cylindrical part of the silo above the cone spout is 12 feet while the height of the entire silo is 16 feet.

How many cubic feet of grain are held in the cone spout of the silo? How many cubic feet of grain can the entire silo hold?



### Student Response

The entire grain silo holds  $\frac{640}{3}\pi$  cubic feet of grain.

The cone holds  $\frac{64}{3}\pi$  cubic feet of grain. Since the radius is 4 feet and the height of the cone is also 4 feet ( $16 - 12$ ), the volume is  $\frac{1}{3}\pi 4^2 \cdot 4$ . Calculate the volume of the cylinder ( $\pi 4^2 \cdot 12$ ), and add it to the volume of the cone to get the volume of the entire silo.

### Activity Synthesis

For the first problem, invite students to explain how they calculated the volume of both figures and have them share the different strategies they used. If not mentioned by students bring up these strategies:

- Calculate the volume of the cylinder, then divide volume of cylinder by 3 to get the volume of the cone.
- Calculate the volume of the cylinder, then multiply volume of cylinder by  $\frac{1}{3}$  to get the volume of the cone.
- Calculate the volume of the cone, then multiply volume of cone by 3 to get the volume of the cylinder.

For the third problem, ask students to share any sketches they came up with to help them calculate the answer. Explain to students that sometimes we encounter problems that don't have a visual example and only a written description. By using sketches to help to visualize what is being described in a problem, we can better understand what is being asked.

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### Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.

*Supports accessibility for: Language; Social-emotional skills; Attention*

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### Access for English Language Learners

*Writing: MLR3 Clarify, Critique, Correct.* Present an incorrect response to the first question that reflects a possible misunderstanding from the class. For example, "The volume is  $4000\pi$  because  $\pi * 4 * 10^2$ ." Prompt students to critique the reasoning (e.g., ask, "Do you agree with the author's reasoning? Why or why not?") and then write feedback to the author that identifies the misconception and how to improve on his/her work. Listen for students who tie their feedback to the difference between the radius and diameter and use the academic vocabulary (e.g., height, radius, diameter, cylinder, cone, volume, etc.). This will help students evaluate, and improve on, the written mathematical arguments of others and highlight the distinction between and the importance of radii when calculating volume of cylinders and cones.

*Design Principle(s): Maximize meta-awareness*

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## Lesson Synthesis

Have students summarize the highlights of the lesson by asking:

- “What is the relationship between the volume of a cylinder and the volume of a cone?” (The volume of a cone is  $\frac{1}{3}$  of the volume of a cylinder or the volume of the cylinder is 3 times the volume of the cone.)
- “If we know the volume of a cone, how do we calculate the volume of a cylinder that has the same height and base area?” (We can multiply the volume of the cone by 3.)
- “If we know the volume of a cylinder, how do we calculate the volume of a cone that has the same height and base area?” (We can multiply the volume of the cylinder by  $\frac{1}{3}$ .)
- “If a cylinder and a cone have the same base, how tall does the cone have to be relative to the cylinder so that they both have the same volume?” (The cone needs to have a height 3 times the height of the cylinder for the two shapes to have the same volume.)

## 15.4 Calculate Volumes of Two Figures

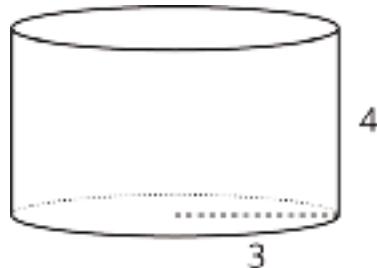
Cool Down: 5 minutes

### Addressing

- 8.G.C.9

#### Student Task Statement

A cone with the same base but a height 3 times taller than the given cylinder exists. What is the volume of each figure? Express your answers in terms of  $\pi$ .



#### Student Response

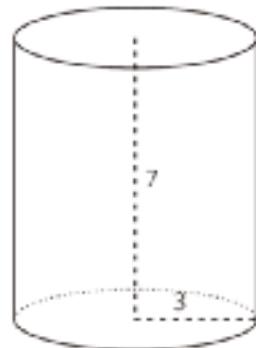
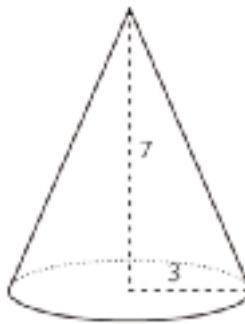
cylinder:  $36\pi$  cubic units because  $\pi \cdot 3^2 \cdot 4 = 36\pi$

cone:  $36\pi$  cubic units because  $\frac{1}{3}\pi \cdot 3^2 \cdot 12 = 36\pi$

#### Student Lesson Summary

If a cone and a cylinder have the same base and the same height, then the volume of the cone is  $\frac{1}{3}$  of the volume of the cylinder. For example, the cylinder and cone shown here both have a base with radius 3 feet and a height of 7 feet.

The cylinder has a volume of  $63\pi$  cubic feet since  $\pi \cdot 3^2 \cdot 7 = 63\pi$ . The cone has a volume that is  $\frac{1}{3}$  of that, or  $21\pi$  cubic feet.



If the radius for both is  $r$  and the height for both is  $h$ , then the volume of the cylinder is  $\pi r^2 h$ . That means that the volume,  $V$ , of the cone is

$$V = \frac{1}{3}\pi r^2 h$$

# Lesson 15 Practice Problems

## Problem 1

### Statement

A cylinder and cone have the same height and radius. The height of each is 5 cm, and the radius is 2 cm. Calculate the volume of the cylinder and the cone.

### Solution

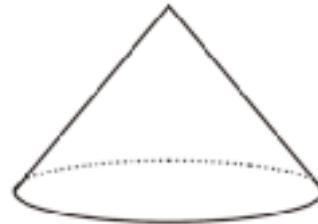
Cylinder:  $20\pi \text{ cm}^3$ , cone:  $\frac{20}{3}\pi \text{ cm}^3$

## Problem 2

### Statement

The volume of this cone is  $36\pi$  cubic units.

What is the volume of a cylinder that has the same base area and the same height?



### Solution

$108\pi$ , about 339 cubic units (The volume of the cylinder is exactly three times the volume of the corresponding cone.)

## Problem 3

### Statement

A cylinder has a diameter of 6 cm and a volume of  $36\pi \text{ cm}^3$ .

- a. Sketch the cylinder.
- b. Find its height and radius in centimeters.
- c. Label your sketch with the cylinder's height and radius.

### Solution

Answers vary. The radius of the cylinder is 3 cm and the height of the cylinder is 4 cm since  $\frac{36}{3^2} = 4$ .

(From Unit 5, Lesson 14.)

## Problem 4

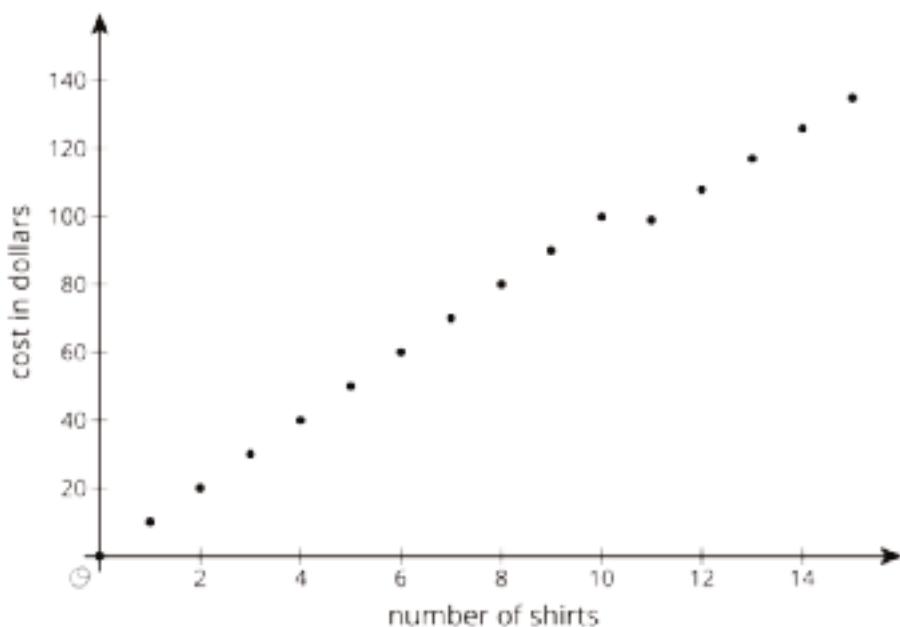
### Statement

Lin wants to get some custom T-shirts printed for her basketball team. Shirts cost \$10 each if you order 10 or fewer shirts and \$9 each if you order 11 or more shirts.

- Make a graph that shows the total cost of buying shirts, for 0 through 15 shirts.
- There are 10 people on the team. Do they save money if they buy an extra shirt? Explain your reasoning.
- What is the slope of the graph between 0 and 10? What does it mean in the story?
- What is the slope of the graph between 11 and 15? What does it mean in the story?

### Solution

a.



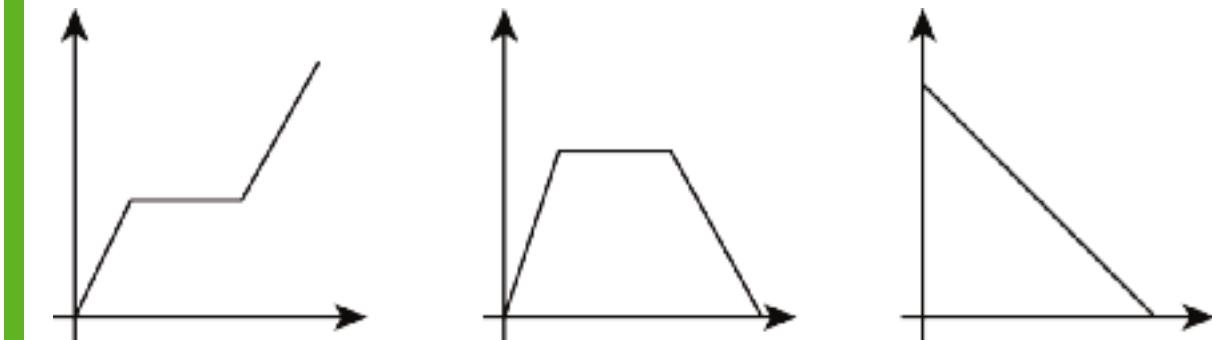
- Yes, 11 shirts cost \$99, and 10 shirts cost \$100. Even if they split the cost of the extra shirt, they still save \$1 altogether, or \$0.10 apiece. If they can find someone to buy the extra shirt, they save \$1 each.
- 10 dollars per shirt: the price per shirt when you buy 10 or fewer
- 9 dollars per shirt: the price per shirt when you buy 11 or more

(From Unit 5, Lesson 10.)

## Problem 5

### Statement

In the following graphs, the horizontal axis represents time and the vertical axis represents distance from school. Write a possible story for each graph.



### Solution

Answers vary. Sample response:

- Left graph: A student leaves school and walks to a friend's house that is halfway between school and their house. After spending some time at a friend's house, the student continues walking home.
- Center graph: An athlete leaves school to go home for a while, then returns to school for a game later in the evening.
- Right graph: A parent starts from their workplace and drives directly to school to pick up their daughter.

(From Unit 5, Lesson 6.)

# Lesson 16: Finding Cone Dimensions

## Goals

- Calculate the value of one dimension of a cylinder, and explain (orally and in writing) the reasoning.
- Compare volumes of a cone and cylinder in context, and justify (orally) which volume is a better value for a given price.
- Create a table of dimensions of cylinders, and describe (orally) patterns that arise.

## Learning Targets

- I can find missing information of about a cone if I know its volume and some other information.

## Lesson Narrative

As they did with cylinders in a previous lesson, students in this lesson use the formula  $V = \frac{1}{3}\pi r^2 h$  to find the radius or height of a cone given its volume and the other dimension. Then they apply their understanding about the volumes of cylinders and cones to decide which popcorn container and price offers the best deal. Depending on the amount of guidance students are given, this last activity can be an opportunity to explain their reasoning and critique the reasoning of others (MP3).

## Alignments

### Building On

- 6.EE.B.5: Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

### Addressing

- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

### Instructional Routines

- MLR5: Co-Craft Questions
- MLR8: Discussion Supports
- Number Talk
- Think Pair Share

### Student Learning Goals

Let's figure out the dimensions of cones.

# 16.1 Number Talk: Thirds

## Warm Up: 5 minutes

The purpose of this number talk is to elicit understandings and review strategies students have for finding the unknown value in an equation that involves the fraction  $\frac{1}{3}$ . These understandings will be helpful later in this lesson when students are solving for the unknown length of the radius or height of a cone given its volume.

While four problems are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem, saving most of the time for the final question.

## Building On

- 6.EE.B.5

## Instructional Routines

- MLR8: Discussion Supports
- Number Talk

## Launch

Reveal one problem at a time. Give students brief quiet think time for each problem, and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

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## Access for Students with Disabilities

*Representation: Internalize Comprehension.* To support working memory, provide students with sticky notes or mini whiteboards.

*Supports accessibility for: Memory; Organization*

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## Anticipated Misconceptions

Students might want to divide 27 by 3 or multiply 27 by  $\frac{1}{3}$  to solve for  $h$ . Ask students what number you can multiply by  $\frac{1}{3}$  (or divide by 3) to result in the number 27.

### Student Task Statement

For each equation, decide what value, if any, would make it true.

$$27 = \frac{1}{3}h$$

$$27 = \frac{1}{3}r^2$$

$$12\pi = \frac{1}{3}\pi a$$

 
$$12\pi = \frac{1}{3}\pi b^2$$

## Student Response

- 81
- 9
- 36
- 6

## Activity Synthesis

Ask students to share their strategies for each problem, in particular highlighting the ways students worked with the fraction  $\frac{1}{3}$ . Record and display their explanations for all to see. To involve more students in the conversation, consider asking:

- "Who can restate \_\_\_'s reasoning in a different way?"
- "Did anyone have the same strategy but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to \_\_\_'s strategy?"
- "Do you agree or disagree? Why?"

---

### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.*.: Display sentence frames to support students when they explain their strategy. For example, "First, I \_\_\_\_ because . . ." or "I noticed \_\_\_\_ so I . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.

*Design Principle(s): Optimize output (for explanation)*

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## 16.2 An Unknown Radius

**Optional: 5 minutes**

The purpose of this activity is for students to calculate the radius of the cone given the volume and height. This activity is similar to an activity in a previous lesson where students calculated the radius of a cylinder given the volume and height. A difference here is that solving for the unknown takes an additional step to deal with the  $\frac{1}{3}$ .

Encourage students to connect the strategies that they used in the warm-up to this problem. Identify students who make the connections and ask them to share during the discussion.

## Addressing

- 8.G.C.9

## Instructional Routines

- MLR8: Discussion Supports

## Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by time to share their strategies with their partner. Follow with a whole-class discussion.

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Display sentence frames for students to use when they share their strategies with their partners. For example, “The radius is \_\_\_\_\_ because \_\_\_\_ .”, or “To find the radius, first I \_\_\_\_\_. Then, I \_\_\_\_\_. ” or “I used the  $\frac{1}{3}$  by \_\_\_\_\_. ” Ask students to further clarify their strategy with operational words, such as “multiply by the reciprocal,” or “divide by.” This will help students practice and develop language for describing their strategies for working with fractions. *Design Principle(s): Support sense-making*

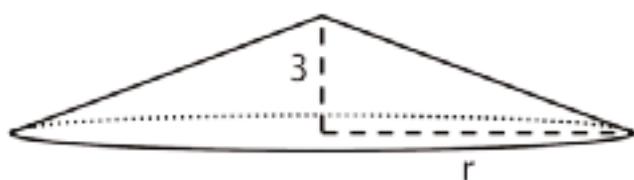
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## Anticipated Misconceptions

Students might struggle with the  $\frac{1}{3}$  while solving for the unknown radius length. Encourage students to think about the strategies they used in the warm-up to work with the  $\frac{1}{3}$ .

### Student Task Statement

The volume  $V$  of a cone with radius  $r$  is given by the formula  $V = \frac{1}{3}\pi r^2 h$ .



The volume of this cone with height 3 units and radius  $r$  is  $V = 64\pi$  cubic units. This statement is true:

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3$$

What does the radius of this cone have to be? Explain how you know.

## Student Response

The radius must be 8 units because the equation simplifies to  $64 = r^2$ , so  $r$  must be 8.

## Activity Synthesis

Select previously identified students to share the strategies they used to calculate the radius. Ask students, “Which do you think is more challenging: calculating the radius of a cone or the radius of the cylinder when the height and volume of the shapes are known? Why?” (I think they are the same, you just have to work with  $\frac{1}{3}$  when calculating the radius of the cone, which adds an extra step.)

# 16.3 Cones with Unknown Dimensions

15 minutes

The purpose of this activity is for students to use the structure of the volume formula for cones to calculate missing dimensions of a cone given other dimensions. Students are given the image of a generic cone with marked dimensions for the radius, diameter, and height to help their reasoning about the different rows in the table.

While completing the table, students work with approximations and exact values of  $\pi$  as well as statements that require reasoning about squared values. The final row of the table asks students to find missing dimensions given an expression representing volume that uses letters to represent the height and the radius. This requires students to manipulate expressions consisting only of variables representing dimensions.

Encourage students to make use of work done in some rows to help find missing information in other rows. Identify students who use this strategy, and ask them to share during the discussion.

## Addressing

- 8.G.C.9

## Instructional Routines

- MLR8: Discussion Supports

## Launch

Give students 6–8 minutes of quiet work time, followed by a whole-class discussion.

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## Access for Students with Disabilities

*Action and Expression: Internalize Executive Functions.* Chunk this task into more manageable parts to support students who benefit from support with organization and problem solving. For example ask students to complete two lines of the table at a time and then assess for accuracy and comprehension.

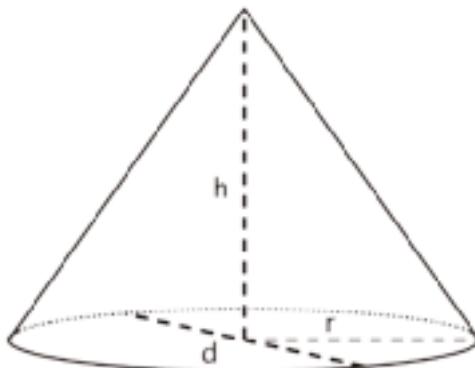
*Supports accessibility for: Organization; Attention*

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## Anticipated Misconceptions

Students might try to quickly fill in the missing dimensions without the proper calculations. Encourage students to use the volume of a cone equation and the given dimensions to figure out the unknown dimensions.

### Student Task Statement



Each row of the table has some information about a particular cone. Complete the table with the missing dimensions.

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
	4		3	
	$\frac{1}{3}$		6	
		$144\pi$	$\frac{1}{4}$	
20				$200\pi$
			12	$64\pi$
			3	3.14

## Student Response

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
8	4	$16\pi$	3	$16\pi$
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}\pi$	6	$\frac{2}{9}\pi$
24	12	$144\pi$	$\frac{1}{4}$	$12\pi$
20	10	$100\pi$	6	$200\pi$
8	4	$16\pi$	12	$64\pi$
2	1	$\pi$	3	3.14

## Are You Ready for More?

A *frustum* is the result of taking a cone and slicing off a smaller cone using a cut parallel to the base.



Find a formula for the volume of a frustum, including deciding which quantities you are going to include in your formula.

## Student Response

Answers vary. Sample response:

Imagine the original cone before the top piece is cut off. Then we will let  $R$  be the larger radius of the frustum,  $r$  be the smaller radius of the frustum,  $H$  be the height of the original cone, and  $h$  be the height of the conical piece cut off. Then the formula for the frustum is the volume of a cone with radius  $R$  and height  $H$  minus the volume of the removed top, which has radius  $r$  and height  $h$ :

$$V = \frac{1}{3}(\pi R^2 H - \pi r^2 h)$$

Alternatively, let  $x$  be the height of the piece cut off, and  $h$  be the height of the frustum. Then we can calculate volume using

$$V = \frac{1}{3}\pi(R^2(h+x) - r^2x)$$

## Activity Synthesis

Select previously identified students to share the strategies they used to fill in the missing information. Ask students:

- “Which information, in your opinion, was the hardest to calculate?”
- “If you had to pick two pieces of information given in the table which information would you want? Why?”

Select a few rows of the table, and ask students how they might find the volume of a cylinder with the same radius and height as the cone. (Multiply by 3)

Make sure students understand that when working with the volume formula for either a cylinder or cone, if they know two out of three for radius, height and volume, they can always calculate the third.

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support whole-class discussion. For each strategy that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will help students to produce and make sense of the language needed to communicate their own ideas when calculating missing dimensions of a cone.

*Design Principle(s): Support sense-making; Optimize output (for explanation)*

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## 16.4 Popcorn Deals

10 minutes

The purpose of this activity is for students to reason about the volume of popcorn that a cone- and cylinder-shaped popcorn cup holds and the price they pay for it. Students start by picking the popcorn container they would buy (without doing any calculations) and then work with a partner to answer the question of which container is a better value. This activity is designed for the discussion to happen around a few different concepts:

- The volume is a lower estimate because there is still some popcorn coming out the top of the containers.
- The fact that one container (the cone) looks like it has more coming out of the top might sway students to think that the cone is a better deal. However, the difference in the volume amounts should support the fact that even with a taller and wider diameter the cone still holds less volume than the cylinder.

- Why would the movie theater purposefully sell a container that has less volume for more price?

Identify groups who see these connections while they are working on the task, and ask them to share their arguments during the discussion.

## Addressing

- 8.G.C.9

## Instructional Routines

- MLR5: Co-Craft Questions
- Think Pair Share

## Launch

Arrange students in groups of 2. Give students 1 minute of quiet think time to decide which popcorn container they would purchase. Students are not asked to use any written calculations to determine this answer. Poll the class and display the results for all to see which containers students chose. Keep this information displayed to refer back to during the discussion.

Give students 2–3 minutes of time to work with their partner to determine which container is a better value. Follow with a whole-class discussion.

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## Access for Students with Disabilities

*Action and Expression: Develop Expression and Communication.* Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, “First, I \_\_\_\_\_ because. . .,” “How did you get. . .?” and “Can you say more about . . .?”

*Supports accessibility for: Language; Organization*

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## Access for English Language Learners

*Writing, Conversing: MLR5 Co-Craft Questions.* Display only the opening statement (i.e., “A movie theater offers two containers.”) and the two images of the popcorn containers. Ask pairs of students to write possible questions that could be answered with mathematics. Invite 2–3 groups to share their questions with the class. Look for questions that ask students to compare the volumes of the two different containers. Next, reveal the question of the activity. This routine helps students consider the context of this problem and increases awareness about language used to talk about volume of cones and cylinders.

*Design Principle(s): Maximize meta-awareness; Support sense-making*

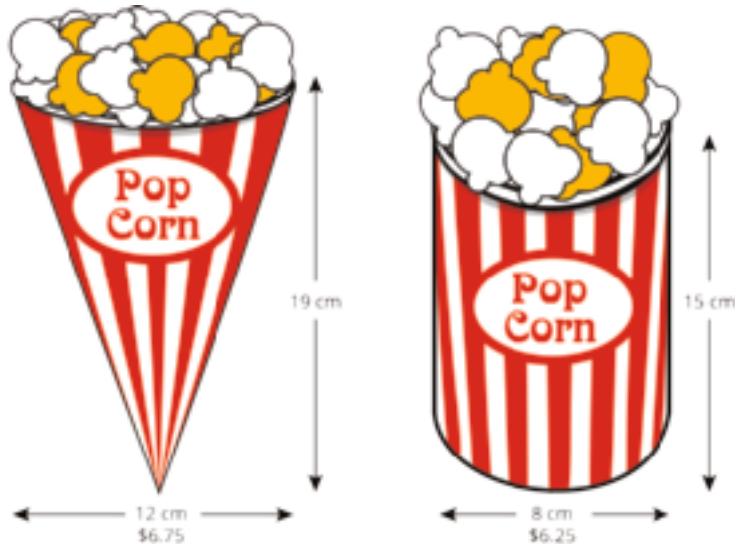
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## Anticipated Misconceptions

Students may try to use just the image to reason about which is a better deal. Make sure they understand that the image of the two containers is not to scale.

### Student Task Statement

A movie theater offers two containers:



Which container is the better value? Use 3.14 as an approximation for  $\pi$ .

### Student Response

The cylinder cup is a better value. The cone's volume is about 715.92 cubic centimeters and using the cost of the conical cup is about 106.07 cubic centimeters per dollar ( $715.92 \div 6.75 \approx 106.07$ ). The cylinder's volume is about 753.6 cubic centimeters and using the cost of the cylindrical cup is about 120.58 cubic centimeters per dollar ( $753.6 \div 6.25 \approx 120.58$ ). Since the cylinder gives you more volume per dollar, it is a better value.

### Activity Synthesis

Poll the class again for which container they would purchase. Display the new results next to the original results for all to see during the discussion.

Select previously identified groups to share their arguments for which container has a better value. Encourage groups to share details of their calculations and record these for all to see in order to mark any similarities and differences between the groups' arguments. Consider asking some of the following questions to help students think deeper about the situation:

- “Do you think your volume calculations overestimate or underestimate the amount of popcorn each container can hold?” (I think the calculations underestimate because the popcorn piles higher than the lip of the container.)

- “Why do you think movie theaters charged more for the cone?” (It may look like it has more volume to some people since it has a larger diameter and height. It may be easier to place into a cup holder.)
- “Do you think a lot of people would buy the cone over the cylinder?” (Yes. The 50 cent difference is not a lot, and since it is taller and wider, people might think the cone is bigger. Although the cylinder is a better value, there may be other considerations like how easy the cup is to hold or put in the cup holder in the seat, or whether you have time in line to calculate the value.)

## Lesson Synthesis

Display the image of the popcorn cone from the activity Popcorn Deals, including the dimensions of the cone. Ask students, “What size of cylinder cup would you need to have the same volume as the cone?”

Working in groups of 2, tell partners to determine the height and radius of a possible cylinder of equivalent volume, including making a sketch with labels on the dimensions. Display sketches and invite students to share the strategies they used to find their cylinders.

## 16.5 A Square Radius

Cool Down: 5 minutes

### Addressing

- 8.G.C.9

### Launch

Provide students with access to calculators.

#### Student Task Statement

Noah and Lin are making paper cones to hold popcorn to hand out at parent math night. They want the cones to hold  $9\pi$  cubic inches of popcorn. What are two different possible values for height  $h$  and radius  $r$  for the cones?

#### Student Response

Answers vary. Sample responses:

- Height and radius both 3 inches, since  $\frac{1}{3}\pi \cdot 3^2 \cdot 3 = 9\pi$ .
- Radius 2 inches and height 6.75 inches, since  $\frac{1}{3}\pi \cdot 2^2 \cdot 6.75 = 9\pi$ .
- Radius 1 inch and height 27 inches, since  $\frac{1}{3}\pi \cdot 1^2 \cdot 27 = 9\pi$ .
- Radius 9 inches and height  $\frac{1}{3}$  inches, since  $\frac{1}{3}\pi \cdot 9^2 \cdot \frac{1}{3} = 9\pi$ . (This cone may look more like a plate, but it solves the problem.)

## Student Lesson Summary

As we saw with cylinders, the volume  $V$  of a cone depends on the radius  $r$  of the base and the height  $h$ :

$$V = \frac{1}{3}\pi r^2 h$$

If we know the radius and height, we can find the volume. If we know the volume and one of the dimensions (either radius or height), we can find the other dimension.

For example, imagine a cone with a volume of  $64\pi \text{ cm}^3$ , a height of 3 cm, and an unknown radius  $r$ . From the volume formula, we know that

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3$$

Looking at the structure of the equation, we can see that  $r^2 = 64$ , so the radius must be 8 cm.

Now imagine a different cone with a volume of  $18\pi \text{ cm}^3$ , a radius of 3 cm, and an unknown height  $h$ . Using the formula for the volume of the cone, we know that

$$18\pi = \frac{1}{3}\pi 3^2 h$$

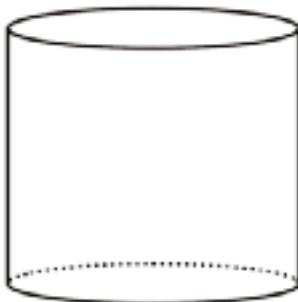
so the height must be 6 cm. Can you see why?

# Lesson 16 Practice Problems

## Problem 1

### Statement

The volume of this cylinder is  $175\pi$  cubic units.



What is the volume of a cone that has the same base area and the same height?

### Solution

$\frac{175}{3}\pi$ , about 183 cubic units (The volume of the cone is exactly one-third the volume of the corresponding cylinder.)

(From Unit 5, Lesson 15.)

## Problem 2

### Statement

A cone has volume  $12\pi$  cubic inches. Its height is 4 inches. What is its radius?

### Solution

3 inches. (The volume of a cone is given by  $V = \frac{1}{3}\pi r^2 h$ . With the information given:  $12\pi = \frac{1}{3}\pi r^2 \cdot 4$ ;  $12 = \frac{4}{3}r^2$ ;  $9 = r^2$  Then  $r = 3$ .)

## Problem 3

### Statement

A cone has volume  $3\pi$ .

- If the cone's radius is 1, what is its height?
- If the cone's radius is 2, what is its height?
- If the cone's radius is 5, what is its height?
- If the cone's radius is  $\frac{1}{2}$ , what is its height?

e. If the cone's radius is  $r$ , then what is the height?

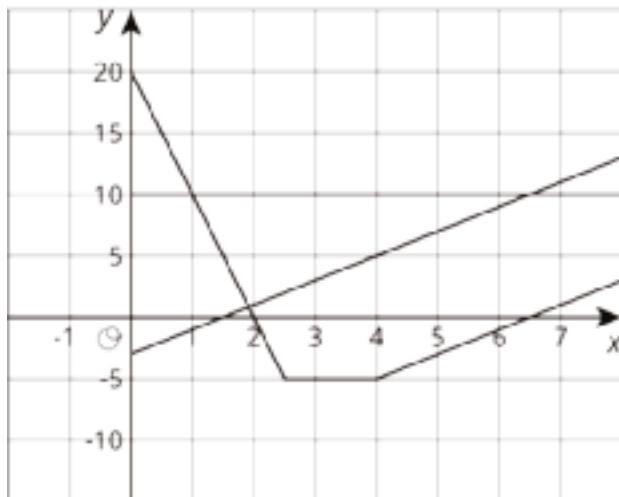
## Solution

- a. 9 units
- b.  $\frac{9}{4}$  units
- c.  $\frac{9}{25}$  units
- d. 36 units
- e.  $\frac{9}{r^2}$  units

## Problem 4

### Statement

Three people are playing near the water. Person A stands on the dock. Person B starts at the top of a pole and ziplines into the water, then climbs out of the water. Person C climbs out of the water and up the zipline pole. Match the people to the graphs where the horizontal axis represents time in seconds and the vertical axis represents height above the water level in feet.



## Solution

A is the constant graph at 10. B is the graph that includes the point  $(0, 20)$ . C is the graph that starts negative and increases. (Students may label the graphs A, B, and C or describe which person's story matches each graph.)

(From Unit 5, Lesson 6.)

## Problem 5

### Statement

A room is 15 feet tall. An architect wants to include a window that is 6 feet tall. The distance between the floor and the bottom of the window is  $b$  feet. The distance between the ceiling and the top of the window is  $a$  feet. This relationship can be described by the equation

$$a = 15 - (b + 6)$$

- a. Which variable is independent based on the equation given?
- b. If the architect wants  $b$  to be 3, what does this mean? What value of  $a$  would work with the given value for  $b$ ?
- c. The customer wants the window to have 5 feet of space above it. Is the customer describing  $a$  or  $b$ ? What is the value of the other variable?

### Solution

- a.  $b$
- b. It means the architect wants the bottom of the window to be 3 feet above the floor.  $a$  is 6.
- c.  $a$ .  $b$  is 4.

(From Unit 5, Lesson 3.)

# Section: Dimensions and Spheres

## Lesson 17: Scaling One Dimension

### Goals

- Create a graph and an equation to represent the function relationship between the volume of a cylinder and its height, and justify (orally) that the relationship is linear.
- Interpret (in writing) a point on a graph representing the volume of a cone as a function of its height, and explain (orally) how changing one dimension affects the other.

### Learning Targets

- I can create a graph the relationship between volume and height for all cylinders (or cones) with a fixed radius.
- I can explain in my own words why changing the height by a scale factor changes the volume by the same scale factor.

### Lesson Narrative

This lesson is optional. This is the first of a series of lessons building up to the formula for the volume of a sphere. In order to understand why that formula has the radius raised to the third power, students start studying how the volume of a three-dimensional figure changes when you scale one or more of its dimensions (length, width, height, radius). In this lesson they consider just one of the dimensions.

In the warm-up, students graph a proportional relationship and recall that in a proportional relationship the two quantities change by the same scale factor: when you multiply one of them by a scale factor the other one gets multiplied by the same scale factor. In the first activity, students consider a rectangular prism with two edges of constant length and one edge of variable length. They graph the volume of the prism as a function of the length and see that the volume is proportional to the length. They conclude that when you double the length the volume doubles. Then they investigate the volume of a cone as a function of its height when you keep the radius constant. Again they see that the volume is proportional to the height, and that when you halve the height you halve the volume. In the final activity they use a graph of this proportional relationship to find the radius.

The main purpose of the lesson is to understand that when you scale just one of the dimensions of a three-dimensional figure by a factor, the volume scales by the same factor. A secondary purpose is to see some examples of linear functions arising out of geometry. (A proportional relationship is a particular kind of linear function.)

## Alignments

### Addressing

- 8.F.A.1:

Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.

- 8.F.B: Use functions to model relationships between quantities.
- 8.G.C: Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.
- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

### Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR5: Co-Craft Questions
- MLR7: Compare and Connect
- Think Pair Share

### Student Learning Goals

Let's see how changing one dimension changes the volume of a shape.

## 17.1 Driving the Distance

### Warm Up: 5 minutes

The purpose of this warm-up is for students to jump back into recognizing functions and determining if two quantities are a function of each other. Students are asked questions similar to these throughout this lesson, and the discussion of this warm-up is meant to get students using the language of functions, which continues throughout the rest of the activities.

### Addressing

- 8.F.A.1

### Launch

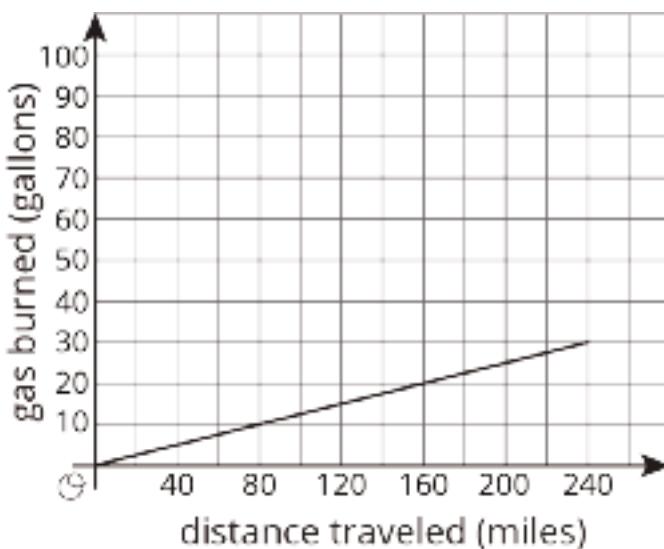
Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

### Anticipated Misconceptions

If students struggle with the second and third questions, ask them if this is an example of a proportional relationship. Remind them what they learned previously about proportional relationships and how that can be used to answer these questions.

### Student Task Statement

Here is a graph of the amount of gas burned during a trip by a tractor-trailer truck as it drives at a constant speed down a highway:



1. At the end of the trip, how far did the truck drive, and how much gas did it use?
2. If a truck traveled half this distance at the same rate, how much gas would it use?
3. If a truck traveled double this distance at the same rate, how much gas would it use?
4. Complete the sentence: \_\_\_\_\_ is a function of \_\_\_\_\_.

### Student Response

1. The truck drove 240 miles and used 30 gallons of gas.
2. 15 gallons. Since it is a proportional relationship, if the miles are halved then the gallons are also halved.
3. 60 gallons. Since it is a proportional relationship, if the miles are doubled then the gallons are also doubled.
4. Gallons of gas burned is a function of miles traveled. The number of miles traveled is also a function of the gallons of gas burned.

### Activity Synthesis

Invite students to share their answers and their reasoning for why gas burned is a function of distance traveled. Questions to further the discussion around functions:

- “Looking at the graph, what information do you need in order to determine how much gas was used?” (We need to know the number of miles traveled.)
- “What is the independent variable? Dependent variable? How can you tell from the graph?” (The independent value is the distance traveled. The dependent value is the gas burned. By convention, the independent is on the  $x$ -axis and the dependent is on the  $y$ -axis.)
- “What are some ways that we can tell from the graph that the relationship between gas burned and distance traveled is proportional?” (The graph is a line that goes through the origin, we can see a constant ratio between  $y$  and  $x$  in some points like  $(80, 10)$ ,  $(160, 20)$ , and  $(240, 30)$ .)

## 17.2 Double the Edge

**Optional: 10 minutes (there is a digital version of this activity)**

This activity is optional. The purpose of this activity is for students to apply what they know about functions and their representations in order to investigate the effect of a change in one dimension on the volume of a rectangular prism. Students use graphs and equations to represent the volume of a rectangular prism with one unknown edge length. They then use these representations to investigate what happens to the volume when one of the edge lengths is doubled. Groups make connections between the different representations by pointing out how the graph and the equation reflect an edge length that is doubled. Identify groups who make these connections and ask them to share during the discussion.

### Addressing

- 8.F.A.1
- 8.F.B

### Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Think Pair Share

### Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time, followed by time to discuss the last question with their partner. Follow with a whole-class discussion.

Students using the digital activity can generate the graph using the digital applet.

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### Access for Students with Disabilities

*Action and Expression: Provide Access for Physical Action.* Provide access to tools and assistive technologies such as a graphing software or applet. Some students may benefit from a checklist or list of steps to be able to use the software or applet.

*Supports accessibility for: Organization; Conceptual processing; Attention*

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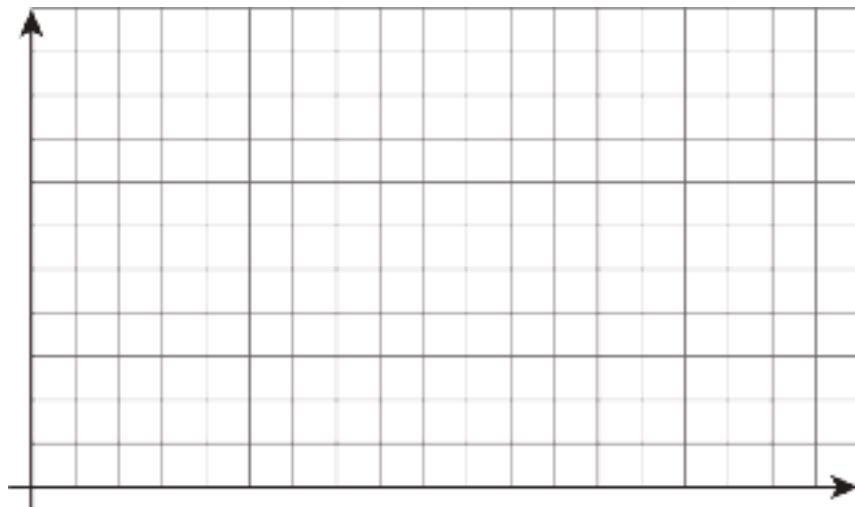
### Anticipated Misconceptions

If students struggle to see how the change in volume is reflected in the equation, have them start with values from the graph and put those into the equation to see how the volume changes.

### Student Task Statement

There are many right rectangular prisms with one edge of length 5 units and another edge of length 3 units. Let  $s$  represent the length of the third edge and  $V$  represent the volume of these prisms.

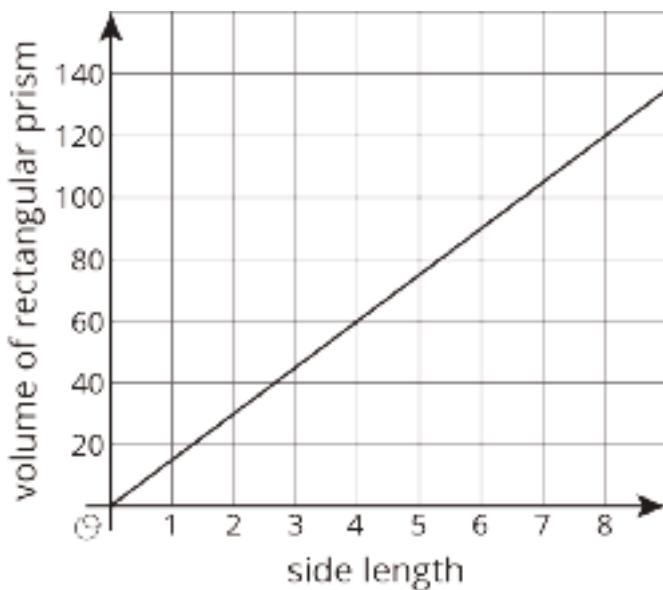
1. Write an equation that represents the relationship between  $V$  and  $s$ .
2. Graph this equation and label the axes.



3. What happens to the volume if you double the edge length  $s$ ? Where do you see this in the graph? Where do you see it algebraically?

### Student Response

1.  $V = 15s$
- 2.



3. When the edge length  $s$  is doubled, the volume is also doubled. Other answers vary. Sample response: In the graph, it can be seen that when the edge length is 4 cm, the volume is  $60 \text{ cm}^3$ . When the edge length doubles to 8 cm, then the volume doubles to be  $120 \text{ cm}^3$  as the graph shows a proportional relationship. Algebraically, if the edge length is doubled from  $s$  to  $2s$ , then the volume goes from  $15s$  to  $15 \cdot 2s$ , or  $30s$ , which is double the original volume.

### Activity Synthesis

The goal of the discussion is to ensure that students use function representations to support the idea that the volume doubles when  $s$  is doubled.

Select previously identified groups to share what happens to the volume when you double  $s$ . Display the graph and equation for all to see and have students point out where they see the effect of doubling  $s$  in the graph (by looking at any two edge lengths that are double each other, their volume will be double also). Ask students:

- “Which of your variables is the independent? The dependent?” (The side length,  $s$ , is the independent variable, and volume,  $V$ , is dependent variable.)
- “Which variable is a function of which?” (Volume is a function of side length.)

If it has not been brought up in students’ explanations, ask what the volume equation looks like when we double the edge length  $s$ . Display volume equation  $V = 15(2s)$  for all to see. Ask, “How can we write this equation to show that the volume doubled when  $s$  doubled?” (Using algebra, we can rewrite  $V = 15(2s)$  as  $V = 2(15s)$ . Since the volume for  $s$  was  $15s$ , this shows that the volume for  $2s$  is twice the volume for  $s$ ).

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### Access for English Language Learners

*Writing, Conversing: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to refine their response to the last question. Ask each student to meet with 2–3 other partners for feedback. Provide students with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “What part of the graph supports your reasoning?” or “Can algebraic calculations be applied to tripling the side length?”). Students can borrow ideas and language from each partner to strengthen the final product.

*Design Principle(s): Optimize output (for explanation)*

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## 17.3 Halve the Height

### Optional: 10 minutes (there is a digital version of this activity)

This activity is optional. This activity is similar to the previous (optional) one from a function point of view, but now students investigate the volume of a cylinder instead of a rectangular prism. Students continue working with functions to investigate what happens to the volume of a cylinder when you halve the height. The exploration and representations resemble what was done in the previous activity, and students continue to identify the effect of the changing dimension on the graph and the equation of this function. As groups work on the task, identify those who make the connection between the graph and equation representations, and encourage groups to look for similarities and differences between what they see in this activity and what they saw in the previous activity. Invite these groups to share during the whole-class discussion.

### Addressing

- 8.F.A.1
- 8.F.B
- 8.G.C.9

### Instructional Routines

- MLR7: Compare and Connect
- Think Pair Share

### Launch

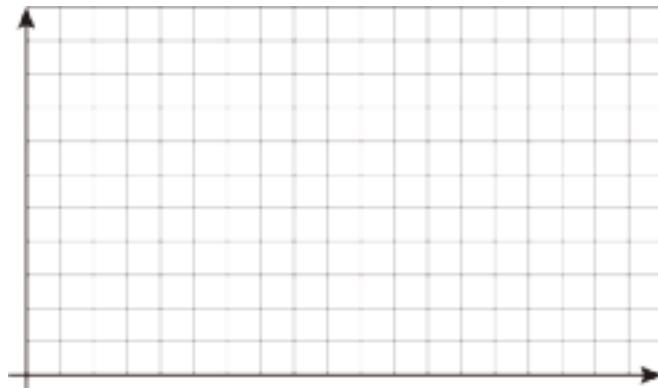
Keep students in the same groups. Tell students that this activity is similar to the previous one, but they will work with a cylinder instead of a rectangular prism. Give students 2–3 minutes of quiet work time followed by time to discuss the last question with their partner. Follow with a whole-class discussion.

Students using the digital activity can graph their equations using the applet.

## Student Task Statement

There are many cylinders with radius 5 units. Let  $h$  represent the height and  $V$  represent the volume of these cylinders.

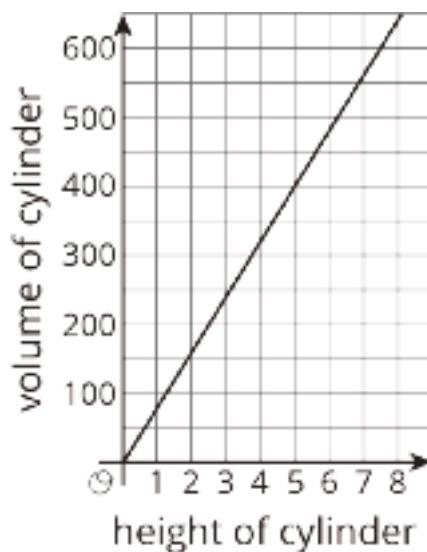
1. Write an equation that represents the relationship between  $V$  and  $h$ . Use 3.14 as an approximation of  $\pi$ .
2. Graph this equation and label the axes.



3. What happens to the volume if you halve the height,  $h$ ? Where can you see this in the graph? How can you see it algebraically?

## Student Response

1. The equation that expresses the relationship between  $V$  and  $h$  is  $V = 78.5h$ .
- 2.



3. When the height  $h$  is halved, the volume is also halved. Other answers vary. Sample response: In the graph, it can be seen that the graph shows a proportional relationship and when the height is  $\frac{1}{2}$  the corresponding point on the line is half as high as when the height is 1.

Algebraically, if the height is halved from  $h$  to  $\frac{1}{2}h$ , then the volume goes from  $78.5 \cdot h$  to  $78.5 \cdot \frac{1}{2}h$ , or  $39.25h$ , which is half the original volume.

### Are You Ready for More?

Suppose we have a rectangular prism with dimensions 2 units by 3 units by 6 units, and we would like to make a rectangular prism of volume 216 cubic units by stretching *one* of the three dimensions.

- What are the three ways of doing this? Of these, which gives the prism with the smallest surface area?
- Repeat this process for a starting rectangular prism with dimensions 2 units by 6 units by 6 units.
- Can you give some general tips to someone who wants to make a box with a certain volume, but wants to save cost on material by having as small a surface area as possible?

### Student Response

- The volume of the starting box is 36 cubic units, so to get to 216 cubic units we need to increase any one of the dimensions by a factor of 6. This gives the following three possibilities:
  - 12 units by 3 units by 6 units, with a surface area of 252 square units.
  - 2 units by 18 units by 6 units, with a surface area of 312 square units.
  - 2 units by 3 units by 36 units, with a surface area of 372 square units.
- The volume of the starting box is 72 cubic units, so to get to 216 cubic units we need to increase any one of the dimensions by a factor of 3. This gives the following two possibilities (since scaling either side of length 6 would give a box of the same dimensions):
  - 6 units by 6 units by 6 units, with a surface area of 216 square units.
  - 2 units by 6 units by 18 units, with a surface area of 504 square units.
- All five boxes we built have a volume of 216 cubic units, but have surfaces areas ranging from 216 to 504 square units. This could make quite a difference to the cost of enclosing 216 cubic units! In general, it seems that there is a lot more surface area when there is a lot of imbalance between the side lengths -- one side significantly shorter or longer than the other two. The best result we found, and indeed the best result possible, is to make all three sides the same length.

### Activity Synthesis

Ask previously identified groups to share their graphs and equations.

If students completed the optional activity, display student graphs from both activities and ask:

- “Compare the graph in this activity to the graph in the last activity. How are they alike? How are they different?”
- “Compare the equation in this activity to the equation in the last activity. How are they alike? How are they different?”
- “How can you tell that this is a linear relationship?”

For the last question, make sure students understand that “looks like a line” is insufficient evidence for saying a relationship is linear since the scale of the axes can make non-linear graphs look linear. It is important students connect that the equation is linear (of the form  $y=mx+b$ ) or relate back to the relationship between the height and volume: that the volume is  $25\pi$ , or about 78.5, times the height.

If students did not complete the optional activity, ask:

- “Which of your variables is the independent? The dependent?” (The height,  $h$ , is the independent variable, and volume,  $V$ , is dependent variable.)
- “Which variable is a function of which?” (Volume is a function of height.)
- “How can you tell that this is a linear relationship?”

See notes above on the last question in this list. If it has not been brought up in students’ explanations, ask what the volume equation looks like when we double the height  $h$ . Display volume equation  $V = 78.5(2h)$  for all to see. Ask how we can write this equation to show that the volume doubled when  $h$  doubled (using algebra, we can rewrite  $V = 78.5(2h)$  as  $V = 2(78.5h)$ . Since the volume for  $h$  was  $78.5h$ , this shows that the volume for  $2h$  is twice the volume for  $h$ ).

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#### Access for English Language Learners

*Representing, Listening, Speaking: MLR7 Compare and Connect.* Invite students to create a visual representation that shows what is the same and what is different between this activity and the previous one. If students did not complete the optional activity, students can create multiple representations that illustrate why volume is a function of height. Give students time to do a gallery walk of the displays. Look for opportunities to highlight approaches that compare the multiple representations of the function of volume. Lead a whole-class discussion that connects features on the displays, such as comparing the equations and graphs. This will help students communicate about the multiple representations of volume functions while using mathematical language.

*Design Principle(s): Optimize output; Maximize meta-awareness*

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## 17.4 Figuring Out Cone Dimensions

Optional: 10 minutes

This activity is optional. The purpose of this activity is for students to use representations of functions to explore more about the volume of a cone. This activity differs from the previous ones because students are given a graph that shows the relationship between the volume of a cone and the height of a cone. They use the graph to reason about the coordinates and their meaning. Students use the equation for the volume of a cone and coordinates from the graph to determine the radius of this cone and discuss the answer to this last question with their partners. Identify students to share their strategies during the discussion.

### **Addressing**

- 8.F.A.1
- 8.F.B
- 8.G.C.9

### **Instructional Routines**

- MLR5: Co-Craft Questions

### **Launch**

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by time to discuss the answer and their strategies for the last question with their partner. Follow with a whole-class discussion.

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### **Access for Students with Disabilities**

*Representation: Internalize Comprehension.* Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, ask students to use different colors to represent the height and volume of the cone in the graph. Ask students to use these same colors and a third color for the radius when writing an equation.

*Supports accessibility for: Visual-spatial processing*

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### Access for English Language Learners

*Writing, Conversing: MLR5 Co-Craft Questions.* Display the graph and the context without revealing the questions that follow. Ask pairs of students to write possible questions that could be answered by the information contained in the graph. Then, invite pairs to share their questions with the class. Look for questions that ask students to interpret the relationship between the two quantities represented in the graph and create an equation to represent that relationship. Next, reveal the questions of the activity. This helps students produce the language of mathematical questions and talk further about the relationships between the height and volume of cones.

*Design Principle(s): Maximize meta-awareness; Support sense-making*

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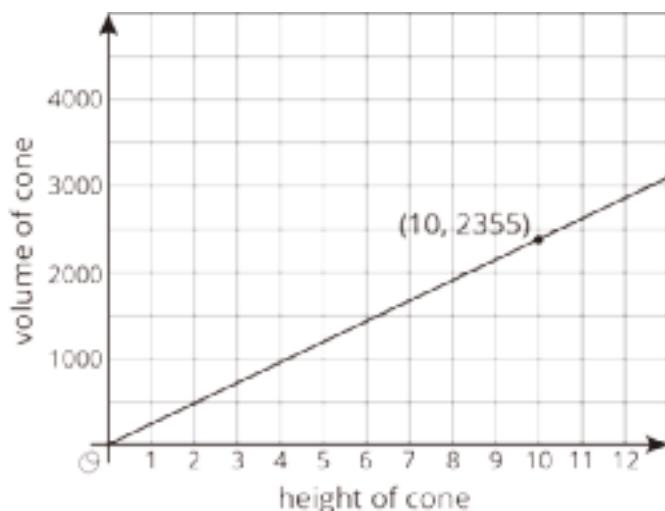
### Anticipated Misconceptions

Students may round  $\pi$  at different points during their work, leading to slightly different answers.

Students can solve the equation  $2,355 = \frac{1}{3}\pi r^2 10$  with different series of steps. If they choose to first multiply the numbers on the side with  $r$ , they may round the expression  $\frac{1}{3}\pi r^2 10$  to  $10.47r^2$ . This would give  $224.9 = r^2$ , making  $r$  slightly less than 15, though it rounds to 15. Solving by multiplying each side by  $3 \cdot \frac{1}{\pi} \cdot \frac{1}{10}$  yields  $r^2 = 225$ , making  $r$  equal 15 exactly (if we accept 3.14 as the value of  $\pi$ ). This is an opportunity to discuss how rounding along the way in a solution can introduce imprecision.

### Student Task Statement

Here is a graph of the relationship between the height and the volume of some cones that all have the same radius:



1. What do the coordinates of the labeled point represent?

2. What is the volume of the cone with height 5? With height 30?
3. Use the labeled point to find the radius of these cones. Use 3.14 as an approximation for  $\pi$ .
4. Write an equation that relates the volume  $V$  and height  $h$ .

### Student Response

1. The volume of a particular cone with a height of 10 units.
2. The approximate volume of the cone with a height of 5 units is 1,177.5 cubic units. The approximate volume of the cone with a height of 30 units is 7,065 cubic units.
3. 15 units. Substituting the values of the labeled point (10, 2,355) into the volume equation, we know that  $2,355 = \frac{1}{3}\pi r^2 10$ , which means  $225 \approx r^2$ .
4.  $V = 75\pi h$

### Activity Synthesis

Ask previously identified students to share their answers and their strategies for finding the radius in the last question.

To further the discussion, consider asking some of the following questions:

- “How did you deal with the  $\frac{1}{3}$  in the equation?”
- “How did you deal with  $\pi$  when you were trying to solve for the radius?”
- “How do you know that the relationship between volume and height for these cones is a function? How is this shown in the graph? In the equation?”
- “Identify the independent and dependent variables in this relationship. If they were switched, would we still have a function? Explain how you know.”

### Lesson Synthesis

Tell students to imagine a cylindrical water tank with a radius  $r$  units and a height  $h$  units. Display the following prompts and ask students to respond to them in writing, encouraging them to include sketches:

- “How can you change a dimension of the water tank so that the volume of the tank increases by a factor of 2?”
- “How can you change a dimension of the water tank so that the volume of the tank increases by a factor of  $a$ ?”

After quiet work time, invite students to share their responses. If students made a sketch, display them for all to see.

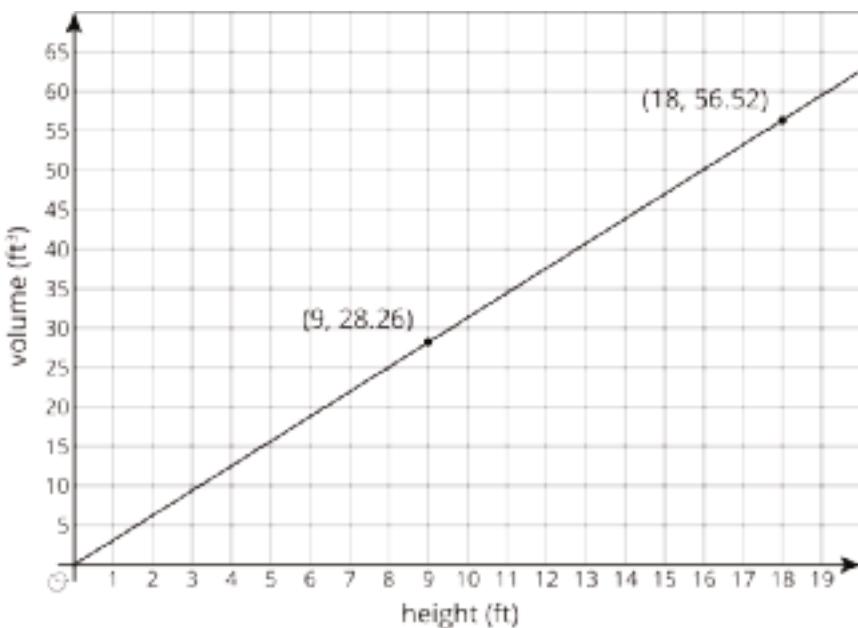
## 17.5 A Missing Radius

Cool Down: 5 minutes

### Addressing

- 8.F.A.1
- 8.G.C

### Student Task Statement



Here is a graph of the relationship between the height and volume of some cylinders that all have the same radius,  $R$ . An equation that represents this relationship is  $V = \pi R^2 h$  (use 3.14 as an approximation for  $\pi$ ).

What is the radius of these cylinders?

### Student Response

The radius is 1 foot. Substituting the information (9, 28.26) into the volume of a cylinder equation produces  $28.26 = 3.14R^29$ . This is equivalent to  $1 = R^2$ , which means  $R = 1$  since  $1^2 = 1$ .

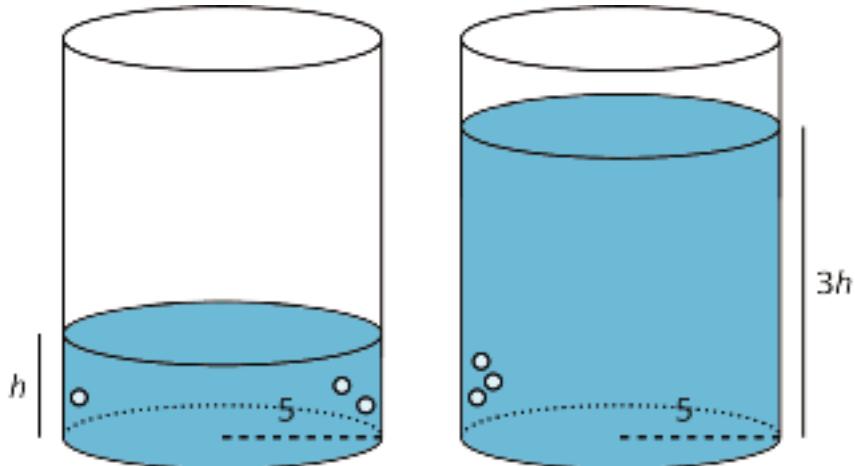
### Student Lesson Summary

Imagine a cylinder with a radius of 5 cm that is being filled with water. As the height of the water increases, the volume of water increases.

We say that the volume of the water in the cylinder,  $V$ , depends on the height of the water  $h$ . We can represent this relationship with an equation:  $V = \pi \cdot 5^2 h$  or just

$$V = 25\pi h$$

This equation represents a *proportional relationship* between the height and the volume. We can use this equation to understand how the volume changes when the height is tripled.



The new volume would be  $V = 25\pi(3h) = 75\pi h$ , which is precisely 3 times as much as the old volume of  $25\pi h$ . In general, when one quantity in a proportional relationship changes by a given factor, the other quantity changes by the same factor.

Remember that proportional relationships are examples of linear relationships, which can also be thought of as functions. So in this example  $V$ , the volume of water in the cylinder, is a function of the height  $h$  of the water.

# Lesson 17 Practice Problems

## Problem 1

### Statement

A cylinder has a volume of  $48\pi \text{ cm}^3$  and height  $h$ . Complete this table for volume of cylinders with the same radius but different heights.

height (cm)	volume ( $\text{cm}^3$ )
$h$	$48\pi$
$2h$	
$5h$	
$\frac{h}{2}$	
$\frac{h}{5}$	

### Solution

height (cm)	volume ( $\text{cm}^3$ )
$h$	$48\pi$
$2h$	$96\pi$
$5h$	$240\pi$
$\frac{h}{2}$	$24\pi$
$\frac{h}{5}$	$\frac{48}{5}\pi$

## Problem 2

### Statement

A cylinder has a radius of 3 cm and a height of 5 cm.

- What is the volume of the cylinder?
- What is the volume of the cylinder when its height is tripled?
- What is the volume of the cylinder when its height is halved?

## Solution

- a.  $45\pi \text{ cm}^3$
- b.  $135\pi \text{ cm}^3$
- c.  $\frac{45}{2}\pi \text{ cm}^3$

## Problem 3

### Statement

A graduated cylinder that is 24 cm tall can hold 1 L of water. What is the radius of the cylinder? What is the height of the 500 ml mark? The 250 ml mark? Recall that 1 liter (L) is equal to 1000 milliliters (ml), and that 1 liter (L) is equal to 1,000  $\text{cm}^3$ .

## Solution

The radius of the cylinder is about 3.64 cm since  $\frac{1000}{24\pi} \approx 13.26$  and  $3.64^2 \approx 13.26$ . The height of the 500 ml mark is 12 cm. The height of the 250 ml mark is 6 cm.

## Problem 4

### Statement

An ice cream shop offers two ice cream cones. The waffle cone holds 12 ounces and is 5 inches tall. The sugar cone also holds 12 ounces and is 8 inches tall. Which cone has a larger radius?

## Solution

The waffle cone (Since its height is smaller, the radius must be larger in order to have the same volume as the sugar cone.)

(From Unit 5, Lesson 16.)

## Problem 5

### Statement

A 6 oz paper cup is shaped like a cone with a diameter of 4 inches. How many ounces of water will a plastic cylindrical cup with a diameter of 4 inches hold if it is the same height as the paper cup?

## Solution

18 oz (Since the cups are the same height and radius, the cylindrical cup must have 3 times the volume of the conical cup.)

(From Unit 5, Lesson 15.)

## Problem 6

### Statement

Lin's smart phone was fully charged when she started school at 8:00 a.m. At 9:20 a.m., it was 90% charged, and at noon, it was 72% charged.

- a. When do you think her battery will die?
- b. Is battery life a function of time? If yes, is it a linear function? Explain your reasoning.

## Solution

- a. Answers vary. Sample response: Approximately 9:20 p.m. (Since 10 percent of battery was lost in 80 minutes, it would take 800 minutes to lose all of the battery. This would give a prediction of 13 hours and 20 minutes after the start time, so approximately 9:20 p.m.)
- b. Battery remaining is a function of time but not a linear function. Explanations vary. Sample response: It took 80 minutes for her phone to lose the first 10 percent of the battery, and then then 160 minutes for her phone to lose another 18 percent. If the function were linear, it would lose exactly twice as much in 160 minutes as it did in 80 minutes. It might, however, be reasonably well-modeled by a linear function.

(From Unit 5, Lesson 9.)

# Lesson 18: Scaling Two Dimensions

## Goals

- Compare and contrast (orally) graphs of linear and nonlinear functions.
- Create an equation and a graph representing the volume of a cone as a function of its radius, and describe (orally and in writing) how a change in radius affects the volume.
- Describe (orally and in writing) how changing the input of a certain nonlinear function affects the output.

## Learning Targets

- I can create a graph representing the relationship between volume and radius for all cylinders (or cones) with a fixed height.
- I can explain in my own words why changing the radius by a scale factor changes the volume by the scale factor squared.

## Lesson Narrative

This is lesson optional. The previous lesson explored some proportional relationships that arise when we consider the volume of a rectangular prism or cone as a function of one of its dimensions, such as side length or height. Students studied what happens to the volume of the figure when you scale that dimension. In this lesson they see what happens to the volume when you scale two of the dimensions. They consider a rectangular prism on a square base where you keep the height constant and vary the side length of base, and a cone where you keep the height constant and vary the radius of the base. In both cases you are really varying two dimensions, because both the length and the width of the base change at the same time. As in the previous lesson, they consider what happens when you scale the side length or the radius by a particular factor, and this time they discover that the volume scales by the square of the factor. For example, if you triple the side length of the square base of the prism, you multiply the volume by 9, which is  $3^2$ . In general, if you scale the side length by  $a$ , you multiply the volume by  $a^2$ .

The main purpose of this lesson is to understand that if you scale two of the dimensions of a three-dimensional figure by the same factor, the volume scales by the square of that factor. A secondary purpose is to see some interesting examples of non-linear functions arising from geometry.

## Alignments

## Building On

- 6.EE.A: Apply and extend previous understandings of arithmetic to algebraic expressions.

## Addressing

- 8.F.A.3: Interpret the equation  $y = mx + b$  as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function  $A = s^2$

giving the area of a square as a function of its side length is not linear because its graph contains the points  $(1, 1)$ ,  $(2, 4)$  and  $(3, 9)$ , which are not on a straight line.

- 8.F.B: Use functions to model relationships between quantities.
- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

### Instructional Routines

- MLR3: Clarify, Critique, Correct
- MLR8: Discussion Supports

### Student Learning Goals

Let's change more dimensions of shapes.

## 18.1 Tripling Statements

### Warm Up: 5 minutes

The purpose of this warm-up is for students to explore how scaling the addends or factors in an expression affects their sum or product. Students determine which statements are true and then create one statement of their own that is true. This will prepare students to see structure in the equations they will encounter in the lesson. Identify students who:

- choose the correct statements (b, c)
- pick numbers to test the validity of statements
- use algebraic structure to show that the statements are true

Ask these students to share during the discussion.

### Building On

- 6.EE.A

### Launch

Arrange students in groups of 2. Give students 1–2 minutes of quiet work time followed by time to discuss their chosen statements with their partner. Follow with a whole-class discussion.

### Student Task Statement

$m$ ,  $n$ ,  $a$ ,  $b$ , and  $c$  all represent positive integers. Consider these two equations:

$$\begin{aligned}m &= a + b + c \\n &= abc\end{aligned}$$

1. Which of these statements are true? Select **all** that apply.
  - If  $a$  is tripled,  $m$  is tripled.
  - If  $a$ ,  $b$ , and  $c$  are all tripled, then  $m$  is tripled.

c. If  $a$  is tripled,  $n$  is tripled.

d. If  $a$ ,  $b$ , and  $c$  are all tripled, then  $n$  is tripled.

2. Create a true statement of your own about one of the equations.

### Student Response

1. b, c

2. Answers vary. Sample response: If  $a$ ,  $b$ , and  $c$  are all tripled, then  $n$  is 27 times as large. When  $a$ ,  $b$ , and  $c$  are tripled, the result is  $3a \cdot 3b \cdot 3c$ , which can be written as  $(3 \cdot 3 \cdot 3)abc$  or  $27abc$ .

### Activity Synthesis

Ask previously identified students to share their reasoning about which statements are true (or not true). Display any examples (or counterexamples) for all to see and have students refer to them while sharing. If using the algebraic structure is not brought up in students' explanations, display for all to see:

- If  $a$ ,  $b$ , and  $c$  are all tripled, the expression becomes  $3a + 3b + 3c$ , which can be written as  $3(a + b + c)$  by using the distributive property to factor out the 3. So if all the addends are tripled, their sum,  $m$ , is also tripled.
- Looking at the third statement, if  $a$  is tripled, the expression becomes  $(3a)bc$ , which, by using the associative property, can be written as  $3(abc)$ . So if just  $a$  is tripled, then  $n$ , the product of  $a$ ,  $b$ , and  $c$  is also tripled.

## 18.2 A Square Base

### Optional: 15 minutes

This activity is optional. The purpose of this activity is for students to examine how changing the input of a non-linear function changes the output. In this activity, students consider how the volume of a rectangular prism with a square base and a known height of 11 units changes if the edge lengths of the base triple. By studying the structure of the equation representing the volume function, students see that tripling the input leads to an output that is 9 times greater. In the following activity, students will continue this thinking with the volume function for cylinders.

Identify students who make sketches of the two rectangular prisms or write expressions of the form  $99s^2$  or  $11(3s)^2$  to describe the volume of the tripled rectangular prism.

### Addressing

- 8.F.A.3
- 8.F.B

### Instructional Routines

- MLR8: Discussion Supports

## Launch

Give students quiet work time. Leave 5–10 minutes for a whole-class discussion and follow-up questions.

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### Access for Students with Disabilities

*Action and Expression: Develop Expression and Communication.* To help get students started, display sentence frames such as “Han is \_\_\_ because \_\_\_” or “I agree with \_\_\_ because \_\_\_”  
*Supports accessibility for: Language; Organization*

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### Anticipated Misconceptions

Some students might think that because you triple  $s$  then you are only tripling one dimension. Encourage these students to make a sketch of the prism before and after the doubling and to label all three edge lengths to help them see that two dimensions are tripling.

#### Student Task Statement

Clare sketches a rectangular prism with a height of 11 and a square base and labels the edges of the base  $s$ . She asks Han what he thinks will happen to the volume of the rectangular prism if she triples  $s$ .

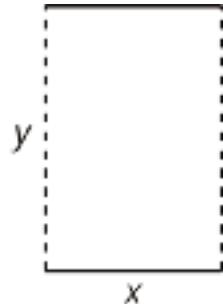
Han says the volume will be 9 times bigger. Is he right? Explain or show your reasoning.

#### Student Response

Yes, Han is right. Sample reasoning: Han is right because when the edge length of the square base is tripled, the area of the base is multiplied by 9. This makes the volume equation,  $99s^2$ , nine times as large as the original since  $V = 11(3s)^2 = 11 \cdot 3s \cdot 3s = 11 \cdot 9s^2 = 99s^2$ .

#### Are You Ready for More?

A cylinder can be constructed from a piece of paper by curling it so that you can glue together two opposite edges (the dashed edges in the figure).



1. If you wanted to increase the volume inside the resulting cylinder, would it make more sense to double  $x$ ,  $y$ , or does it not matter?
2. If you wanted to increase the surface area of the resulting cylinder, would it make more sense to double  $x$ ,  $y$ , or does it not matter?

3. How would your answers to these questions change if we made a cylinder by gluing together the solid lines instead of the dashed lines?

### Student Response

1. Double length  $x$ . Since  $x$  represents the circumference of the circular base, this would result in doubling the radius of the cylinder as well. Doubling the radius will result in 4 times the volume. Doubling  $y$ , the height of the cylinder, results in doubling the volume.
2. Double length  $x$ . Doubling  $x$  will quadruple the area of the circular bases as well as double the area of the curved surface. Doubling  $y$  will only double the curved surface.
3. If the cylinder was created by connecting the solid edges, doubling length  $y$  would result in both a larger volume and surface area because  $y$  would be the length related to the radius of the cylinder.

### Activity Synthesis

Select previously identified students to share whether they think Han is correct. If possible, begin with students who made sketches of the two rectangular prisms to make sense of the problem.

Ask students: “If this equation was graphed with edge length  $s$  on the horizontal axis and the volume of the prism on the vertical axis, what would the graph look like?” Suggest that they complete a table showing the volume of the rectangular prism when  $s$  equals 1, 2, 3, 4, and 5 units (the corresponding values of volume are 11, 44, 99, 176, and 275 cubic units) and then sketch a graph using these points. Give students quiet work time and then select a student to display their table and graph for all to see. Ask students what they notice about the graph when compared to the graphs from the previous lesson (the graph is non-linear—the volume increases by the square of whatever the base edge-length increases by).

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#### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* As students explain whether they agree with Han, invite students to restate their reasoning using mathematical language. Press for details inviting other students to challenge or elaborate on an idea. This will help students produce and make sense of the language needed to communicate their own ideas when reasoning about volume and the effects of changing dimensions.

*Design Principle(s): Support sense-making; Optimize output (for generalization)*

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## 18.3 Playing with Cones

**Optional: 15 minutes (there is a digital version of this activity)**

This activity is optional. In this activity, students continue working with function representations to investigate how changing the radius affects the volume of a cone with a fixed height. Students represent the relationship between the volume of the cone and the length of its radius with an

equation and graph. They use these representations to justify what they think will happen when the radius of the cylinder is tripled. The work students did in the previous activity prepared them to use the equation to see the effect that tripling the radius has on the volume of the cone.

Identify students who use the equation versus the graph to answer the last question.

## Addressing

- 8.F.A.3
- 8.F.B
- 8.G.C.9

## Instructional Routines

- MLR3: Clarify, Critique, Correct

## Launch

Give students 4–7 minutes of quiet work time followed by a whole-class discussion.

For students using the digital activity, they can generate their graph using an applet.

## Anticipated Misconceptions

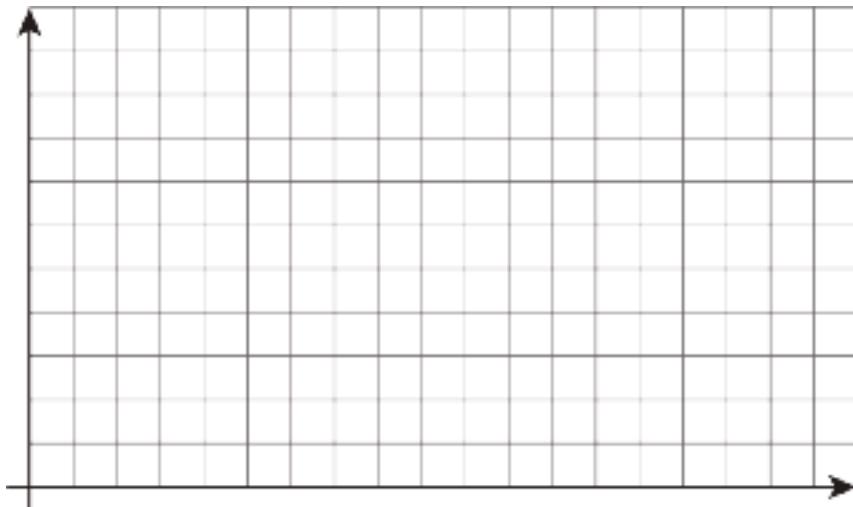
While students calculate the volume (or write an equation) they might mistake  $(3r)^2$  as  $2 \cdot 3r$ , remind students what squaring a term involves and encourage them to expand the term if they need to see that  $(3r)^2$  is  $3r \cdot 3r$ .

If students struggle to use the equation to see how the volume changes, encourage students to make a sketch of the cone.

### Student Task Statement

There are many cones with a height of 7 units. Let  $r$  represent the radius and  $V$  represent the volume of these cones.

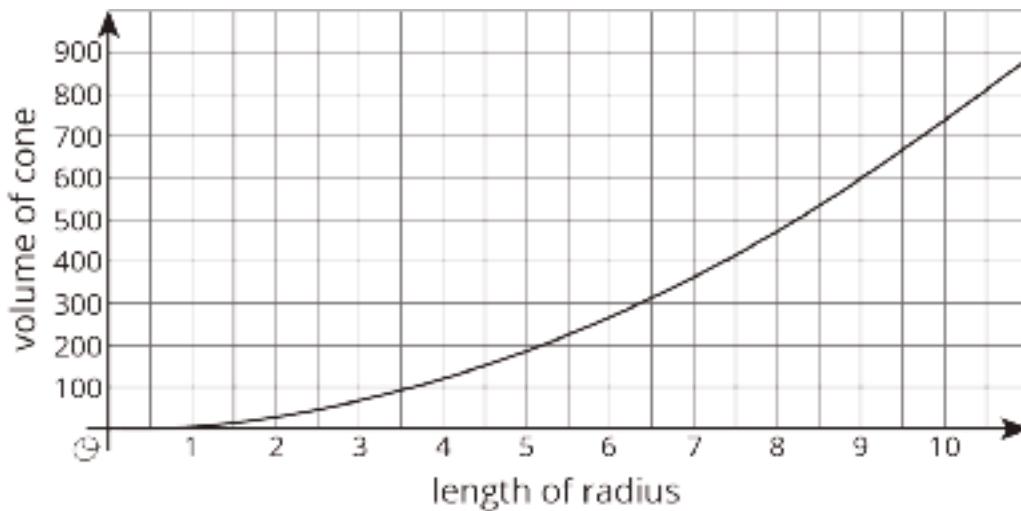
1. Write an equation that expresses the relationship between  $V$  and  $r$ . Use 3.14 as an approximation for  $\pi$ .
2. Predict what happens to the volume if you triple the value of  $r$ .
3. Graph this equation.



4. What happens to the volume if you triple  $r$ ? Where do you see this in the graph? How can you see it algebraically?

### Student Response

1.  $V = 7.33r^2$
2. Answers vary. Sample response: The volume is 9 times bigger if the value of  $r$  is tripled.
- 3.



4. Answers vary. Sample response: When the radius,  $r$ , is tripled, the volume is 9 times as large. Since this is not a proportional relationship, in the graph it can be seen that when the radius triples from 1 cm to 3 cm, the volume changes from  $7.33 \text{ cm}^3$  to  $65.94 \text{ cm}^3$ , a value 9 times as large. Algebraically, if the radius triples from  $r$  to  $3r$ , then the volume changes from  $7.33r^2$  to  $7.33(3r)^2 = 9 \cdot 7.33r^2 = 65.94r^2$ .

## Activity Synthesis

The purpose of this discussion is for students to use the graph and equation to see that when you triple the radius you get a volume that is 9 times as large.

Ask previously identified students to share their graphs and equations. Display both representations for all to see, and ask students to point out where in each representation we see that the volume is 9 times as large. Ask students:

- “If the radius was quadrupled (made 4 times as large), how many times as large would the volume be?” (The volume would be 16 times as large since  $\frac{1}{3}\pi(4r)^2 = \frac{1}{3}\pi r^2 4^2 = 16 \cdot \frac{1}{3}\pi r^2$ .)
- “If the radius was halved, how many times as large would the volume be?” (The volume would be  $\frac{1}{4}$  times as large since  $\frac{1}{3}\pi\left(\frac{1}{2}r\right)^2 = \frac{1}{3}\pi r^2 \left(\frac{1}{2}\right)^2 = \frac{1}{4} \cdot \frac{1}{3}\pi r^2$ .)
- “If the radius was scaled by an unknown factor  $a$ , how many times as large would the volume be?” (The volume would be  $a^2$  times as large since  $\frac{1}{3}\pi(ar)^2 = \frac{1}{3}\pi r^2 a^2 = a^2 \frac{1}{3}\pi r^2$ .)

If students do not see the connection between scaling the radius length with a known value like 4 and an unknown value  $a$ , use several known values to help students generalize that scaling the radius by  $a$  scales the volume by  $a^2$ .

If time allows, ask students to compare this activity to the previous. How do the equations compare? How do the graphs compare? (In the last activity, the graph was sketched during the discussion.)

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### Access for English Language Learners

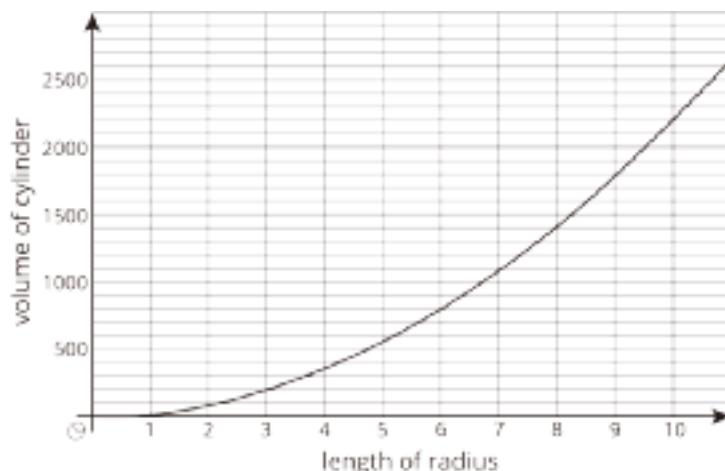
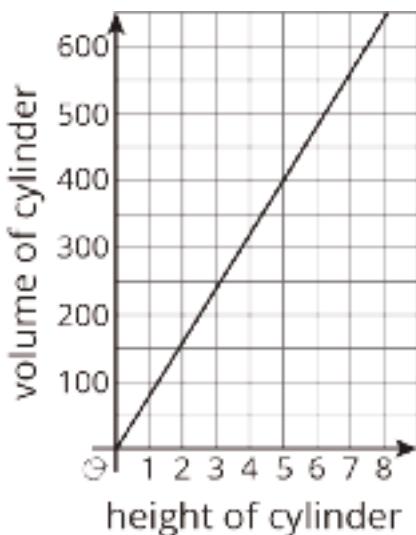
*Writing: MLR3 Clarify, Critique, Correct.* Present an incorrect prediction describing what happens to the volume of the cone when the radius is tripled that reflects a possible misunderstanding from the class. For example, “The volume will be 3 times greater because that’s what ‘triples’ means.” Prompt students to critique the reasoning by asking, “Do you agree with the reasoning? Why or why not?” Invite students to write feedback to the author that identifies the misconception and how to improve on his/her work. Listen for students who tie their feedback to the formula and the describe the distinction between  $(3r)^2$  and  $3 \cdot r^2$ . This will help students evaluate, and improve on, the written mathematical arguments of others and reason about the effect that tripling the radius has on the volume of the cone.

*Design Principle(s): Maximize meta-awareness; Optimize output (for generalization)*

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## Lesson Synthesis

Display these graphs for all to see and give students a minute to consider what they represent.



Ask students:

- "What do these graphs represent? How are these graphs similar? Different?" (The first graph shows the relationship between the height and volume of all the cylinders with a fixed radius. The second graph shows the relationship between the radius and volume of all the cylinders with a fixed height. The first is linear, the second is non-linear.)
- "Think about what happens when a cube's edge lengths are doubled or tripled. What happens to the volume?" (The volume is increased by  $2^3 = 8$ , or by  $3^3 = 27$ .)
- "Why do you think changing the radius of a cylinder results in a graph that is not proportional?" (Two dimensions are changing when you change the radius of a cylinder.)

## 18.4 Halving Dimensions

Cool Down: 5 minutes

### Addressing

- 8.F.A.3
- 8.F.B
- 8.G.C.9

### Student Task Statement

There are many cylinders for which the height and radius are the same value. Let  $c$  represent the height and radius of a cylinder and  $V$  represent the volume of the cylinder.

1. Write an equation that expresses the relationship between the volume, height, and radius of this cylinder using  $c$  and  $V$ .
2. If the value of  $c$  is halved, what must happen to the value of the volume  $V$ ?

## Student Response

1.  $V = \pi c^3$
2. If the value of  $c$  is halved, then the value of the volume would be  $\frac{1}{8}$  of the original volume since  $\pi\left(\frac{1}{2}c\right)^3 = \pi c^3 \left(\frac{1}{2}\right)^3 = \frac{1}{8}\pi c^3$

## Student Lesson Summary

There are many rectangular prisms that have a length of 4 units and width of 5 units but differing heights. If  $h$  represents the height, then the volume  $V$  of such a prism is

$$V = 20h$$

The equation shows us that the volume of a prism with a base area of 20 square units is a linear function of the height. Because this is a proportional relationship, if the height gets multiplied by a factor of  $a$ , then the volume is also multiplied by a factor of  $a$ :

$$V = 20(ah)$$

What happens if we scale *two* dimensions of a prism by a factor of  $a$ ? In this case, the volume gets multiplied by a factor of  $a$  twice, or  $a^2$ .

For example, think about a prism with a length of 4 units, width of 5 units, and height of 6 units. Its volume is 120 cubic units since  $4 \cdot 5 \cdot 6 = 120$ . Now imagine the length and width each get scaled by a factor of  $a$ , meaning the new prism has a length of  $4a$ , width of  $5a$ , and a height of 6. The new volume is  $120a^2$  cubic units since  $4a \cdot 5a \cdot 6 = 120a^2$ .

A similar relationship holds for cylinders. Think of a cylinder with a height of 6 and a radius of 5. The volume would be  $150\pi$  cubic units since  $\pi \cdot 5^2 \cdot 6 = 150\pi$ . Now, imagine the radius is scaled by a factor of  $a$ . Then the new volume is  $\pi \cdot (5a)^2 \cdot 6 = \pi \cdot 25a^2 \cdot 6$  or  $150a^2\pi$  cubic units. So scaling the radius by a factor of  $a$  has the effect of multiplying the volume by  $a^2$ !

Why does the volume multiply by  $a^2$  when only the radius changes? This makes sense if we imagine how scaling the radius changes the base area of the cylinder. As the radius increases, the base area gets larger in two dimensions (the circle gets wider and also taller), while the third dimension of the cylinder, height, stays the same.

# Lesson 18 Practice Problems

## Problem 1

### Statement

There are many cylinders with a height of 18 meters. Let  $r$  represent the radius in meters and  $V$  represent the volume in cubic meters.

- Write an equation that represents the volume  $V$  as a function of the radius  $r$ .
- Complete this table, giving three possible examples.

$r$	$V$
1	

- If the radius of a cylinder is doubled, does the volume double? Explain how you know.
- Is the graph of this function a line? Explain how you know.

### Solution

a.  $V = 18\pi r^2$

b. Answers vary. Sample response:

$r$	$V$
1	$18\pi$
2	$72\pi$
4	$288\pi$

- No, the volume does not double, it is multiplied by four. Explanations vary.
- It is *not* a line. The three points in the table do not lie on a straight line.

## Problem 2

### Statement

As part of a competition, Diego must spin around in a circle 6 times and then run to a tree. The time he spends on each spin is represented by  $s$  and the time he spends running is  $r$ . He gets to the tree 21 seconds after he starts spinning.

- Write an equation showing the relationship between  $s$  and  $r$ .
- Rearrange the equation so that it shows  $r$  as a function of  $s$ .
- If it takes Diego 1.2 seconds to spin around each time, how many seconds did he spend running?

### Solution

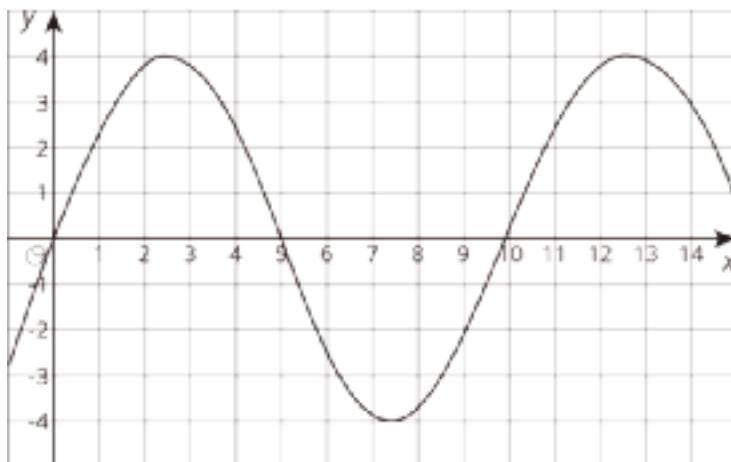
- $6s + r = 21$
- $r = 21 - 6s$
- 13.8 seconds

(From Unit 5, Lesson 3.)

## Problem 3

### Statement

The table and graph represent two functions. Use the table and graph to answer the questions.



$x$	1	2	3	4	5	6
$y$	3	-1	0	4	5	-1

- a. For which values of  $x$  is the output from the table less than the output from the graph?
- b. In the graphed function, which values of  $x$  give an output of 0?

## Solution

- a. 2 and 3
- b. 0, 5, and 10

(From Unit 5, Lesson 7.)

## Problem 4

### Statement

A cone has a radius of 3 units and a height of 4 units.

- a. What is the volume of this cone?
- b. Another cone has quadruple the radius, and the same height. How many times larger is the new cone's volume?

## Solution

- a.  $12\pi$  cubic units
- b. 16 times larger (The new cone's volume is  $V = 192\pi$ , 4 times larger. Quadrupling the radius makes the volume  $4^2$  times larger.)

# Lesson 19: Estimating a Hemisphere

## Goals

- Calculate the volume of a cylinder and cone with the same radius and height, and justify (orally and in writing) that the volumes are an upper and lower bound for the volume of a hemisphere of the same radius.
- Estimate the volume of a hemisphere using the formulas for volume of a cone and cylinder, and explain (orally) the estimation strategy.

## Learning Targets

- I can estimate the volume of a hemisphere by calculating the volume of shape I know is larger and the volume of a shape I know is smaller.

## Lesson Narrative

The purpose of this lesson is to introduce students to working with spheres by using shapes they are now familiar with—cubes, cones, and cylinders—to estimate the volume of a hemisphere. In the previous lesson, students saw that changing the radius of a cone by a factor of  $a$  scales the volume by a factor of  $a^2$ . Here, the connection between spheres and cubes is made in the first activity to help them build understanding for why changing the radius of a sphere (or, in the case of this lesson, hemisphere) by a factor of  $a$  changes the volume by a factor of  $a^3$ . For example, think about a cube with side length 2 units that has a volume of

$$2^3 = 8.$$

If the length, width, and height are all scaled by a factor of  $a$ , then each edge length would be  $2a$  units and the new volume would be

$$(2a)^3 = 8a^3$$

cubic units, which is  $a^3$  times the original volume. The first activity sets the expectation that spheres work the same way, so that the  $r^3$  in the formula for the volume of a sphere of radius  $r$ , given in the next lesson, makes sense.

In the second activity, students fit a hemisphere inside a cylinder, and use the volume of the cylinder to make an estimate of the volume of the hemisphere. Then they do the same thing with a cone that fits inside the hemisphere. The volume of the hemisphere has to be between the volume of the cone and the volume of the cylinder, both of which students can calculate from work in previous lessons. So this activity gives a range of possibilities for volume of the hemisphere. In the next lesson, students will see the exact formula.

## Alignments

### Addressing

- 8.G.C: Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR6: Three Reads
- Notice and Wonder
- Think Pair Share

## Required Materials

### Spherical objects

## Required Preparation

If possible, have some physical examples of hemispheres on hand for students to see. Examples could be glass paperweights or dome lids. Alternatively, have a sphere, such as a globe or basketball, with a marked equator to clearly divide it into two hemispheres.

### Student Learning Goals

Let's estimate volume of hemispheres with figures we know.

## 19.1 Notice and Wonder: Two Shapes

### Warm Up: 5 minutes

The purpose of this warm-up is for students to review how to manipulate the formulas for volume of a cylinder and cone and consider what they look like when the height and radius are the same. Students will encounter these shapes again later in the lesson.

### Addressing

- 8.G.C.9

## Instructional Routines

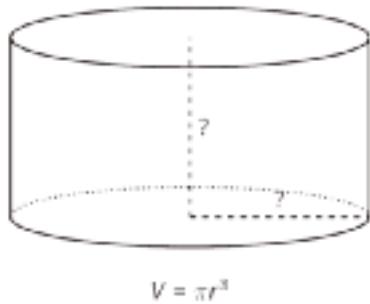
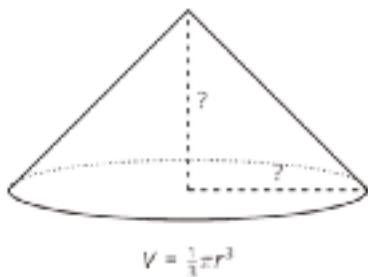
- Notice and Wonder

### Launch

Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

### Student Task Statement

Here are two shapes.



What do you notice? What do you wonder?

### Student Response

Things students may notice:

- If the height and radius are the same for both the cylinder and cone, then the volume of the cone is one-third the volume of the cylinder.
- When the radius is the same as the height, the cone and cylinder seem much wider than tall.
- When the height and radius are the same, the volume acts as a function of one variable.

Things students may wonder:

- Why would we want a cone or cylinder where the height and radius are the same?
- Does the volume of this type of cone or cylinder change in the same way a cube's volume does?
- Could we put the cone inside the cylinder to create a new shape, since their dimensions match?

### Activity Synthesis

Select students to share things they have noticed and things they have wondered. Encourage students to think about the relationship with the dimension  $r$  and the volume  $V$  of each of these shapes. Ensure students understand that the two equations have no variable  $h$  for height since the  $h$  was replaced by  $r$  due to the height and radius being the same for both shapes.

## 19.2 Hemispheres in Boxes

15 minutes

In this activity, students think about a hemisphere fitting inside the smallest possible box (or rectangular prism). Students reason that the smallest box in which a hemisphere can fit has a square base with edge length that is the same as the diameter of the hemisphere, and its height will be the radius of the hemisphere. Further, if we calculate the volume of the box, we have an upper bound for the volume of the hemisphere. This activity prepares students for the next, where they will calculate the upper and lower bounds for the volume of a hemisphere by considering cylinders and cones that fit outside or inside the hemisphere.

## Addressing

- 8.G.C

## Instructional Routines

- MLR6: Three Reads

## Launch

Ask students if they are familiar with the word *sphere*, and if they can think of examples of spheres. Some examples they might come up with are: ping pong balls, soap bubbles, baseballs, volleyballs, a globe. If students don't come up with many examples, offer your own, or perhaps display an Internet image search for "sphere." If you brought in any examples of physical spheres, display these or allow students to hold them. Explain to students that the radius of a sphere is the distance from the center of the sphere (the point in the exact middle) to any point on the sphere. If you brought in physical examples of spheres, ask students to rank them from smallest diameter to largest diameter.

Ask students if they are familiar with the word *hemisphere*, and if they can think of examples of hemispheres. Some examples are hemispherical (or 'half') balance boards, half of Earth, a dome or planetarium, and convex security mirrors. Display an example, illustration, or diagram of a hemisphere for all to see, pointing out how the radius is the distance from the center of the flat side of the sphere to any point on the curved surface of the sphere.

Tell students that in this activity, they will think about building a box (or rectangular prism) around a hemisphere. Arrange students in groups of 2. Ask students to quietly work through the first question and then share their reasoning with their partner. Pause for a whole-class discussion, and invite students to share their responses. The purpose of this problem is to see how the measurements of a hemisphere determine the dimensions of the prism. Ensure everyone understands the box must have edge lengths 6, 6, and 3 inches, and that the hemisphere must have a volume that is less than 108 cubic inches (since  $6^2 \cdot 3 = 108$ ). Discuss how much less students *think* the volume of the sphere is and their reasoning, then have students move onto the second problem with their partner.

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## Access for Students with Disabilities

*Representation: Develop Language and Symbols.* Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of: sphere and hemisphere.

*Supports accessibility for: Memory; Language*

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### Access for English Language Learners

*Reading, Speaking, Listening: MLR6 Three Reads.* Use this routine to support comprehension of the situation before looking at the questions. In the first read, students read the problem with the goal of comprehending the situation (e.g., Mai has a paperweight that is a dome. She wants to design a box for it.). If needed, discuss the meaning of unfamiliar terms at this time. Use the second read to identify the important quantities by asking students what can be counted or measured (e.g., the radius of the dome, the area of the base of the dome, the side lengths of the box). In the third read, ask students to brainstorm possible strategies to answer the questions. This helps students connect the language in the word problem and the reasoning needed to solve the problem keeping the intended level of cognitive demand in the task.

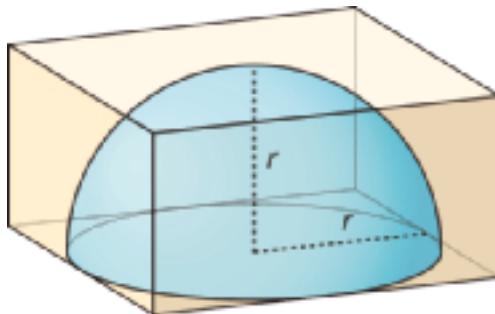
*Design Principle(s): Support sense-making*

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### Anticipated Misconceptions

Students may believe that since this box is not a cube, but a rectangular prism, the volume will not increase by 8 when the side lengths are doubled. Since volume of a rectangular prism is the product of length, width, and height, the side lengths do not need to be equal; if every side length is doubled, then the volume will be increased by  $2 \cdot 2 \cdot 2 = 8$ .

### Student Task Statement



1. Mai has a dome paperweight that she can use as a magnifier. The paperweight is shaped like a hemisphere made of solid glass, so she wants to design a box to keep it in so it won't get broken. Her paperweight has a radius of 3 cm.
  - a. What should the dimensions of the inside of box be so the box is as small as possible?
  - b. What is the volume of the box?
  - c. What is a reasonable estimate for the volume of the paperweight?
2. Tyler has a different box with side lengths that are twice as long as the sides of Mai's box. Tyler's box is just large enough to hold a different glass paperweight.
  - a. What is the volume of the new box?
  - b. What is a reasonable estimate for the volume of this glass paperweight?

c. How many times bigger do you think the volume of the paperweight in this box is than the volume of Mai's paperweight? Explain your thinking.

### Student Response

1.
  - a. 6 cm by 6 cm by 3 cm. The radius of the hemispherical paperweight is only half of the box's side length.  $3 \cdot 2$  gives us the entire length of the side. The height is the same as the radius.  
b. 108 cubic centimeters. 108 is  $6^2 \cdot 3$ .  
c. Answers vary. Correct responses should be less than 108 cubic centimeters. The box's volume is  $6^2 \cdot 3$ , and since there is space in the box that the hemisphere does not take up, the volume of the hemisphere has to be less than the volume of the box.
2.
  - a. 864 cubic centimeters. If every side length is doubled then the volume gets 8 times larger, and 864 is  $108 \cdot 8$ .  
b. Answers vary. Correct responses should be less than 864 cubic centimeters. There is still space in the box that the hemisphere does not take up, so the volume of the hemisphere has to be less than the volume of the box.  
c. Answers vary. Sample response: Eight times bigger. If the volume of a box gets 8 times bigger and the hemisphere has to fit inside the new box, then the hemisphere's volume must also get 8 times bigger.

### Activity Synthesis

Invite students to share their answers to the second question. The purpose of this discussion is for students to see that since a rectangular prism's volume gets larger by a factor of  $2^3$  when edge lengths are doubled then it would make sense for a hemisphere's volume to do the same. Ensure that students understand the volume calculated for the box holding the hemisphere is greater than the actual volume of the hemisphere because of the space left around the paperweight in the box and that this will be the upper bound of the volume of a sphere.

Tell students that in the next activity they are going to investigate a better way to estimate the volume of a hemisphere. If time allows, ask students to suggest shapes they are already familiar with that they could use to find the volume of a hemisphere.

## 19.3 Estimating Hemispheres

**15 minutes**

In this activity, students use different solid figures to estimate an upper and lower bound for the volume of a hemisphere. For the upper bound, the hemisphere fits snugly inside a cylinder whose height and radius are equal to the radius of the hemisphere. For the lower bound, the cone fits snugly inside the hemisphere, and its radius and height also equal the radius and height of the hemisphere.

Select students who use the reasoning from the first problem to assist them in answering the second problem to share during the Activity Synthesis. For example, since the cylinder and cone have the same dimensions, the volume of the cone must be  $\frac{1}{3}$  that of the cylinder.

## Addressing

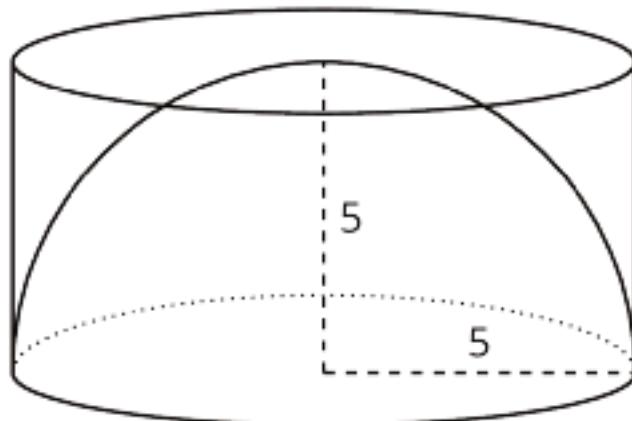
- 8.G.C.9

## Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Think Pair Share

## Launch

Keep students in the same groups. Display the image of a hemisphere of radius 5 that fits snugly inside a cylinder of the same radius and height.



Ask “How does the hemisphere affect the height of the cylinder?” and then give students one minute of quiet think time, then one minute to discuss their response with a partner. Ask partners to share their responses. If not mentioned by students, point out that the height of the cylinder is equal to the radius of the hemisphere.

Give students work time for the activity followed by a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Provide examples of actual 3-D models of cylinders, hemispheres, and cones for students to view or manipulate. Ask students to use the 3-D models and the volumes calculated in each question to estimate the volume of the hemisphere.

*Supports accessibility for: Visual-spatial processing; Conceptual processing*

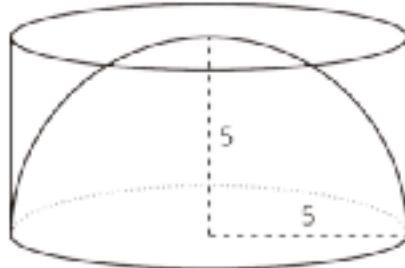
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## Anticipated Misconceptions

Students may not realize that the radius of the hemisphere determines the height of the cone and cylinder. If students struggle with identifying the needed information, remind them of the previous activity in which the radius of the cylinder was also the height of the box (rectangular prism). Ask students how that can help them determine the dimensions of the cone or cylinder.

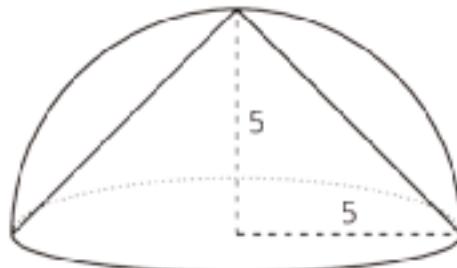
### Student Task Statement

1. A hemisphere with radius 5 units fits snugly into a cylinder of the same radius and height.



- a. Calculate the volume of the cylinder.
- b. Estimate the volume of the hemisphere. Explain your reasoning.

2. A cone fits snugly inside a hemisphere, and they share a radius of 5.



- a. What is the volume of the cone?
- b. Estimate the volume of the hemisphere. Explain your reasoning.

3. Compare your estimate for the hemisphere with the cone inside to your estimate of the hemisphere inside the cylinder. How do they compare to the volumes of the cylinder and the cone?

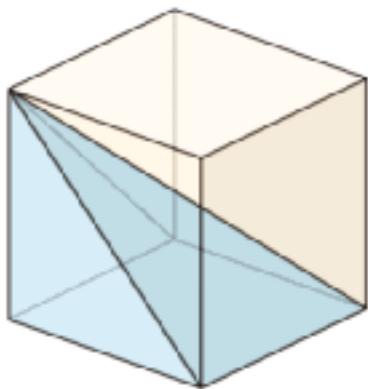
### Student Response

1. a.  $125\pi$ . The cylinder's volume can be calculated using  $V = \pi(5)^2(5)$ .  
b. Answers vary. Correct responses should be less than the volume of the cylinder.
2. a.  $\frac{125}{3}\pi$ . The cone's volume can be calculated using  $V = \frac{1}{3}\pi(5)^2(5)$ .  
b. Answers vary. Correct responses should be greater than the volume of the cone and less than the volume of the cylinder from the previous question.

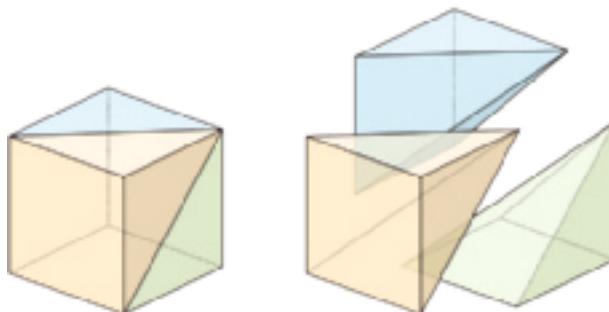
3. Answers vary. Sample response: The volume of the hemisphere will be greater than the volume of the cone but less than the volume of the cylinder.

### Are You Ready for More?

Estimate what fraction of the volume of the cube is occupied by the pyramid that shares the base and a top vertex with the cube, as in the figure.



### Student Response



In fact, the pyramid is precisely  $\frac{1}{3}$  of the volume of the cube. One way to see this is by decomposing the cube into three pyramids each congruent to the original.

### Activity Synthesis

Ask previously identified students to share their responses to the first two problems. Draw attention to any connections made between the two problems.

The purpose of this discussion is for students to recognize how the upper and lower bounds for the volume of a hemisphere are established by the cylinder and cone. Consider asking the following question:

- “What do the volumes of the cone and cylinder tell us about the volume of the hemisphere?” (The volume of the hemisphere has to be between the two volumes.)
- “Did anyone revise their original estimate for the hemisphere based on the calculation of the volume of the cone?” (Answers vary. If students estimated low values after calculating the volume of a cylinder, some may have needed to adjust their estimates.)

- “Compare the equations for volume of a cylinder and cone where radius and height are equal. If the volume of the hemisphere has to be between these two, what might an equation for the volume of a hemisphere look like?” (Cylinder volume:  $V = \pi r^3$ ; Cone volume:  $V = \frac{1}{3}\pi r^3$ . A possible hemisphere volume might be the average of these two, or  $V = \frac{2}{3}\pi r^3$ .)

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### Access for English Language Learners

*Writing, Speaking, Listening: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise and refine their response to the last question. Ask students to meet with 2–3 partners in a row for feedback. Provide students with prompts for feedback that will help them to strengthen their ideas and clarify their language (e.g., “Why do you think...?”, “How did you compare the volume of the two shapes?”, and “How does the first/second question help?”, etc.). Students can borrow ideas and language from each partner to strengthen their final version.

*Design Principle(s): Optimize output (for explanation)*

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## Lesson Synthesis

To have students describe some of the important highlights of the lesson, ask:

- “How did we use a cylinder to estimate a volume of a hemisphere that is an overestimate?”
- “How did we use a cone to estimate a volume of a hemisphere that is an underestimate?”
- “How did we get a closer estimate for the volume of a hemisphere?”
- “How did we use today’s work to estimate the volume of a sphere?”

Explain to students that we used figures we know how to find the volume of (a cone and cylinder) to try and estimate the volume of a figure we do not know how to find the volume of (a sphere). In the next lesson, we will do something similar to learn how to find the volume of a sphere and see how close our reasoning today was to the actual calculation.

## 19.4 A Mirror Box

**Cool Down: 5 minutes**

### Addressing

- 8.G.C

### Student Task Statement

A hemisphere-shaped security mirror fits exactly inside a rectangular prism box with a square base that has edge length 10 inches. What is a reasonable estimate for the volume of this mirror?

## Student Response

Answers vary. Sample responses include:

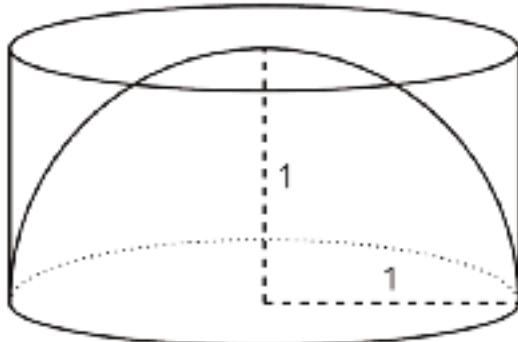
- Less than 500 cubic inches. The volume of the box that the mirror fits in is  $10^2 \cdot 5$ , and the mirror does not take up all the space in the box.
- Less than  $125\pi$  cubic inches. The volume of the cylinder that the mirror fits in is  $\pi(5)^2 \cdot 5$ , and the mirror does not take up all the space in the cylinder.
- More than  $\frac{125}{3}\pi$  cubic inches. The volume of the cone that fits in the mirror is  $\frac{1}{3}\pi(5)^2 \cdot 5$ , and the mirror is larger than the cone.

## Student Lesson Summary

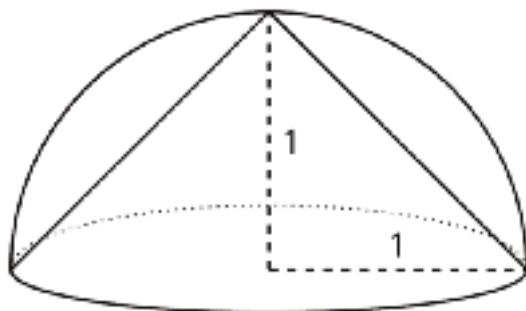
We can estimate the volume of a hemisphere by comparing it to other shapes for which we know the volume. For example, a hemisphere of radius 1 unit fits inside a cylinder with a radius of 1 unit and height of 1 unit.

Since the hemisphere is *inside* the cylinder, it must have a smaller volume than the cylinder making the cylinder's volume a reasonable over-estimate for the volume of the hemisphere.

The volume of this particular cylinder is about 3.14 units<sup>3</sup> since  $\pi(1)^2(1) = \pi$ , so we know the volume of the hemisphere is less than 3.14 cubic units.



Using similar logic, a cone of radius 1 unit and height 1 unit fits inside of the hemisphere of radius 1 unit.



Since the cone is *inside* the hemisphere, the cone must have a smaller volume than the hemisphere making the cone's volume a reasonable under-estimate for the volume of the hemisphere.

The volume of this particular cone is about 1.05 units<sup>3</sup> since  $\frac{1}{3}\pi(1)^2(1) = \frac{1}{3}\pi \approx 1.05$ , so we know the volume of the hemisphere is more than 1.05 cubic units.

Averaging the volumes of the cylinder and the cone, we can estimate the volume of the hemisphere to be about 2.10 units<sup>3</sup> since  $\frac{3.14+1.05}{2} \approx 2.10$ . And, since a hemisphere is half of

a sphere, we can also estimate that a sphere with radius of 1 would be double this volume, or about 4.20 units<sup>3</sup>.

# Lesson 19 Practice Problems

## Problem 1

### Statement

A baseball fits snugly inside a transparent display cube. The length of an edge of the cube is 2.9 inches.

Is the baseball's volume greater than, less than, or equal to  $2.9^3$  cubic inches? Explain how you know.

### Solution

Less than  $2.9^3$  cubic inches. The baseball fits inside the cube, and the cube's volume is  $2.9^3$  cubic inches. Therefore, the baseball's volume is less than  $2.9^3$  cubic inches.

## Problem 2

### Statement

There are many possible cones with a height of 18 meters. Let  $r$  represent the radius in meters and  $V$  represent the volume in cubic meters.

- Write an equation that represents the volume  $V$  as a function of the radius  $r$ .
- Complete this table for the function, giving three possible examples.

$r$	$V$
2	

- If you double the radius of a cone, does the volume double? Explain how you know.
- Is the graph of this function a line? Explain how you know.

### Solution

a.  $V = 6\pi r^2$

b. Answers vary. Sample response:

$r$	$V$
2	$24\pi$
4	$96\pi$
5	$150\pi$

c. No, the volume does not double. It is multiplied by four. Explanations vary.

d. It is *not* a line. The three points in the table do not lie on a straight line.

(From Unit 5, Lesson 18.)

## Problem 3

### Statement

A hemisphere fits snugly inside a cylinder with a radius of 6 cm. A cone fits snugly inside the same hemisphere.

- a. What is the volume of the cylinder?
- b. What is the volume of the cone?
- c. Estimate the volume of the hemisphere by calculating the average of the volumes of the cylinder and cone.

### Solution

- a.  $216\pi \text{ cm}^3$
- b.  $72\pi \text{ cm}^3$
- c.  $144\pi \text{ cm}^3$

## Problem 4

### Statement

- a. Find the hemisphere's diameter if its radius is 6 cm.
- b. Find the hemisphere's diameter if its radius is  $\frac{1000}{3}$  m.
- c. Find the hemisphere's diameter if its radius is 9.008 ft.
- d. Find the hemisphere's radius if its diameter is 6 cm.

e. Find the hemisphere's radius if its diameter is  $\frac{1000}{3}$  m.

f. Find the hemisphere's radius if its diameter is 9.008 ft.

## Solution

a. 12 cm

b.  $\frac{2000}{3}$  m (or equivalent)

c. 18.016 ft

d. 3 cm

e.  $\frac{500}{3}$  m (or equivalent)

f. 4.504 ft

## Problem 5

### Statement

After almost running out of space on her phone, Elena checks with a couple of friends who have the same phone to see how many pictures they have on their phones and how much memory they take up. The results are shown in the table.

number of photos	2,523	3,148	1,875
memory used in MB	8,072	10,106	6,037

- Could this information be reasonably modeled with a linear function? Explain your reasoning.
- Elena needs to delete photos to create 1,200 MB of space. Estimate the number of photos she should delete.

## Solution

a. Yes, all the points are close to  $y = 3.2x$ , where  $y$  represents the memory usage and  $x$  represents the number of photos.

b. About 375 photos ( $1,200 \div 3.2 = 375$ )

(From Unit 5, Lesson 9.)

# Lesson 20: The Volume of a Sphere

## Goals

- Calculate the volume of a sphere, cylinder, and cone which have a radius of  $r$  and height of  $2r$ , and explain (orally) the relationship between their volumes.
- Create an equation to represent the volume of a sphere as a function of its radius, and explain (orally and in writing) the reasoning.

## Learning Targets

- I can find the volume of a sphere when I know the radius.

## Lesson Narrative

The purpose of this lesson is for students to recognize that the volume of a sphere with radius  $r$  is  $\frac{4}{3}\pi r^3$  and begin to use the formula. Students inspect an image of a sphere that snugly fits inside a cylinder (they each have the same radius, and the height of the cylinder is equal to the diameter of the sphere), and use their intuition to guess about how the volume of the sphere relates to the volume of the cylinder, building on the work in the previous lesson. Then, they watch a video that shows a sphere inside a cylinder, and the contents of a cone (with the same base and height as the cylinder) are poured into the remaining space. This demonstration shows that for these figures, the cylinder contains the volumes of the sphere and cone together. From this observation, the volume of a specific sphere is computed. Then, the formula  $\frac{4}{3}\pi r^3$  for the volume of a sphere is derived. (At this point, this is taken to be true for any sphere even though we only saw a demonstration involving a particular sphere, cone, and cylinder. A general proof of the formula for the volume of a sphere would require mathematics beyond grade level.)

## Alignments

### Addressing

- 8.G.C: Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.
- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## Instructional Routines

- MLR5: Co-Craft Questions
- MLR7: Compare and Connect
- Notice and Wonder

## Required Preparation

For the A Sphere in a Cylinder activity, students will need to view a video.

## Student Learning Goals

Let's explore spheres and their volumes.

# 20.1 Sketch a Sphere

### Warm Up: 5 minutes

The purpose of this activity is for students to practice sketching spheres and labeling the radius and diameter of the sphere.

### Addressing

- 8.G.C

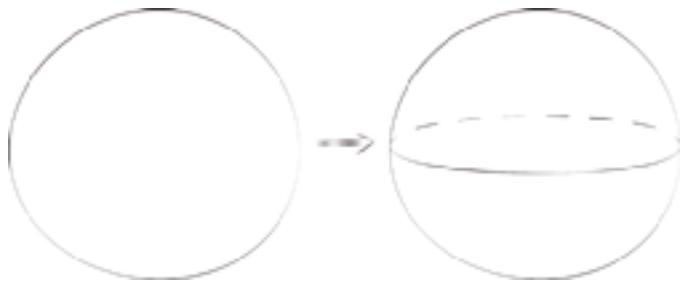
### Launch

Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

### Student Task Statement

Here is a method for quickly sketching a sphere:

- Draw a circle.
- Draw an oval in the middle whose edges touch the sphere.



1. Practice sketching some spheres. Sketch a few different sizes.

2. For each sketch, draw a radius and label it  $r$ .

### Student Response

1. Answers vary.
2. Answers vary.

### Activity Synthesis

Invite students to share their sketches. Ask students to share what the diameter would look like if they did not already draw one in. Remind students that sketches can be used to help visualize a problem where an image might not be provided. In today's lesson, they will be working with activities that might or might not have images provided and they should sketch or label any images provided to use as a tool to help understand the problem thoroughly.

# 20.2 A Sphere in a Cylinder

20 minutes

In this activity, students begin by looking at an image of a sphere in a cylinder. The sphere and cylinder have the same radius and the height of the cylinder is equal to the diameter of the sphere. Students consider the image and reason about how the volumes of the two figures compare to get a closer estimate of the volume of the sphere.

Then students watch a video that shows a sphere inside a cylinder set up like the image. A cone with the same base and height as the cylinder is introduced and its contents poured into the sphere, completely filling the empty space between the sphere and the cylinder. Students are asked to record anything they notice and wonder as they watch the video and a list is created as a class.

Once students are told that the cylinder, cone, and sphere all have the same radius length and the cylinder and cone have equal heights, they are asked to figure out the volumes of the cylinder and cone and are given time to discuss with a partner how to find the volume of the sphere. They are asked to write an explanation in words first before doing actual calculations so that they can reason about the relationships shown in the video and how they can use those relationships to figure out the unknown volume.

Identify students who discuss either method for calculating the volume of the sphere:

- subtract the volume of the cone from the volume of the cylinder.
- make the connection that a cone is  $\frac{1}{3}$  of the cylinder so the sphere must be the  $\frac{2}{3}$  that fills up the rest of the cylinder.

### **Addressing**

- 8.G.C

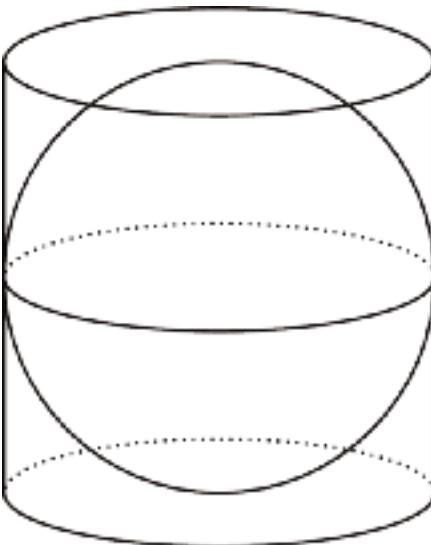
### **Instructional Routines**

- MLR7: Compare and Connect
- Notice and Wonder

### **Launch**

Arrange students in groups of 2. Display for all to see:

A sphere fits snugly into a cylinder so that its circumference touches the curved surface of the cylinder and the top and bottom touch the bases of the cylinder.



Ask, "In the previous lesson we thought about hemispheres in cylinders. Here is a sphere in a cylinder. Which is bigger, the volume of the cylinder or the volume of the sphere? Do you think the bigger one is twice as big, more than twice as big, or less than twice as big?" then give students 1 minute of quiet think time. Invite students to share their responses and keep their answers displayed for all to see throughout the lesson so that they can be referred to during the Lesson Synthesis.

Show the video and tell students to write down anything they notice or wonder while watching. Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If not mentioned by students, be sure these things are brought up:

- Notice:
  - The sphere fits inside the cylinder.
  - The sphere is filled up.
  - There is space around the sphere inside the cylinder.
  - It takes the volume of one cone to fill up the remaining spaces in the cylinder.
- Wonder:
  - Do the sphere and the cylinder have the same radius?
  - Do the cone and cylinder have the same radius?
  - Do the cone and cylinder have the same height?

Tell students that the sphere inside the cylinder seen in the video is the same as the one in the picture shown previously. Ask students: "does this give us any answers to the list of wonders?" (Yes, this tells us that the sphere and cylinder have the same radius.)

Tell students that the cone and cylinder have the same height and base area. Ask students:

- “Does this give us any more answers to the list of wonders?” (Yes, the cone and cylinder have the same height and radius.)
- “What does that mean about the volume of the cone and the volume of the cylinder in the video?” (The volume of the cone is  $\frac{1}{3}$  the volume of the cylinder.)

Show the video one more time and ask students to think about how we might calculate the volume of the sphere if we know the radius of the cone or cylinder. Give students 1 minute of quiet think time followed by time for a partner discussion. Give students time to work on the task followed by a whole-class discussion.

Video 'Volume of a Cylinder, Sphere, and Cone' available here: <https://player.vimeo.com/video/304138133>.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, ask students to color code the radii and heights in each of the figures and when calculating volume.

*Supports accessibility for: Visual-spatial processing*

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### Access for English Language Learners

*Representing: MLR7 Compare and Connect.* Use this routine to compare and contrast the different ways students calculated the volume of the sphere. Ask students to consider what is the same and what is different about each method used. Draw students' attention to the different calculations (e.g.,  $2(250/3)\pi$  or  $250\pi - (250/3)\pi = (500/3)\pi$ ) that equate to the volume of the sphere and how these calculations are related to the volume of the cone and cylinder. In this discussion, emphasize language used to help students make sense of strategies used to calculate the volume of the sphere. These exchanges strengthen students' mathematical language use and reasoning of volume.

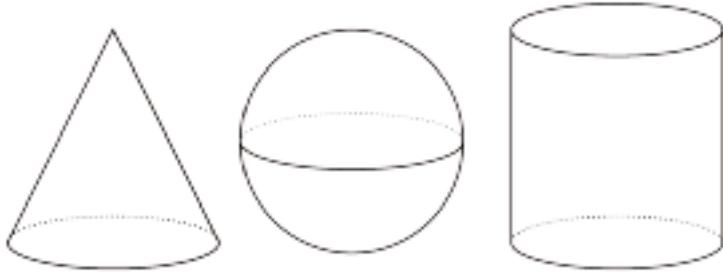
*Design Principle(s): Maximize meta-awareness; Support sense-making*

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### Anticipated Misconceptions

If students struggle to keep track of all the dimensions of the different figures, encourage them to label the images with the appropriate dimensions.

## Student Task Statement



Here are a cone, a sphere, and a cylinder that all have the same radii and heights. The radius of the cylinder is 5 units. When necessary, express all answers in terms of  $\pi$ .

1. What is the height of the cylinder?
2. What is the volume of the cylinder?
3. What is the volume of the cone?
4. What is the volume of the sphere? Explain your reasoning.

## Student Response

1. 10 units. The top of the sphere touches the top of the cylinder, so the diameter of the sphere is the height of the cylinder.
2.  $250\pi$  cubic units because  $V = \pi r^2 \cdot h$ .
3.  $\frac{250}{3}\pi$  cubic units because  $V = \frac{1}{3}\pi r^2 \cdot h$ .
4.  $\frac{500}{3}\pi$  cubic units. Answers vary. Sample response: Subtracting the volume of the cone from the volume of the cylinder gives the volume of the sphere. So the volume of the sphere is  $\frac{500}{3}\pi$  cubic units because  $250\pi - \frac{250}{3}\pi = \frac{500}{3}\pi$ .

## Activity Synthesis

Select previously identified students to share their methods for calculating the volume of the sphere. Ask students to compare the two methods mentioned in the narrative:

- “What is different about these two methods?” (One is using the fact that the volume of the cone is  $\frac{1}{3}$  of the volume of the cylinder so the sphere’s volume must make up the other  $\frac{2}{3}$ . The other subtracts the volumes we know in order to get the unknown volume of sphere.)
- “What do the two methods have in common?” (Both methods are calculating the same amount but in different ways.)

Display for all to see.

$$\pi r^2 \cdot h - \frac{1}{3}\pi r^2 \cdot h$$

Ask students: "What does this expression represent?" (The volume of the cylinder minus the volume of the cone.)

Draw students' attention back to the guesses they made at the start of the activity about how much bigger the cylinder's volume is than the sphere. Ask students if we can answer that question now. (Note: if students do not make the connection that the sphere's volume is  $\frac{2}{3}$  the volume of the cylinder, they will have another chance to look at the relationship in the next activity.)

## 20.3 Spheres in Cylinders

**10 minutes**

The purpose of this activity is to build from the concrete version in the previous activity to a generalized formula of a sphere with an unknown radius. The previous activity prepared students with strategies to work through this task where they must manipulate the variables in the volume equations. Students first calculate the volume of the cylinder and cone in the activity and use what they learned in the previous activity to calculate the volume of the sphere. Finally, they are asked about the relationship between the volume of the cylinder and sphere, which connects back to the discussion of the previous activity.

Identify students who:

- recognize that the volume of the sphere is  $\frac{2}{3}$  the volume of the cylinder and use that to easily come up with the general formula for volume of a sphere  $\frac{4}{3}\pi r^3$ .
- use the subtraction method discussed in the previous activity.

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR5: Co-Craft Questions

### Launch

Tell students that they are going to consider a different sphere inside of a cylinder along with a cone of the same height and radius as the sphere. This is similar to the previous activity; however, in this activity, the length of the radius  $r$  is unknown. Give students 4–6 minutes of quiet work time followed by a whole-class discussion.

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### Access for English Language Learners

*Writing, Conversing: MLR5 Co-Craft Questions.* Display the three images of the shapes along with the task statement without revealing the questions that follow. Ask students to write possible questions that could be answered about this situation. Invite students to share their questions with a partner, and select 2–3 groups to share with the class. Listen for questions that require students to reason about the relationships between the volume of the three different shapes. Next, reveal the questions of the activity. This helps students produce the language of mathematical questions and talk about the relationships between the volumes of the different shapes in this task.

*Design Principle(s): Maximize meta-awareness; Support sense-making*

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### Student Task Statement



Here are a cone, a sphere, and a cylinder that all have the same radii and heights. Let the radius of the cylinder be  $r$  units. When necessary, express answers in terms of  $\pi$ .

1. What is the height of the cylinder in terms of  $r$ ?
2. What is the volume of the cylinder in terms of  $r$ ?
3. What is the volume of the cone in terms of  $r$ ?
4. What is the volume of the sphere in terms of  $r$ ?
5. A volume of the cone is  $\frac{1}{3}$  the volume of a cylinder. The volume of the sphere is what fraction of the volume of the cylinder?

### Student Response

1.  $2r$  because the diameter of the sphere is the height of the cylinder
2.  $2\pi r^3$  because  $V = \pi r^2 2r$
3.  $\frac{2}{3}\pi r^3$  because  $V = \frac{1}{3}\pi r^2 2r$
4.  $\frac{4}{3}\pi r^3$  because  $2\pi r^3 - \frac{2}{3}\pi r^3$
5.  $\frac{2}{3}$

## Activity Synthesis

Select previously identified students to share their methods for calculating the volume of a sphere. Display for all to see the two different strategies side by side and ask students:

- “Which method did you use to calculate the volume of the sphere?”
- “Look at the method that you did not use. Explain to a partner why that method works.”
- “Which method do you prefer? Why?”

If students did not use both of the methods described in the narrative and outlined below, add them to the list of methods for students to compare before asking the questions. Display both methods side by side for all to see and ask the same questions.

The  $\frac{2}{3}$  method:

$$\begin{aligned}\text{volume of the sphere} &= \frac{2}{3}(\text{volume of the cylinder}) \\ &= \frac{2}{3}(2\pi r^3) \\ &= \frac{4}{3}\pi r^3\end{aligned}$$

Subtraction method:

$$\begin{aligned}\text{volume of the sphere} &= \text{volume of the cylinder} - \text{volume of the cone} \\ &= 2\pi r^3 - \frac{2}{3}\pi r^3 \\ &= (2 - \frac{2}{3})\pi r^3 \\ &= \frac{4}{3}\pi r^3\end{aligned}$$

Although either method works, there are reasons students might choose one over the other. The subtraction method is a bit more involved as it requires the distributive property to combine like terms and subtracting  $\frac{2}{3}$  from 2. It might make more sense to students, however, since it describes the video they saw in that the volume of the sphere is the difference between the volumes of the cylinder and cone. The  $\frac{2}{3}$  method is a bit simpler in terms of manipulating expressions, but students might not fully understand why the volume of the sphere is  $\frac{2}{3}$  the volume of the cylinder.

Add the formula  $V = \frac{4}{3}\pi r^3$  and a diagram of a sphere to your classroom displays of the formulas being developed in this unit.

## Lesson Synthesis

Display for all to see the equation  $V \approx 4r^3$ . Tell students “A quick estimate for the volume of a sphere of radius  $r$  that you can use if you don’t have a calculator is  $V \approx 4r^3$ . (No fraction or  $\pi$ !) How good of an approximation do you think this is? Can you come up with a better one?” Ask students to calculate the volume of a sphere with a radius of 10 inches using:

- the actual volume formula  $V = \frac{4}{3}\pi r^3$  (4188.79 cubic inches)
- the approximation formula  $V \approx 4r^3$  (4000 cubic inches)
- their own approximation formula. (Possible formula:  $V \approx 4 \cdot r^3 \cdot 1.05$ . 4200 cubic inches)

Give students quiet think time, then time to compare their improved approximations with a partner and decide which of their formulas is the ‘best approximation.’ Invite partners to share their choices with the class. Record and display student-created volume of a sphere approximation formulas for all to see.

## 20.4 Volumes of Spheres

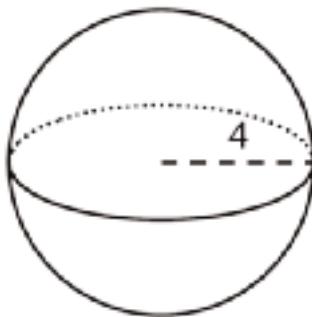
Cool Down: 5 minutes

### Addressing

- 8.G.C.9

#### Student Task Statement

Recall that the volume of a sphere is given by the formula  $V = \frac{4}{3}\pi r^3$ .



1. Here is a sphere with radius 4 feet. What is the volume of the sphere? Express your answer in terms of  $\pi$ .
2. A spherical balloon has a diameter of 4 feet. Approximate how many cubic feet of air this balloon holds. Use 3.14 as an approximation for  $\pi$ , and give a numerical answer.

### Student Response

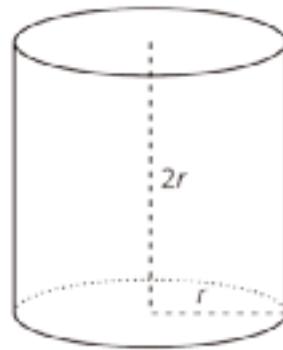
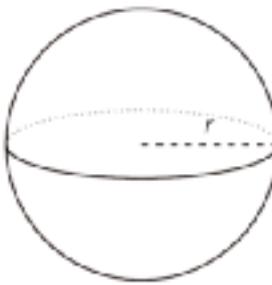
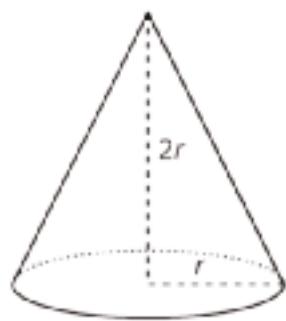
1.  $\frac{256}{3}\pi$  or  $85.33\pi$  cubic feet because  $V = \frac{4}{3}\pi(4)^3$

2. 33.49 cubic feet because  $V = \frac{4}{3}\pi(2)^3$

## Student Lesson Summary

Think about a sphere with radius  $r$  units that fits snugly inside a cylinder. The cylinder must then also have a radius of  $r$  units and a height of  $2r$  units. Using what we have learned about volume, the cylinder has a volume of  $\pi r^2 h = \pi r^2 \cdot (2r)$ , which is equal to  $2\pi r^3$  cubic units.

We know from an earlier lesson that the volume of a cone with the same base and height as a cylinder has  $\frac{1}{3}$  of the volume. In this example, such a cone has a volume of  $\frac{1}{3} \cdot \pi r^2 \cdot 2r$  or just  $\frac{2}{3}\pi r^3$  cubic units.



If we filled the cone and sphere with water, and then poured that water into the cylinder, the cylinder would be completely filled. That means the volume of the sphere and the volume of the cone add up to the volume of the cylinder. In other words, if  $V$  is the volume of the sphere, then

$$V + \frac{2}{3}\pi r^3 = 2\pi r^3$$

This leads to the formula for the volume of the sphere,

$$V = \frac{4}{3}\pi r^3$$

# Lesson 20 Practice Problems

## Problem 1

### Statement

- A cube's volume is 512 cubic units. What is the length of its edge?
- If a sphere fits snugly inside this cube, what is its volume?
- What fraction of the cube is taken up by the sphere? What percentage is this? Explain or show your reasoning.

## Solution

- 8 units
- $\frac{256}{3}\pi$  cubic units
- $\frac{\pi}{6}$ , which is slightly more than 50%. Sample explanation: The volume of the sphere as a fraction of the volume of the cube is  $\frac{\frac{4}{3}\pi \cdot 4^3}{512} = \frac{4}{3}\pi \cdot \left(\frac{1}{2}\right)^3 = \frac{4\pi}{24} = \frac{\pi}{6}$ . To make this fraction easier to work with, note that

$$\frac{\frac{4}{3}\pi \cdot 4^3}{512} = \frac{\frac{4}{3}\pi \cdot 4^3}{8^3} = \frac{4}{3}\pi \cdot \left(\frac{1}{2}\right)^3 = \frac{4\pi}{24} = \frac{\pi}{6}.$$

Since  $\pi$  is slightly more than 3, then  $\frac{\pi}{6}$  is slightly more than 50%.

## Problem 2

### Statement

Sphere A has radius 2 cm. Sphere B has radius 4 cm.

- Calculate the volume of each sphere.
- The radius of Sphere B is double that of Sphere A. How many times greater is the volume of B?

## Solution

- Sphere A:  $\frac{32}{3}\pi \text{ cm}^3$ , Sphere B:  $\frac{256}{3}\pi \text{ cm}^3$
- The volume of Sphere B is 8 times greater than the volume of Sphere A, which is  $2^3$  times.

## Problem 3

### Statement

Three cones have a volume of  $192\pi \text{ cm}^3$ . Cone A has a radius of 2 cm. Cone B has a radius of 3 cm. Cone C has a radius of 4 cm. Find the height of each cone.

### Solution

- Cone A has a height of 144 cm.
- Cone B has a height of 64 cm.
- Cone C has a height of 36 cm.

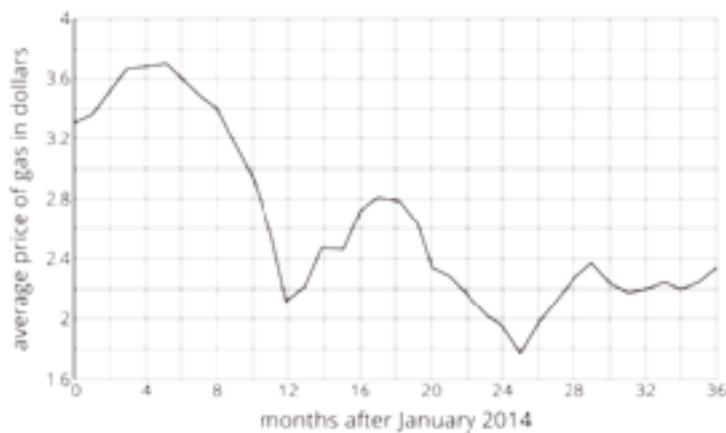
In each case, the height can be found by solving the formula  $192\pi = \frac{1}{3}\pi \cdot r^2 h$  for  $h$ .

(From Unit 5, Lesson 16.)

## Problem 4

### Statement

The graph represents the average price of regular gasoline in the United States in dollars as a function of the number of months after January 2014.



- a. How many months after January 2014 was the price of gas the greatest?
- b. Did the average price of gas ever get below \$2?
- c. Describe what happened to the average price of gas in 2014.

### Solution

- a. 5
- b. Yes, in the 24th, 25th, and 26th months

c. Answers vary. Sample response: The average price of gas rose from January until 5 months later (June) and then decreased for the rest of the year.

(From Unit 5, Lesson 5.)

## Problem 5

### Statement

Match the description of each sphere to its correct volume.

A. Sphere A: radius of 4 cm	1. $288\pi \text{ cm}^3$
B. Sphere B: diameter of 6 cm	2. $\frac{256}{3}\pi \text{ cm}^3$
C. Sphere C: radius of 8 cm	3. $36\pi \text{ cm}^3$
D. Sphere D: radius of 6 cm	4. $\frac{2048}{3}\pi \text{ cm}^3$

### Solution

- A: 2
- B: 3
- C: 4
- D: 1

## Problem 6

### Statement

While conducting an inventory in their bicycle shop, the owner noticed the number of bicycles is 2 fewer than 10 times the number of tricycles. They also know there are 410 wheels on all the bicycles and tricycles in the store. Write and solve a system of equations to find the number of bicycles in the store.

### Solution

$b = 10t - 2$ ,  $3t + 2b = 410$ . There are 178 bicycles in the store.

(From Unit 4, Lesson 15.)

# Lesson 21: Cylinders, Cones, and Spheres

## Goals

- Calculate the value of the radius of a sphere with a given volume using the structure of the equation, and explain (orally) the solution method.
- Determine what information is needed to solve a problem involving volumes of cones, cylinders, and spheres. Ask questions to elicit that information.

## Learning Targets

- I can find the radius of a sphere if I know its volume.
- I can solve mathematical and real-world problems about the volume of cylinders, cones, and spheres.

## Lesson Narrative

In this lesson, students use the formula for the volume of a sphere to solve various problems. They have opportunities to analyze common errors that people make when using this formula. They also use the structure of an equation to find the radius of a sphere when they know its volume. Finally, they have opportunities to practice using all of the new volume formulas they have learned in this unit to solve mixed problems with spheres, cylinders, and cones, reasoning about the effect of different dimensions on the volume of different figures.

## Alignments

### Addressing

- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

## Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR4: Information Gap Cards
- MLR8: Discussion Supports

## Required Materials

**Pre-printed slips, cut from copies of the Instructional master**

paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

**Tools for creating a visual display**

Any way for students to create work that can be easily displayed to the class. Examples: chart

## Required Preparation

Print and cut up slips from the Info Gap: Unknown Dimensions Instructional master. Prepare 1 set of slips for every 2 students. Provide students with access to make a visual display during the Lesson Synthesis.

### Student Learning Goals

Let's find the volume of shapes.

## 21.1 Sphere Arguments

### Warm Up: 5 minutes

The purpose of this warm-up is to catch errors students are making in calculating the volume of a sphere. Monitor for students who use the formula for a cylinder or cone, who use  $r^2$  instead of  $r^3$ , or who forget to include  $\pi$  as a factor in the computation.

### Addressing

- 8.G.C.9

### Launch

Arrange students in groups of 2. Tell students that if they have a hard time visualizing this sphere, they can sketch it. Give students 1–2 minutes of quiet work time followed by time to discuss their responses with their partner.

### Student Task Statement

Four students each calculated the volume of a sphere with a radius of 9 centimeters and they got four different answers.

- Han thinks it is 108 cubic centimeters.
- Jada got  $108\pi$  cubic centimeters.
- Tyler calculated 972 cubic centimeters.
- Mai says it is  $972\pi$  cubic centimeters.

Do you agree with any of them? Explain your reasoning.

### Student Response

Mai's calculation is correct. Explanations vary. Sample explanation: The volume of a sphere is found with the formula  $V = \frac{4}{3}\pi r^3$ . Using 9 for the radius, the volume is  $\frac{4}{3}\pi(9^3) = \frac{4}{3}\pi(729) = 972\pi$ .

### Activity Synthesis

For each answer, ask students to indicate whether or not they agree. Display the number of students who agree with each answer all to see. Invite someone who agreed with  $972\pi$  to explain their reasoning. Ask students if they think they know what the other students did incorrectly to get

their answers. (To get 108, Han and Jada likely used  $r^2$  instead of  $r^3$ , and Tyler probably didn't realize that multiplying  $\frac{4}{3}r^3$  by  $\pi$  is necessary.)

## 21.2 Sphere's Radius

### Optional: 5 minutes

The purpose of this activity is for students to think about how to find the radius of a sphere when its volume is known. Students can examine the structure of the equation for volume and reason about a number that makes the equation true. They can also notice that  $\pi$  is a factor on each side of the equation and divide each side by  $\pi$ . Both strategies simplify the solution process and minimize the need for rounding. Watch for students who substitute a value for  $\pi$  to each side of the equation, who use the structure of the equation to reason about the solution, or who solve another way, so strategies can be shared and compared in the whole-class discussion.

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR1: Stronger and Clearer Each Time

### Launch

Allow students 3–4 minutes work time followed by a whole-class discussion.

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### Access for English Language Learners

*Writing, Speaking: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise and refine their response to the question, "What is the value of  $r$  for this sphere? Explain how you know." Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them to strengthen their ideas and clarify their language (e.g., "What did you do first?", "Why did you...?", and "How did you deal with  $r^3$ ?", etc.). Students can borrow ideas and language from each partner to strengthen their final version.

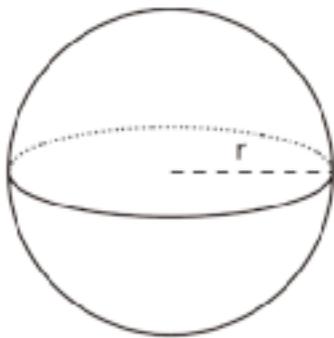
*Design Principle(s): Optimize output (for explanation)*

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### Anticipated Misconceptions

Students who substituted a value for  $\pi$  and solved the resulting equation might have rounded along the way, making the value for the radius slightly less than 6 while the actual value is exactly 6. This is a good opportunity to talk about the effects of rounding and how to minimize the error that rounding introduces.

## Student Task Statement



The volume of this sphere with radius  $r$  is  $V = 288\pi$ . This statement is true:

$$288\pi = \frac{4}{3}r^3\pi.$$

What is the value of  $r$  for this sphere? Explain how you know.

### Student Response

6 units. Explanations vary. Sample responses: Examine the equation  $288\pi = \frac{4}{3}r^3\pi$ .  $\pi$  appears on both sides of the equation. This means that the remaining factors on each side must be equal, so  $288 = \frac{4}{3}r^3$ . Multiplying each side by  $\frac{3}{4}$  gives  $216 = r^3$ . The number 6 yields 216 when cubed.

### Activity Synthesis

The purpose of the discussion is to examine how students reasoned through each step in solving for the unknown radius. Ask previously identified students to share their responses.

Consider asking students the following questions to help clarify the different approaches students took:

- “ $\pi$  appears on both sides of the volume equation. Did you deal with this as a first step or later in the solution process?”
- “How did you deal with the fraction in the equation?”
- “If the final step in your solution was solving for  $r$  when  $r^3 = 216$ , how did you solve? If you found that  $r^3$  was a different number, how did you solve?”

---

### Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.

*Supports accessibility for: Attention; Social-emotional skills*

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## 21.3 Info Gap: Unknown Dimensions

20 minutes

In this info gap activity, students determine and request the information needed to answer questions related to volume equations of cylinders, cones, and spheres.

The info gap structure requires students to make sense of problems by determining what information is necessary, and then to ask for information they need to solve it. This may take several rounds of discussion if their first requests do not yield the information they need (MP1). It also allows them to refine the language they use and ask increasingly more precise questions until they get the information they need (MP6).

Here is the text of the cards for reference and planning:

Info Gap: Unknown Dimensions  
Problem Card 1

A cone and a sphere have the same dimensions. What is the volume of the sphere?

Info Gap: Unknown Dimensions  
Data Card 1

- The volume of the cone is  $V = 144\pi \text{ cm}^3$ .
- The radius of the cone is the same as the radius of the sphere.
- $4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$

Info Gap: Unknown Dimensions  
Problem Card 2

A cone and a sphere have the same height. What is the volume of the sphere?

Info Gap: Unknown Dimensions  
Data Card 2

- The volume of the cone is  $V = 18\pi \text{ cm}^3$ .
- The radius of the sphere is half the height of the cone.
- The height of the cone is twice the value of the radius of the cone.
- $4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$

### Addressing

- 8.G.C.9

## Instructional Routines

- MLR4: Information Gap Cards

### Launch

Tell students that they will continue to refresh their skills of working with proportional relationships. Explain the Info Gap structure and consider demonstrating the protocol if students are unfamiliar with it.

Arrange students in groups of 2. In each group, distribute a problem card to one student and a data card to the other student. After you review their work on the first problem, give them the cards for the second problem and instruct them to switch roles.

---

### Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Display or provide students with a physical copy of the written directions. Check for understanding by inviting students to rephrase directions in their own words. Keep the display of directions visible throughout the activity.

*Supports accessibility for: Memory; Organization*

---

### Access for English Language Learners

*Conversing:* This activity uses MLR4 Information Gap to give students a purpose for discussing information necessary to solve problems involving the volume of shapes. Display questions or question starters for students who need a starting point such as: “Can you tell me . . . (specific piece of information)”, and “Why do you need to know . . . (that piece of information)?”

*Design Principle(s): Cultivate Conversation*

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### Anticipated Misconceptions

Students may have a difficulty making sense of the relationships between the dimensions of the two figures. Encourage these students to sketch the figures and label them carefully.

#### Student Task Statement

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.

Continue to ask questions until you have enough information to solve the problem.

4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner “*What specific information do you need?*” and wait for them to *ask* for information.

If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don’t have that information.
3. Before sharing the information, ask “*Why do you need that information?*” Listen to your partner’s reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

### Student Response

Problem card 1: The volume of the sphere is  $288\pi \text{ cm}^3$ . Students can calculate this value by finding the radius of the cone and then using the volume formula for a sphere, or they can use the fact that the volume of a sphere is twice that of the volume of a cone with the same dimensions.

Problem Card 2: Possible solution paths:

- Use the volume of the cone and  $h = 2r$  to find  $r = 3$ , so the volume of the sphere is  $36\pi \text{ cm}^3$ .
- Since the radius and height of the cone and sphere are equal, the volume of the sphere must be twice the volume of the cone, or  $36\pi \text{ cm}^3$ .

### Activity Synthesis

Select several groups to share their answers and reasoning for each of the problems. If any students made sketches with labeled dimensions, display these for all to see. In particular, contrast students who used volume formulas versus those who remembered that the volume of a cone is half the volume of a sphere with the same dimensions (radius and height).

Consider asking these discussion questions:

- For students who had a Problem Card:
  - “How did you decide what information to ask for? How did the information on your card help?”
  - “How easy or difficult was it to explain why you needed the information you were asking for?”
  - “Give an example of a question that you asked, the clue you received, and how you made use of it.”
  - “How many questions did it take for you to be able to solve the problem? What were those questions?”
  - “Was anyone able to solve the problem with a different set of questions?”
- For students who had a Data Card:
  - “When you asked your partner why they needed a specific piece of information, what kind of explanations did you consider acceptable?”
  - “Were you able to tell from their questions what volume question they were trying to answer? If so, how? If not, why might that be?”

## 21.4 The Right Fit

### 10 minutes

In this activity, students once again consider different figures with given dimensions, this time comparing their capacity to contain a certain amount of water. The goal is for students to not only apply the correct volume formulas, but to slow down and think about how the dimensions of the figures compare and how those measurements affect the volume of the figures.

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR8: Discussion Supports

### Launch

Arrange students in groups of 2. Allow 5 minutes of quiet work time, then a partner discussion followed by a whole-class discussion.

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### Access for Students with Disabilities

*Representation: Internalize Comprehension.* Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems and other text-based content.

*Supports accessibility for: Language; Conceptual processing*

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Provide sentence frames to support students as they work with their partner and explain their reasoning. For example, "I think \_\_\_\_ , because \_\_\_\_ .", or "I (agree/disagree) because \_\_\_\_ .", or "The volume is (greater than/less than) \_\_\_\_ , which means \_\_\_\_ ."

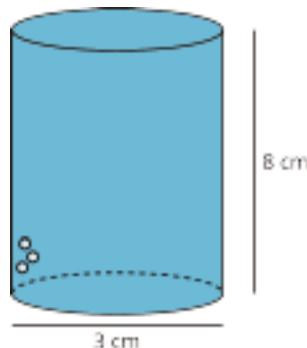
*Design Principle(s): Support sense-making; Optimize output for (explanation)*

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### Anticipated Misconceptions

Students might think that doubling a dimension doubles the volume. Help students recall and reason that changes in the radius, because it is squared or cubed when calculating volume, have a greater affect.

#### Student Task Statement



A cylinder with diameter 3 centimeters and height 8 centimeters is filled with water. Decide which figures described here, if any, could hold all of the water from the cylinder. Explain your reasoning.

1. Cone with a height of 8 centimeters and a radius of 3 centimeters.
2. Cylinder with a diameter of 6 centimeters and height of 2 centimeters.
3. Rectangular prism with a length of 3 centimeters, width of 4 centimeters, and height of 8 centimeters.

4. Sphere with a radius of 2 centimeters.

### Student Response

The cone, cylinder, and rectangular prism can hold the water. The sphere cannot. Explanations vary.  
Sample response:

The volume of the given cylinder is  $18\pi$ , or around 56.5 cubic centimeters:

$$(V = \pi(\frac{3}{2})^2(8) = \pi(\frac{9}{4})(8) = 18\pi).$$

1. Cone:  $V = \frac{1}{3}\pi(3^2)(8) = \frac{1}{3}\pi(9)(8) = 24\pi$ . Since the volume is greater than  $18\pi$ , the cone can hold the water.
2. Cylinder:  $V = \pi(3^2)(2) = \pi(9)(2) = 18\pi$ . Since the volume is equal to  $18\pi$ , the cylinder can hold the water.
3. Rectangular prism:  $V = (3)(4)(8) = 96$ . Since the volume is greater than 56.5, the rectangular prism can hold the water.
4. Sphere:  $V = \frac{4}{3}\pi(2^3) = \frac{4}{3}\pi(8) = \frac{32}{3}\pi$ . Since the volume is less than  $18\pi$ , the sphere cannot hold the water.

### Are You Ready for More?

A thirsty crow wants to raise the level of water in a cylindrical container so that it can reach the water with its beak.

- The container has diameter of 2 inches and a height of 9 inches.
- The water level is currently at 6 inches.
- The crow can reach the water if it is 1 inch from the top of the container.

In order to raise the water level, the crow puts spherical pebbles in the container. If the pebbles are approximately  $\frac{1}{2}$  inch in diameter, what is the fewest number of pebbles the crow needs to drop into the container in order to reach the water?

### Student Response

The current volume of water is  $V_1 = \pi \cdot 1^2 \cdot 6$  cubic inches and the volume required is  $V_2 = \pi \cdot 1^2 \cdot 8$  cubic inches. This means that the difference in volumes is  $\pi \cdot 1^2 \cdot 2 = 6.28$  cubic inches. Each pebble has volume  $V_p = \frac{4}{3} \cdot \pi \cdot (\frac{1}{4})^3 = 0.065$  cubic inches. In order to raise the water by a volume of 6.28 cubic inches,  $6.28/0.065 = 96.62$  pebbles need to be added. Since fractional pebbles are not possible, the crow needs to add 97 pebbles.

### Activity Synthesis

The purpose of this discussion is to compare volumes of different figures by computation and also by considering the effect that different dimensions have on volume.

Ask students if they made any predictions about the volumes before directly computing them and, if yes, how they were able to predict. Students might have reasoned, for example, that the second cylinder had double the radius of the first, which would make the volume 4 times as great, but the height was only  $\frac{1}{4}$  as great so the volume would be the same. Or they might have seen that the cone would have a greater volume since the radius was double and the height the same, making the volume (if it were another cylinder) 4 times as great, so the factor of  $\frac{1}{3}$  for the cone didn't bring the volume down below the volume of the cylinder.

## Lesson Synthesis

In this unit, students have learned how to find the volume of cylinders, cones, and spheres, how to find an unknown dimension when the volume and another dimension are known, and how to reason about the effects of different dimensions on volume. Assign groups of 2–3 students one of the questions shown here and provide them with the tools to make a visual display explaining their response. Encourage students to make their displays as though they are explaining the answer to the question to someone who is not in the class and to make up values for dimensions to use to illustrate their ideas. Suggest sketches of figures where appropriate.

- “Describe some relationships between the volumes of cylinders, cones, and spheres.”
- “How do we find a missing dimension when we know the volume and another dimension of a cylinder, cone, or sphere (or just the volume in the case of the sphere)?”
- “What happens to the volume of a cylinder or cone when its height is doubled? Tripled?”
- “What happens to the volume of a sphere when its height is doubled? Tripled?”
- “What happens to the volume of a cylinder, cone, or sphere when its radius is doubled? Tripled?”
- “What happens to the volume of a cylinder or cone when its height is doubled and its radius is halved?”
- “What happens to the volume of a cylinder or cone when its radius is doubled and its height is halved?”

## 21.5 New Four Spheres

**Cool Down:** 5 minutes

Students synthesize the material they have learned about using the volume of a sphere formula by sorting different representations of the equation.

### Addressing

- 8.G.C.9

## Student Task Statement

Some information is given about each sphere. Order them from least volume to greatest volume. You may sketch a sphere to help you visualize if you prefer.

Sphere A: Has a radius of 4

Sphere B: Has a diameter of 6

Sphere C: Has a volume of  $64\pi$

Sphere D: Has a radius double that of sphere B.

## Student Response

B, C, A, D

Sphere A: Has a radius of 4, so its volume is  $\frac{256}{3}\pi$ .

Sphere B: Has a diameter of 6, so its radius is 3, and its volume is  $36\pi$ .

Sphere C: Has a volume of  $64\pi$ .

Sphere D: Has a radius twice as large as sphere B, so its radius is 6, and its volume is  $288\pi$ .

## Student Lesson Summary

The formula

$$V = \frac{4}{3}\pi r^3$$

gives the volume of a sphere with radius  $r$ . We can use the formula to find the volume of a sphere with a known radius. For example, if the radius of a sphere is 6 units, then the volume would be

$$\frac{4}{3}\pi(6)^3 = 288\pi$$

or approximately 904 cubic units. We can also use the formula to find the radius of a sphere if we only know its volume. For example, if we know the volume of a sphere is  $36\pi$  cubic units but we don't know the radius, then this equation is true:

$$36\pi = \frac{4}{3}\pi r^3$$

That means that  $r^3 = 27$ , so the radius  $r$  has to be 3 units in order for both sides of the equation to have the same value.

Many common objects, from water bottles to buildings to balloons, are similar in shape to rectangular prisms, cylinders, cones, and spheres—or even combinations of these shapes! Using the volume formulas for these shapes allows us to compare the volume of different types of objects, sometimes with surprising results.

For example, a cube-shaped box with side length 3 centimeters holds less than a sphere with radius 2 centimeters because the volume of the cube is 27 cubic centimeters ( $3^3 = 27$ ), and the volume of the sphere is around 33.51 cubic centimeters ( $\frac{4}{3}\pi \cdot 2^3 \approx 33.51$ ).

# Lesson 21 Practice Problems

## Problem 1

### Statement

A scoop of ice cream has a 3-inch diameter. How tall should the ice cream cone of the same diameter be in order to contain all of the ice cream inside the cone?

## Solution

6 inches (The volume of the ice cream is  $\frac{4}{3}\pi(1.5)^3$ , which must be the same as the volume of the cone given by  $\frac{1}{3}\pi(1.5)^2 h$ . Set these two expressions equal to each other and solve for  $h$ .)

## Problem 2

### Statement

Calculate the volume of the following shapes with the given information. For the first three questions, give each answer both in terms of  $\pi$  and by using 3.14 to approximate  $\pi$ . Make sure to include units.

- a. Sphere with a diameter of 6 inches
- b. Cylinder with a height of 6 inches and a diameter of 6 inches
- c. Cone with a height of 6 inches and a radius of 3 inches
- d. How are these three volumes related?

## Solution

- a.  $36\pi$ , about 113.04 cubic inches
- b.  $54\pi$ , about 169.56 cubic inches
- c.  $18\pi$ , about 56.52 cubic inches
- d. Answers vary. Sample response: The volume of the cone plus the volume of the sphere equals the volume of the cylinder. This is like the video in a previous lesson where the sphere fits snugly inside of the cylinder, and after pouring a cone-ful of water, the cylinder fills completely to the top. The cone takes up  $\frac{1}{3}$  of the cylinder, and the sphere takes up the other  $\frac{2}{3}$ .

## Problem 3

### Statement

A coin-operated bouncy ball dispenser has a large glass sphere that holds many spherical balls. The large glass sphere has a radius of 9 inches. Each bouncy ball has radius of 1 inch and sits inside the dispenser.

If there are 243 bouncy balls in the large glass sphere, what proportion of the large glass sphere's volume is taken up by bouncy balls? Explain how you know.

## Solution

About 33% (or  $\frac{1}{3}$ ). The large glass sphere has radius 9 inches, so its volume in cubic inches is  $\frac{4}{3}\pi(9)^3$ . This is about 3052 cubic inches. Each bouncy ball has radius 1 inch. The volume of 243 bouncy balls in cubic inches is  $243 \cdot \frac{4}{3}\pi(1)^3$ . This is about 1017 cubic inches. To find the proportion taken up by bouncy balls, divide  $\frac{1017}{3052} \approx 0.33$ . About 33% of the space is taken up by bouncy balls. By using the exact volume, the result is exactly  $\frac{1}{3}$ .

## Problem 4

### Statement

A farmer has a water tank for cows in the shape of a cylinder with radius of 7 ft and a height of 3 ft. The tank comes equipped with a sensor to alert the farmer to fill it up when the water falls to 20% capacity. What is the volume of the tank be when the sensor turns on?

## Solution

About 92 cubic feet (The volume of the cylinder is given by  $V = \pi \cdot 7^2 \cdot 3$ , which is approximately 462 cubic feet. The sensor turns on when 20% of this volume remains, and 20% of 462 is a little more than 92 cubic feet.)

(From Unit 5, Lesson 13.)

# Section: Let's Put It to Work

## Lesson 22: Volume As a Function of . . .

### Goals

- Describe (orally) how a change in the radius of a sphere affects the volume.
- Interpret (orally and in writing) functions that represent the volume of a sphere, cone, and cylinder, using different representations.

### Learning Targets

- I can compare functions about volume represented in different ways.

### Lesson Narrative

The purpose of this optional culminating lesson is to give students more experience working with non-linear functions that arise out of the work students have been doing with the volume of cylinders, cones, and spheres. In the first activity, students reason about how scaling the radius affects the volume of a sphere, similar to their earlier work considering how volume is affected by changing one or two dimensions of cylinder or cone.

In the second activity, students work with three different functions (represented three different ways) showing the height of water in three different shapes as a function of the volume of water. They consider questions such as:

- Which container is largest?
- At what volume of water poured is the height of water in the containers the same?
- For what range of water volume poured does a particular container have the greatest height of water?
- How do the representations show the maximum height of each container?

### Alignments

#### Addressing

- 8.F.A: Define, evaluate, and compare functions.
- 8.G.C.9: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

### Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR8: Discussion Supports
- Notice and Wonder

- Think Pair Share

## Required Materials

### Straightedges

A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a

straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

## Required Preparation

Provide access to straightedges for the activity A Cylinder, a Cone, and a Sphere.

### Student Learning Goals

Let's compare water heights in different containers.

## 22.1 Missing Information?

### Warm Up: 5 minutes

In this warm-up, students reason about the volume and dimensions of a cylinder based on information given about a sphere of the same height. The purpose of this warm-up is for students to recognize when they do not have enough information to reach a single answer. While they can determine the height of the cylinder, the radius is completely undefined, which means the volume of the cylinder could be anything.

### Addressing

- 8.G.C.9

### Launch

Give students quiet work time followed by a whole-class discussion.

### Anticipated Misconceptions

Some students may not understand why the second problem talks about “possible” volumes if they assume the radius of the cylinder is the same as the sphere. Ask these students to explain how they found the radius of the cylinder to help them notice that the problem never gives that information.

### Student Task Statement

A cylinder and sphere have the same height.

1. If the sphere has a volume of  $36\pi$  cubic units, what is the height of the cylinder?
2. What is a possible volume for the cylinder? Be prepared to explain your reasoning.

### Student Response

1. 6 units
2. Answers vary. Students could answer anything here as the radius is undetermined. Knowing the height, though, creates a function with variables  $V$  and  $r$ . Students just need to pick an  $r$ ,

and they have a volume that works. Going the other way is harder but possible. Teachers can hint here that there is a way to figure out what number was squared, looking ahead to 8.8.

### Activity Synthesis

Ask students to share how they calculated the height of the cylinder. If any students made a sketch, display these for all to see.

Select at least 5 students to give a possible volume for the cylinder and record these for all to see. Ask students,

- “Why do we not know what the volume of the cylinder is?” (We don’t know the radius, only the height, so the volume could be anything.)
- “Is knowing the height of a sphere enough information to determine the volume?” (Yes. The volume of a sphere is based on the radius, which is half the height.)

Tell students that in the next activity, they will investigate how changes to the radius of a sphere changes the volume of the sphere.

## 22.2 Scaling Volume of a Sphere

**Optional: 15 minutes**

Building on work in previous lessons where students investigated how changing one or two dimensions affects the volume of a shape, in this activity, students scale the radius of a sphere and compare the resulting volumes. They will predict how doubling or halving the radius of a sphere affects the volume.

Identify students using different reasoning to answer the last question. For example, some students may calculate the radius of the larger sphere in order to find  $\frac{1}{5}$  of that value to get to the volume of the smaller sphere while others may reason about the volume formula and how volume changes when the radius goes from  $r$  to  $\frac{1}{5}r$ .

### Addressing

- 8.G.C.9

### Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Think Pair Share

### Launch

Arrange students in groups of 2. Give 2–3 minutes of quiet work time to complete the first problem on their own, then time to discuss their solutions with a partner. Groups then finish the remaining problems together followed by a whole-group discussion.

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### Access for Students with Disabilities

*Action and Expression: Develop Expression and Communication.* Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, “It looks like...,” “Is it always true that...?” and “We can agree that...”

*Supports accessibility for: Language; Organization*

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### Access for English Language Learners

*Writing, Speaking, Representing: MLR1 Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise and refine their response to the question, “What happens to the volume of this sphere if its radius is halved?” Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them to strengthen their ideas and clarify their language (e.g., “How did you determine it was \_\_\_ times larger?”, “Why do you think...?”, and “How does the previous question help us to solve this question?”, etc.). Students can borrow ideas and language from each partner to strengthen their final version.

*Design Principle(s): Optimize output (for explanation)*

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### Anticipated Misconceptions

Students may not correctly use exponent rules for  $(2r)^3$ . Students also may need to be reminded of exponent rules for fractions.

### Student Task Statement

1. Fill in the missing volumes in terms of  $\pi$ . Add two more radius and volume pairs of your choosing.

radius	1	2	3	$\frac{1}{2}$	$\frac{1}{3}$	100			r
volume	$\frac{4}{3}\pi$								

- a. How does the volume of a sphere with radius 2 cm compare to the volume of a sphere with radius 1 cm?
- b. How does the volume of a sphere with radius  $\frac{1}{2}$  cm compare to the volume of a sphere with radius 1 cm?

2. A sphere has a radius of length  $r$ .

a. What happens to the volume of this sphere if its radius is doubled?

b. What happens to the volume of this sphere if its radius is halved?

3. Sphere Q has a volume of  $500 \text{ cm}^3$ . Sphere S has a radius  $\frac{1}{5}$  as large as Sphere Q. What is the volume of Sphere S?

### Student Response

1. Answers vary for the radius and volume pairs of students' choosing. The rest of the table should look as follows:

radius	1	2	3	$\frac{1}{2}$	$\frac{1}{3}$	100			$r$
volume	$\frac{4}{3}\pi$	$\frac{32}{3}\pi$	$36\pi$	$\frac{1}{6}\pi$	$\frac{4}{81}\pi$	$\frac{4,000,000}{3}\pi$			$\frac{4}{3}\pi r^3$

a. The volume of a sphere with radius 2 is 8 times as large as the volume of a sphere with radius 1.

b. The volume of a sphere with radius  $\frac{1}{2}$  is  $\frac{1}{8}$  as large as the volume of a sphere with radius 1.

2. a. A sphere with two times the radius will have volume  $V = \frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3$ , which is 8 times larger than the original.

b. A sphere with a radius half the original will have  $V = \frac{4}{3}\pi\left(\frac{r}{2}\right)^3 = \frac{1}{6}\pi r^3$ , which is  $\frac{1}{8}$  of the original volume.

3.  $4 \text{ cm}^3$ . If the radius is  $\frac{1}{5}$  as large, the volume will be  $\left(\frac{1}{5}\right)^3$  as large, or  $\frac{1}{125}$ . Therefore, sphere S will be  $500\left(\frac{1}{125}\right)$ , or  $4 \text{ cm}^3$ .

### Activity Synthesis

Display the completed table for all to see, and invite groups to share a pair they added to the table along with their responses to the first two problems.

Select previously identified students to share their responses to the last question. If students did not reason about the question using the formula for volume of a sphere (that is, by reasoning that if the radius is  $\frac{1}{5}$  as large then  $V = \frac{4}{3}\pi\left(\frac{1}{5}r\right)^3 = \frac{1}{125}\left(\frac{4}{3}\pi r^3\right)$ ), ask students to consider their responses to the second part of the second question, but replace  $\frac{1}{2}$  with  $\frac{1}{5}$ .

## 22.3 A Cylinder, a Cone, and a Sphere

**Optional: 25 minutes**

The purpose of this activity is for students to bring together several ideas they have been working with. In particular: calculating volume and dimensions of round objects, comparing functions represented in different ways, interpreting the slope of a graph in context, reasoning about specific function values, and reasoning about when functions have the same value.

While students are only instructed to add a graph of the cylinder to the given axes, watch for students who plot the values for the sphere as well and ask to share their graphs during the Activity Synthesis.

### Addressing

- 8.F.A
- 8.G.C.9

### Instructional Routines

- MLR8: Discussion Supports
- Notice and Wonder

### Launch

Tell students to close their books or devices and display the graph in the activity for all to see. Tell students that the graph represents water filling a container.

Tell students that their job is to think of at least one thing they notice and at least one thing they wonder. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner. Possible responses:

Things students may Notice:

- The graph is not a linear function.
- The graph is a piecewise function.
- The maximum height of the function is 6 inches.

Things students may Wonder:

- What is the shape of the container?
- What is happening when the height stops at 6 inches?
- Why does the volume increase but the height stay constant on the far right?

Discuss possible responses for questions that students wondered. If not suggested by students, ask what would happen if you kept pouring water into a container even though the water level had reached the top. Ensure students understand that the height stops at 6 inches because that is how tall the container is. Any more water poured into the container at that point just overflows.

Arrange students in groups of 2–3. Provide access to straightedges. Give groups work time followed by a whole-class discussion.

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### Access for Students with Disabilities

*Action and Expression: Internalize Executive Functions.* Chunk this task into more manageable parts to support students who benefit from support with organization and problem solving. For example, present one question at a time.

*Supports accessibility for: Organization; Attention*

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### Anticipated Misconceptions

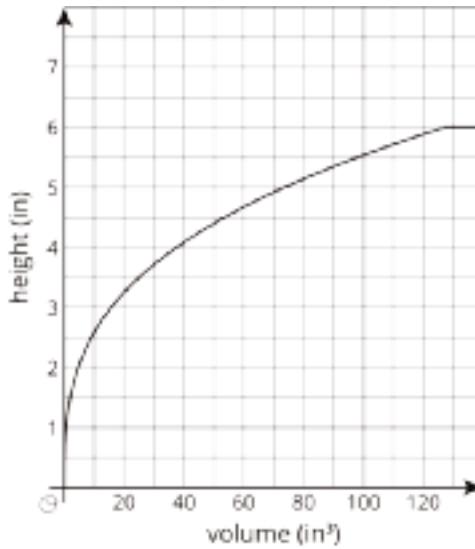
Watch for students struggling to compare in different representations and suggest graphing the points in the table.

For the fourth question, if students calculate volume of water when the height is 6 inches for each shape, direct them to determine whether there is a time when any of the containers have the same height and the same volume of water. (Another way to think of this is an intersection in the graph.)

### Student Task Statement

Three containers of the same height were filled with water at the same rate. One container is a cylinder, one is a cone, and one is a sphere. As they were filled, the relationship between the volume of water and the height of the water was recorded in different ways, shown here:

- Cylinder:  $h = \frac{V}{4\pi}$
- Cone:



- Sphere:

volume (in <sup>3</sup> )	height (in)
0	0
8.38	1
29.32	2
56.55	3
83.76	4
104.72	5
113.04	6
120	6
200	6

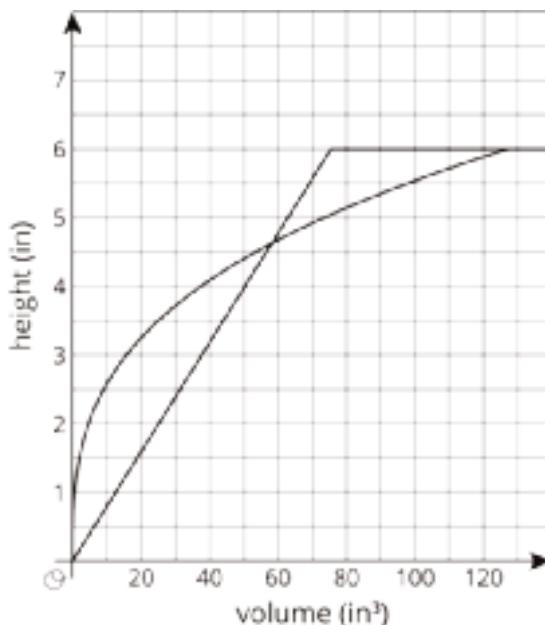
1. The maximum volume of water the cylinder can hold is  $24\pi$ . What is the radius of the cylinder?
2. Graph the relationship between the volume of water poured into the cylinder and the height of water in the cylinder on the same axes as the cone. What does the slope of this line represent?
3. Which container can fit the largest volume of water? The smallest?

4. About how much water does it take for the cylinder and the sphere to have the same height? The cylinder and the cone? Explain how you know.
5. For what approximate range of volumes is the height of the water in the cylinder greater than the height of the water in the cone? Explain how you know.
6. For what approximate range of volumes is the height of the water in the sphere less than the height of the water in the cylinder? Explain how you know.

### Student Response

1. 2 in. We know the maximum volume of water is  $24\pi$  in<sup>3</sup> and the height of the cylinder is 6 in since it is the same as the sphere. As such,  $r = 2$  since  $24\pi = \pi r^2 \cdot 6$  must be true.

2.



The slope of the line is  $\frac{1}{4\pi}$ . This means that for each volume increase of 1 in<sup>3</sup>, the height increases by  $\frac{1}{4\pi}$  in.

3. The cone can hold the most water since the graph shows approximately 127.23 in<sup>3</sup> when the cone is full. The cylinder can hold the smallest amount of water,  $24\pi$  in<sup>3</sup>.
4. Since the data for volume and height of the spherical container is discrete, we can only say that the volumes and heights must be equal somewhere between the two points at (8.38, 1) and (29.32, 2). An approximation between these two points is that when the volume is about 20 in<sup>3</sup>, the height is about 1.5 in for both the cylinder and sphere. For the cylinder and cone, we can see the intersection of the two points on the graph from problem 2, which occurs approximately when the volume is 58 in<sup>3</sup> and the height is 4.5 in.

5. When the volume is between  $58 \text{ in}^3$  and  $125 \text{ in}^3$ , the height of water in the cylinder is greater than the height of water in the cone.

6. When the volume is between about  $20 \text{ in}^3$  (or the data point chosen for the first part of problem 4) and  $113 \text{ in}^3$ , the height of water in the cylinder is greater than the height of water in the sphere.

### Activity Synthesis

Select previously identified students who graphed the data from the table in order to complete the problems. Display one or two of these representations for all to see.

Consider asking the following questions:

- “If I showed you this graph without telling you which function represented each shape, how could you figure out which one represents the cone, the cylinder, and the sphere?” (The graph of the cylinder must be linear, since the shape of the container as the water level rises never changes. The graph of the cone would first fill quickly, but then fill more slowly as it got wider near the top. (Note: it helps to know the cone is tip down here.) The graph of the sphere would change partway through, since it would start fast, slow down toward the middle where the sphere is widest, then speed up again as the sphere narrows, and the table data, or discrete points, is the only representation that does that.)
- “Can you use the information provided about each function to determine the radius of the sphere and cone?” (The radius of the sphere is half the height, so the radius is 3 in. The cone has a volume of  $127.23 \text{ in}^3$ , so we can tell that  $r^2$  is about 20.25. We can use guess-and-check to find  $r$ , since 20.25 is between 16 and 25. We can check what happens when  $r = 4.5$ , which does give a value of 20.25 for  $r^2$ . So the radius of the cone is about 4.5 in.)
- “For each of these graphs, is volume a function of height?” (No, when the height reaches 6 inches, the extra water spills out, so the volume keeps increasing. Therefore, if height is the input, there are multiple outputs of volume for each graph.)
- “What could we change in order to make volume a function of height for this scenario?” (If we stopped pouring when height was 6 inches, there would not be multiple volumes for one height.)

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### Access for English Language Learners

*Speaking: MLR8 Discussion Supports.* Use this routine to support student responses to the question: "If I showed you the graph (of the cone) without telling you which function represented each shape, how could you figure out which one represents the cone, the cylinder, and the sphere?" Provide students with a sentence frame such as: "The graph represents the volume of the \_\_\_\_\_ because \_\_\_\_\_ ." or "The graph of a \_\_\_\_\_ would \_\_\_\_\_ ." This will help students produce and make sense of the language needed to reason about the relationship between volume and height of different shapes.

*Design Principle(s): Support sense-making; Optimize output (for justification)*

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# Family Support Materials

# Family Support Materials

## Functions and Volume

Here are the video lesson summaries for Grade 8, Unit 5: Functions and Volume. Each video highlights key concepts and vocabulary that students learn across one or more lessons in the unit. The content of these video lesson summaries is based on the written Lesson Summaries found at the end of lessons in the curriculum. The goal of these videos is to support students in reviewing and checking their understanding of important concepts and vocabulary. Here are some possible ways families can use these videos:

- Keep informed on concepts and vocabulary students are learning about in class.
- Watch with their student and pause at key points to predict what comes next or think up other examples of vocabulary terms (the bolded words).
- Consider following the Connecting to Other Units links to review the math concepts that led up to this unit or to preview where the concepts in this unit lead to in future units.

Grade 8, Unit 5: Functions and Volume	Vimeo	YouTube
Video 1: Inputs and Outputs (Lessons 1–3)	<a href="#">Link</a>	<a href="#">Link</a>
Video 2: Representing and Interpreting Functions (Lessons 4–7)	<a href="#">Link</a>	<a href="#">Link</a>
Video 3: Linear Functions and Rates of Change (Lessons 8–10)	<a href="#">Link</a>	<a href="#">Link</a>
Video 4: Cylinders and Cones (Lessons 11–16)	<a href="#">Link</a>	<a href="#">Link</a>
Video 5: Spheres (Lessons 19–21)	<a href="#">Link</a>	<a href="#">Link</a>

### Video 1

Video 'VLS G8U5V1 Inputs and Outputs (Lessons 1–3)' available here:  
<https://player.vimeo.com/video/493392446>.

### **Video 2**

Video 'VLS G8U5V2 Representing and Interpreting Functions (Lessons 4–7)' available here:  
<https://player.vimeo.com/video/498502033>.

### **Video 3**

Video 'VLS G8U5V3 Linear Functions and Rates of Change (Lessons 8–10)' available here:  
<https://player.vimeo.com/video/490206352>.

### **Video 4**

Video 'VLS G8U5V4 Cylinders and Cones (Lessons 11–16)' available here:  
<https://player.vimeo.com/video/493397357>.

### **Video 5**

Video 'VLS G8U5V5 Spheres (Lessons 19–21)' available here: <https://player.vimeo.com/video/498158048>.

### **Connecting to Other Units**

- *Coming soon*

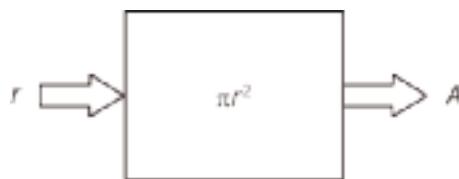
# Inputs and Outputs

## Family Support Materials 1

This week, your student will be working with **functions**. A function is a rule that produces a single output for a given input.

Not all rules are functions. For example, here's a rule: the input is "first letter of the month" and the output is "the month." If the input is J, what is the output? A function must give a single output, but in this case the output of this rule could be January, June, or July, so the rule is not a function.

Here is an example of a rule that is a function: input a number, square it, then multiply the result by  $\pi$ . Using  $r$  for the input and  $A$  for the output, we can draw a diagram to represent the function:



We could also represent this function with an equation,  $A = \pi r^2$ . We say that the input of the function,  $r$ , is the **independent variable** and the output of the function,  $A$ , is the **dependent variable**. We can choose any value for  $r$ , and then the value of  $A$  depends on the value of  $r$ . We could also represent this function with a table or as a graph. Depending on the question we investigate, different representations have different advantages. You may recognize this rule and know that the area of a circle depends on its radius.

Here is a task to try with your student:

Jada can buy peanuts for \$0.20 per ounce and raisins for \$0.25 per ounce. She has \$12 to spend on peanuts and raisins to make trail mix for her hiking group.

1. How much would 10 ounces of peanuts and 16 ounces of raisins cost? How much money would Jada have left?
2. Using  $p$  for pounds of peanuts and  $r$  for pounds of raisins, an equation relating how much of each they buy for a total of \$12 is  $0.2p + 0.25r = 12$ . If Jada wants 20 ounces of raisins, how many ounces of peanuts can she afford?
3. Jada knows she can rewrite the equation as  $r = 48 - 0.8p$ . In Jada's equation, which is the independent variable? Which is the dependent variable?

Solution:

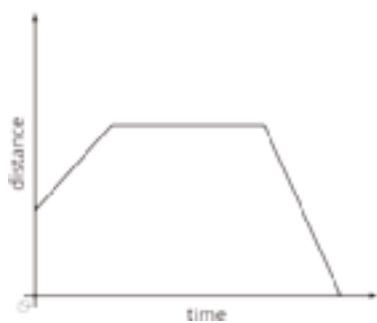
1. 10 ounces of peanuts would cost \$2 since  $0.2 \cdot 10 = 2$ . 16 ounces of raisins would cost \$4 since  $0.25 \cdot 16 = 4$ . Together, they would cost Jada \$6, leaving her with \$6.
2. 35 ounces of peanuts. If Jada wants 20 ounces of raisins, then  $0.2p + 0.25 \cdot 20 = 12$  must be true, which means  $p = 35$ .
3.  $p$  is the independent variable and  $r$  is the dependent variable for Jada's equation.

# Linear Functions and Rates of Change

## Family Support Materials 2

This week, your student will be working with graphs of functions. The graph of a function is all the pairs (input, output), plotted in the coordinate plane. By convention, we always put the input first, which means the inputs are represented on the horizontal axis and the outputs on the vertical axis.

For a graph representing a context, it is important to specify the quantities represented on each axis. For example this graph shows Elena's distance as a function of time. If it is distance from home, then Elena starts at some distance from home (maybe at her friend's house), moves further away from her home (maybe to a park), stays there a while, and then returns home. If it is distance from school, the story is different.

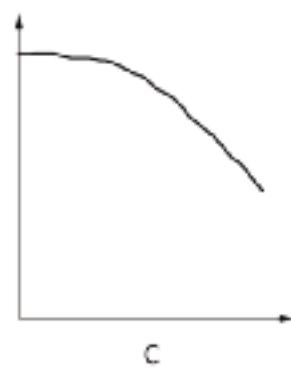
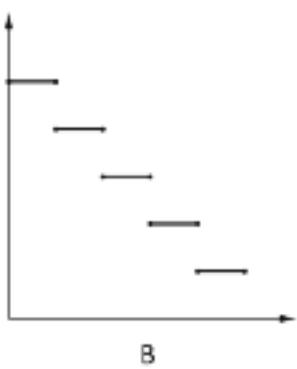


The story also changes depending on the scale on the axes: is distance measured in miles and time in hours, or is distance measured in meters and time in seconds?

Here is a task to try with your student:

Match each of the following situations with a graph (you can use a graph multiple times). Define possible inputs and outputs, and label the axes.

1. Noah pours the same amount of milk from a bottle every morning.
2. A plant grows the same amount every week.
3. The day started very warm but then it got colder.
4. A cylindrical glass contains some partially melted ice. The more water you pour in, the higher the water level.



Solution:

1. Graph B, input is time in days, output is amount of milk in the bottle
2. Graph A, input is time in weeks, output is height of plant
3. Graph C, input is time in hours, output is temperature
4. Graph A, input is volume of water, output is height of water

In each case, the horizontal axis is labeled with the input, and the vertical axis is labeled with the output.

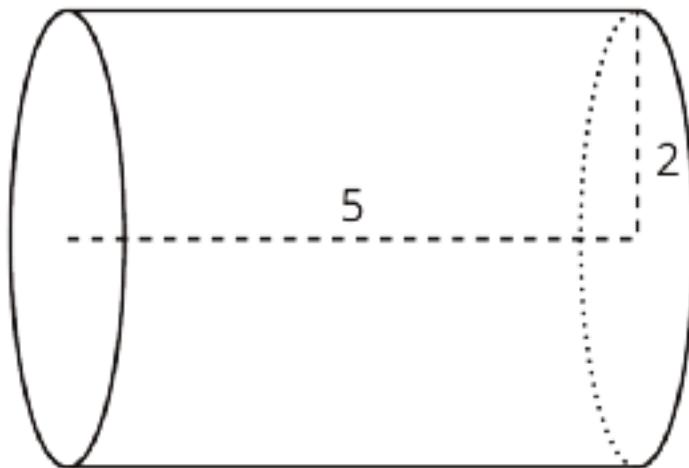
# Cylinders and Cones

## Family Support Materials 3

This week your student will be working with volumes of three-dimensional objects. We can determine the volume of a cylinder with radius  $r$  and height  $h$  using two ideas that we've seen before:

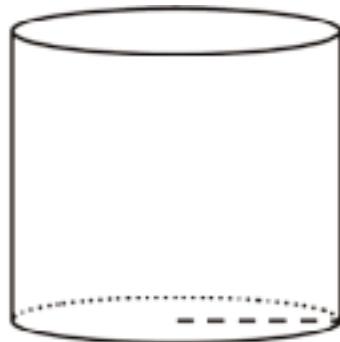
- The volume of a rectangular prism is a result of multiplying the area of its base by its height.
- The base of the cylinder is a circle with radius  $r$ , so the base area is  $\pi r^2$ .

Just like a rectangular prism, the volume of a cylinder is the area of the base times the height. For example, let's say we have a cylinder whose radius is 2 cm and whose height is 5 cm like the one shown here:



The base has an area of  $\pi 2^2 = 4\pi \text{ cm}^2$ . Using this, we can calculate the volume to be  $20\pi \text{ cm}^3$  since  $4\pi \cdot 5 = 20$ . If we use 3.14 as an approximation for  $\pi$ , we can say that the volume of the cylinder is approximately  $62.8 \text{ cm}^3$ . Students will also investigate the volume of cones and how their volume is related to the volume of a cylinder with the same radius and height.

Here is a task to try with your student:



This cylinder has a height and radius of 5 cm. Leave your answers in terms of  $\pi$ .

1. What is the diameter of the base?
2. What is the area of the base?
3. What is the volume of the cylinder?

Solution:

1. 10 cm. The diameter is  $2 \cdot r$ , and  $2 \cdot 5 = 10$ .
2.  $25\pi \text{ cm}^2$ . The area is  $\pi$  times the radius squared, or  $5^2 \cdot \pi$ .
3.  $125\pi \text{ cm}^3$ . The volume is the area of the base times the height. The area of the base here is  $25\pi$ , so the volume is  $125\pi \text{ cm}^3$  since  $25\pi \cdot 5 = 125\pi$ .

# Dimensions and Spheres

## Family Support Materials 4

This week, your student will compare the volumes of different objects. Many common objects, from water bottles to buildings to balloons, are similar in shape to rectangular prisms, cylinders, cones, and spheres—or even combinations of these shapes. We can use the volume formulas for these shapes to compare the volume of different types of objects.

For example, let's say we want to know which has more volume: a cube-shaped box with an edge length of 3 centimeters or a sphere with a radius of 2 centimeters.

The volume of the cube is 27 cubic centimeters since  $\text{edge}^3 = 3^3 = 27$ . The volume of the sphere is about 33.51 cubic centimeters since  $\frac{4}{3}\pi \cdot \text{radius}^3 = \frac{4}{3}\pi \cdot 2^3 \approx 33.51$ . Therefore, we can tell that the cube-shaped box holds less than the sphere.

Here is a task to try with your student:

A globe fits tightly inside a cubic box. The box has an edge length of 8 cm.

1. What is the volume of the box?
2. Estimate the volume of the globe: is it more or less than the volume of the box? How can you tell?
3. What is the diameter of the globe? The radius?
4. The formula for the volume of a sphere (like a globe) is  $V = \frac{4}{3}\pi r^3$ . What is the actual volume of the sphere? How close was your estimate in the previous problem?

Solution:

1.  $512 \text{ cm}^3$ . The box is a cube, so its volume is  $8^3$  cubic centimeters.
2. Answers vary. The number should be less than  $512 \text{ cm}^3$  since the volume of the globe must be less than the volume of the box. Possible explanation: it fits entirely inside the box, so it takes up less space. Since you can fit the globe inside the box and there is still space left over, the box has more volume.
3. Since the globe fits tightly inside the cubic box, the diameter of the globe must be the same as the edge length of the box, 8 cm. This means the radius is 4 cm.

4.  $\frac{256}{3}\pi$  or about 268 cm<sup>3</sup>. Since the side length of the cube is 8 cm, the radius of the globe is half of that, or 4 cm. The volume of the globe is therefore  $\frac{4}{3}\pi \cdot 4^3 = \frac{256}{3}\pi$ .



## Unit Assessments

Check Your Readiness A and B

Mid-Unit Assessment A and B

End-of-Unit Assessment A and B

# Functions and Volume: Check Your Readiness (A)

1. Jada is three years older than twice her brother's age.

Select **all** the equations that correctly represent the relationship between Jada's age  $j$  and her brother's age  $b$ .

A.  $j = 2b + 3$

B.  $j = 2(b + 3)$

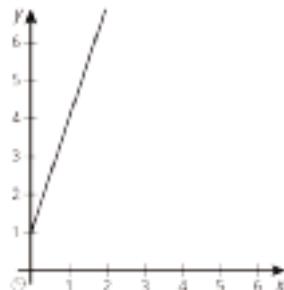
C.  $j = \frac{b}{2} - 3$

D.  $b = 2j + 3$

E.  $b = \frac{j-3}{2}$

F.  $b = \frac{j}{2} - 3$

2. Select **all** the proportional relationships.



A.

B. A train is traveling at a constant speed of 60 miles per hour. The number of hours the train has been traveling is  $t$ . The number of miles the train has traveled is  $d$ .

C. The relationship is represented by this table:

x	y
3	6
4	12
5	24

D.  $y = 3x$ , where  $x$  and  $y$  are both positive numbers.

E.  $y = \frac{1}{x}$

3. There are 16 cups in a gallon. The equation  $c = 16g$  gives the number of cups in terms of the number of gallons. Write another equation for this situation, giving the number of gallons in terms of the number of cups:

$$g =$$

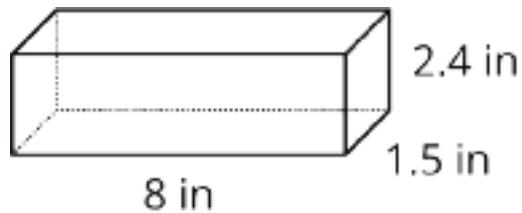
4. Given the equation  $y = -3x + 2.5$ :

a. When  $x$  is 1, what value of  $y$  makes the equation true?

b. When  $x$  is -1.5, what value of  $y$  makes the equation true?

c. When  $y$  is 8.5, what value of  $x$  makes the equation true?

5. Here is a rectangular prism.



a. What is the surface area of the prism?

b. What is the volume of the prism?

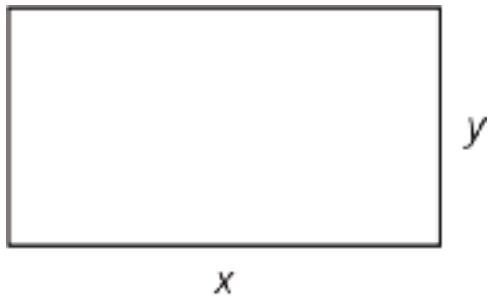
6. A circular field has area  $14400\pi$  square feet.

a. What is the radius of the field?

b. What is the diameter of the field?

c. What is the circumference of the field, to the nearest foot?

7. A rectangle has length  $x$  and width  $y$ .



Select **all** the statements that must be true.

- A. The perimeter is  $x + y$ .
- B. The perimeter is  $xy$ .
- C. The perimeter is  $2(x + y)$ .
- D. The perimeter is  $2xy$ .
- E. The perimeter is  $2x + 2y$ .
- F. The area is  $x + y$ .
- G. The area is  $xy$ .
- H. The area is  $2xy$ .

# Functions and Volume: Check Your Readiness (B)

1. Noah is three years younger than four times the age of his brother. Select **all** the equations that correctly represent the relationship between Noah's age  $n$  and his brother's age  $b$ .

A.  $n = 3 - 4b$

B.  $n = 4b - 3$

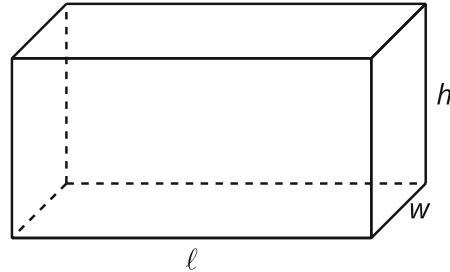
C.  $n = 4(b - 3)$

D.  $b = \frac{n+3}{4}$

E.  $b = 4n - 3$

F.  $b = \frac{n}{4} + 3$

2. A rectangular prism has length  $\ell$ , width  $w$ , and height  $h$ . Select **all** the statements that must be true.



A. The volume is  $(\ell wh)^2$

B. The volume is  $2(\ell wh)$

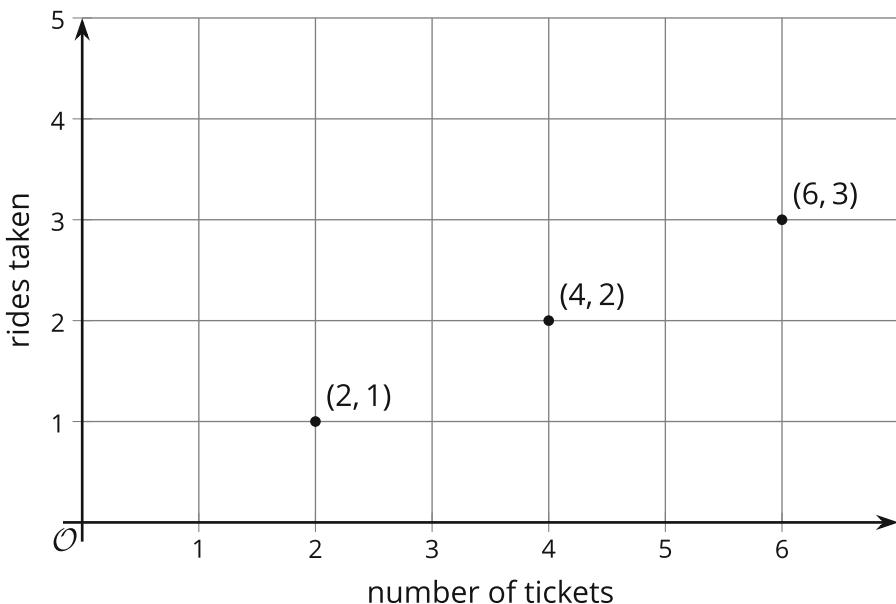
C. The volume is  $(\ell wh)$

D. The surface area is  $2(\ell wh)$

E. The surface area is  $2(\ell w) + 2(wh) + 2(\ell h)$

F. The surface area is  $2(\ell w) \cdot 2(wh) \cdot 2(\ell h)$

3. Select **all** the proportional relationships.



B.

$x$	$y$
0	6
3	9
6	12
9	15
12	18

C.

day	1	2	3	4
number of cell phones sold	5	10	15	20

D.  $y = \frac{1}{3}x$ , where  $x$  and  $y$  are both positive numbers.

E. Clare borrowed \$100 from her father. She pays him back \$10 each week.  $M$  is the amount of money Clare still owes, and  $W$  is the number of weeks:  
 $100 - 10W = M$

4. A recipe for salad dressing calls for  $\frac{1}{3}$  tablespoon of vinegar for every tablespoon of oil. The equation  $V = \frac{1}{3}L$  gives the amount of vinegar needed ( $V$ ) in terms of the amount of olive oil used ( $L$ ). Write another equation describing the recipe, this time giving  $L$  in terms of  $V$ .

$$L =$$

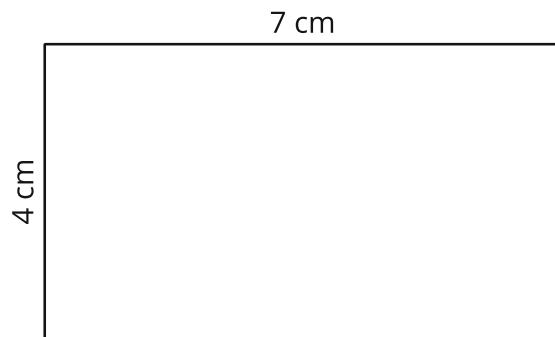
5. Given the equation  $y = -5x - (1.5)$ :

a. When  $x$  is 2, what value of  $y$  makes the equation true?

b. When  $x$  is -2.5, what value of  $y$  makes the equation true?

c. When  $y$  is -16.5, what value of  $x$  makes the equation true?

6. Here is a rectangle:



a. What is the area of the rectangle?

b. What is the perimeter of the rectangle?

7. The area of a circular table is  $225\pi$  in<sup>2</sup>.

a. What is the radius of the table?

b. What is the circumference of the table, to the nearest inch?

c. What is the diameter of the table?

# Functions and Volume: Mid-Unit Assessment (A)

You may use graph paper and a four-function or scientific calculator, but not a graphing calculator.

1. Select **all** the functions whose graphs include the point  $(16, 4)$ .
  - $y = 2x$
  - $y = x^2$
  - $y = x + 12$
  - $y = x - 12$
  - $y = \frac{1}{4}x$
2. This graph shows the temperature in Diego's house between noon and midnight one day.

Time (hours after noon)	Temperature (degrees F)
0	70
4	75
5	72
9	72
11	68
12	67

Select **all** the true statements.

- Time is a function of temperature.
- The lowest temperature occurred between 4:00 and 5:00.
- The temperature was increasing between 9:00 and 10:00.
- The temperature was 74 degrees twice during the 12-hour period.
- There was a four-hour period during which the temperature did not change.

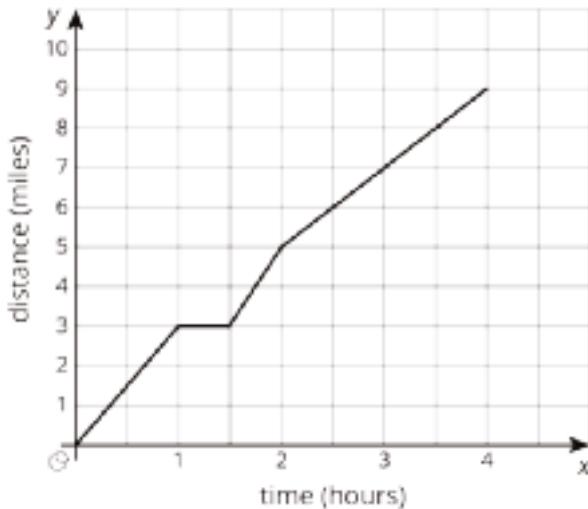
3. This table shows a linear relationship between the amount of water in a tank and time.

time (minutes)	water (gallons)
0	30
5	20
10	10

Which of these statements is true?

- A. The water in the tank is increasing at a rate of 2 gallons per minute.
- B. The water in the tank is increasing at a rate of 10 gallons per minute.
- C. The water in the tank is decreasing at a rate of 2 gallons per minute.
- D. The water in the tank is decreasing at a rate of 10 gallons per minute.

4. Elena goes for a long walk. This graph shows her time and distance traveled throughout the walk.



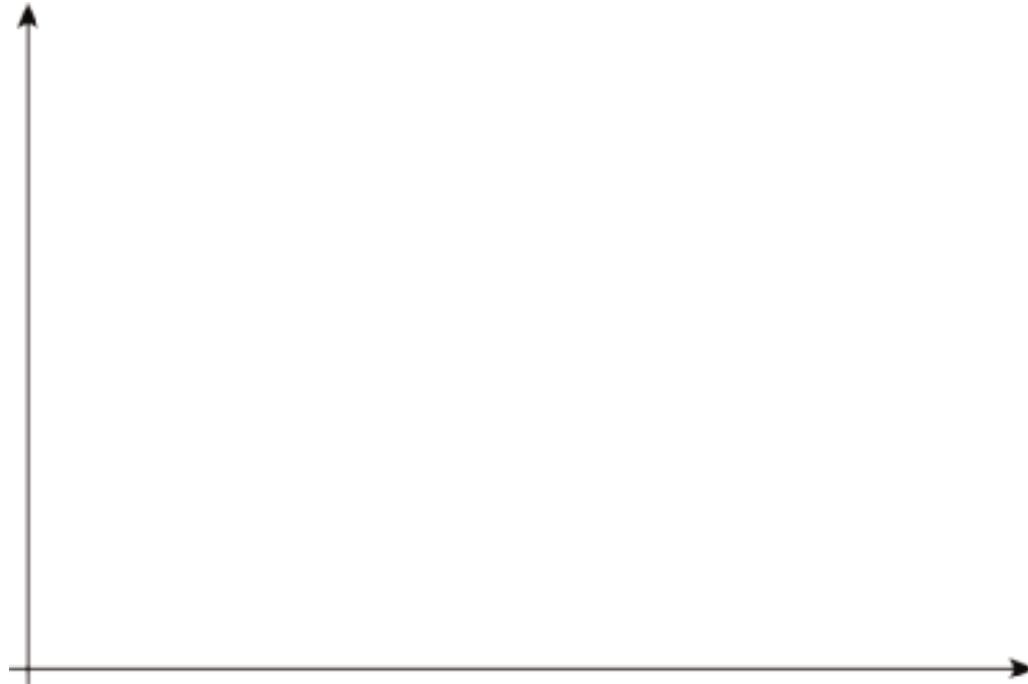
What was her fastest speed, in miles per hour?

5. Lin counts 5 bacteria under a microscope. She counts them again each day for four days, and finds that the number of bacteria doubled each day—from 5 to 10, then from 10 to 20, and so on.

Is the population of bacteria a function of the number of days? If so, is it linear? Explain your reasoning.

6. Draw a graph of Andre's distance as a function of time for this situation:

When the football play started, Andre ran forward 20 yards, then turned around and ran 5 yards back. He stood in that spot for 3 seconds, then walked back to where he began.



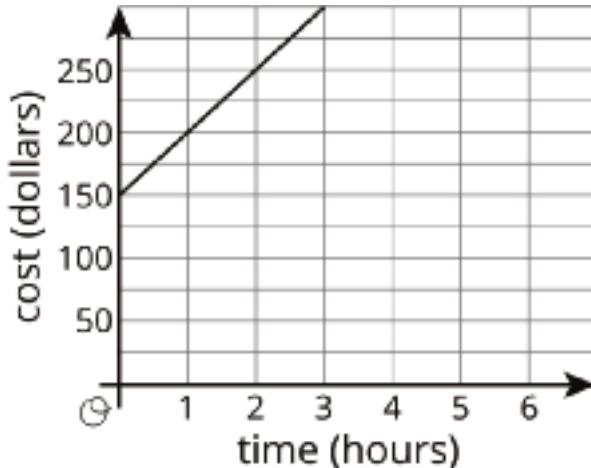
Label the axes appropriately. You do *not* have to include numbers on the axes or the coordinates of points on your graph.

7. Two plumbing companies charge money for each hour of work, plus a one-time fee.

A Plus Plumbing charges according to this table:

time (hours)	cost (dollars)
1	140
4	320
6	440

Quality Plumbing charges according to this graph:



- How much does A Plus Plumbing cost for each hour of work, and what is the one-time fee? Explain or show your reasoning.
- How much does Quality Plumbing charge for each hour of work, and what is the one-time fee? Explain or show your reasoning.
- Can A Plus Plumbing and Quality Plumbing ever charge the same total for the same amount of time? Explain or show your reasoning.

# Functions and Volume: Mid-Unit Assessment (B)

You may use graph paper and a four-function or scientific calculator, but not a graphing calculator.

1. Which point is on the graph of the function  $y = 3x + 2$ ?

- A. (1, 5)
- B. (2, 3)
- C. (3, 2)
- D. (5, 1)

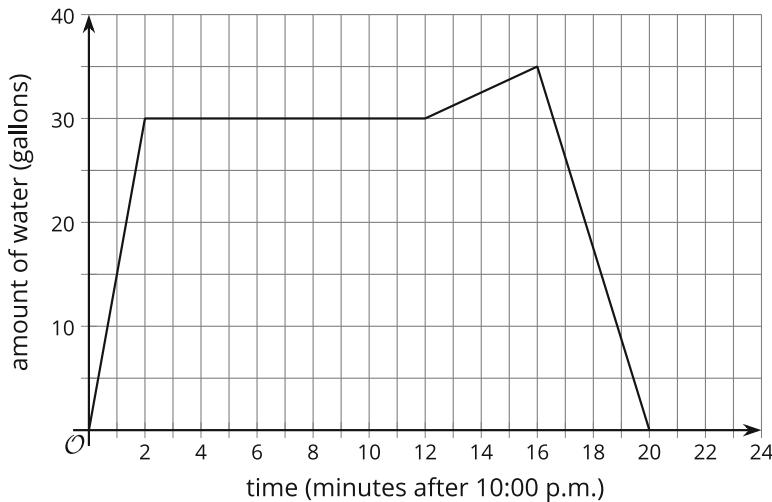
2. This table shows a linear relationship between the age of a newborn baby in weeks and their weight.

age (weeks)	weight (pounds)
1	6
3	8
10	15
12	17

Select all the true statements.

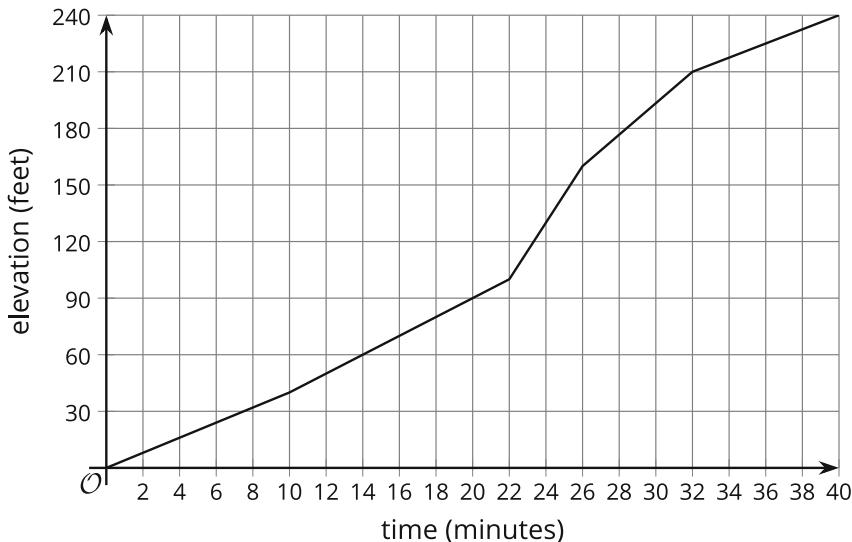
- A. The weight of the baby increases at a constant rate of 2 lbs every week.
- B. The weight of the baby increases at a constant rate of 1 lb every week.
- C. The weight of the baby increases at a constant rate of  $\frac{1}{2}$  lb every week.
- D. Age is a function of weight.
- E. Weight is a function of age.

3. The graph shows the amount of water in a bathtub starting at 10:00. Select **all** true statements.



- A. The tub was filling faster at 10:14 than at 10:01.
- B. At 10:06, the tub was neither filling nor draining.
- C. The maximum amount of water in the tub was about 35 gallons.
- D. The amount of water in the tub stayed the same from 10:16 to 10:20.
- E. It took 4 minutes for the tub to drain.

4. Mai hiked up a trail for 40 minutes. The graph shows the elevation in feet that she reached throughout her hike. Name the time period where Mai gained elevation at the fastest rate.



5. A tree planted today has a height of 5 feet and grows one foot each month.

Is the height of the tree a function of the number of months? If so, is it a linear or a nonlinear function? Explain your reasoning.

6. Draw a graph of Lin's distance as a function of time for this situation:

Lin walked a half-mile to school at a constant rate. Five minutes after arriving at school, Lin realized that she had left her permission slip at home. Lin began sprinting home, but when she was halfway there she got tired and walked the rest of the way.



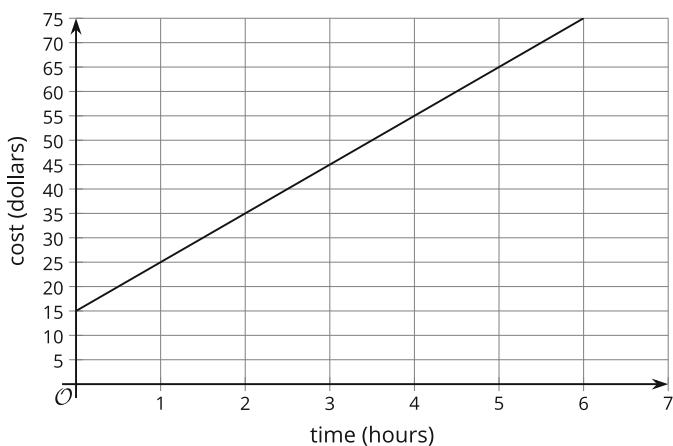
Label the axes appropriately. You do *not* have to include numbers on the axes or the coordinates of points on your graph.

7. Two cleaning services charge money for each hour of work, plus a one-time fee.

Sparkle Team Cleaners charges according to this table:

time (hours)	cost (dollars)
1	28
2	36
4	52
5	60
8	84

So Fresh & So Clean charges according to this graph:



- A customer would like a company to visit their home and give a quote for their services, without doing further work. Which company is the most cost effective for this customer? Explain your answer.
- Another customer would like the company to perform seven hours of work. Which company should this customer choose?
- What is the smallest number of hours of work for which Sparkle Team Cleaners is cheapest?

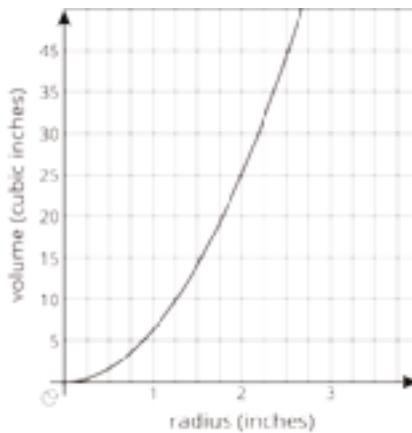
# Functions and Volume: End-of-Unit Assessment (A)

You may use any type of calculator.

1. A cylinder has volume  $78 \text{ cm}^3$ . What is the volume of a cone with the same radius and height?

- A.  $26 \text{ cm}^3$
- B.  $39 \text{ cm}^3$
- C.  $156 \text{ cm}^3$
- D.  $234 \text{ cm}^3$

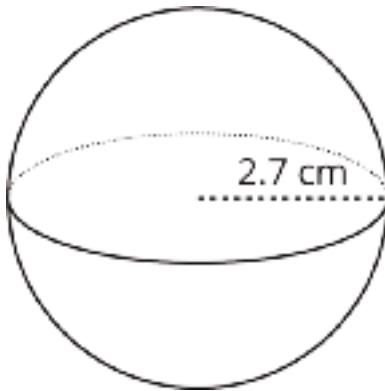
2. The graph shows the relationship between the radius and volume for many cones whose height is 6 inches.



Select **all** the true statements about such cones.

- A. The relationship between radius and volume is linear.
- B. The relationship between radius and volume is not linear.
- C. If the radius of the cone doubles, the volume of the cone doubles.
- D. If the radius of the cone doubles, the volume of the cone is multiplied by 4.
- E. If the radius of the cone is 2 inches, the volume of the cone is about 25 cubic inches.

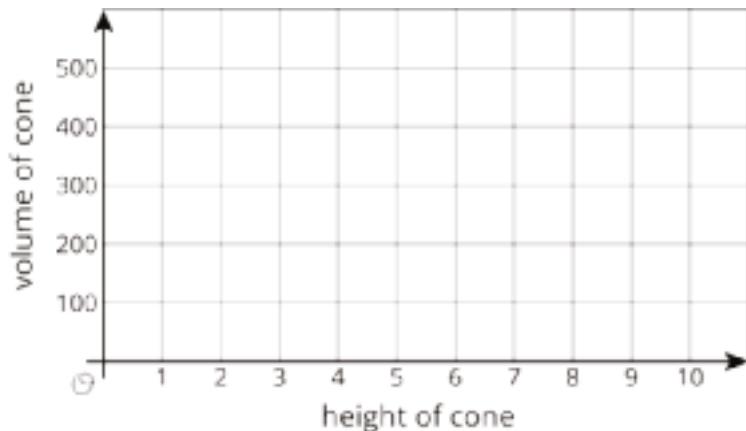
3. A sphere has radius 2.7 centimeters.



What is its volume, to the nearest cubic centimeter?

- A. 23
- B. 26
- C. 62
- D. 82

4. For cones with radius 6 units, the equation  $V = 12\pi h$  relates the height  $h$  of the cone, in units, and the volume  $V$  of the cone, in cubic units.



- a. Sketch the graph of this equation on the axes.
- b. Is there a linear relationship between height and volume? Explain how you know.

5. A cylinder has radius 1.6 meters. Its volume is 95 cubic meters. Find its height to the nearest tenth of a meter.

6. Cones A and B both have volume  $48\pi$  cubic units, but have different dimensions.

Cone A has radius 6 units and height 4 units. Find one possible radius and height for Cone B. Explain how you know Cone B has the same volume as Cone A.

7. There are many cylinders with a height of 9 inches. Let  $r$  represent the radius in inches and  $V$  represent the volume in cubic inches.

$r$	$V$
1	
2	
3	

a. Complete the table relating the radius and volume of cylinders with height 9 inches. Write each volume as a multiple of  $\pi$ , or round to the nearest cubic inch.

b. Is there a linear relationship between the radius and the volume of these cylinders? Explain how you know.

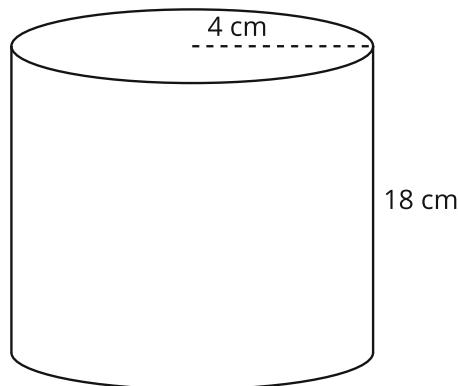
c. If a cylinder with height 9 inches and radius  $r$  is filled with water, it can fill a certain pitcher. How many of these pitchers can a cylinder with height 9 inches and radius  $2r$  fill? Explain how you know.

# Functions and Volume: End-of-Unit Assessment (B)

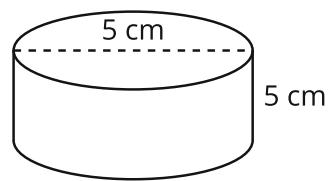
You may use any type of calculator.

1. Which is closest to the difference in the volume of the two cylinders?

A



B

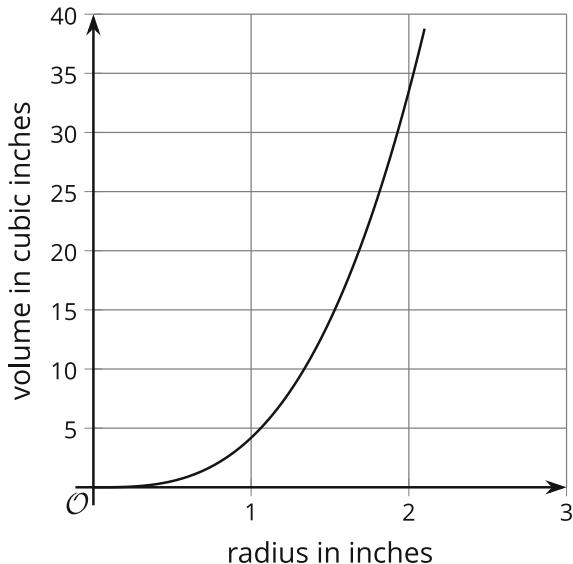


- A.  $15,795 \text{ cm}^3$
- B.  $2,534 \text{ cm}^3$
- C.  $806 \text{ cm}^3$
- D.  $512 \text{ cm}^3$

2. A sphere has a volume of  $972\pi \text{ cm}^3$ . What is its radius in centimeters?

- A. 4.5
- B. 9
- C. 11
- D. 18

3. The graph shows the relationship between a sphere's radius and volume.



Select **all** the statements that are true about this relationship.

- A. If the radius doubles, the volume doubles.
- B. If the radius doubles the volume becomes about 4 times bigger.
- C. If the radius doubles, the volume becomes about 8 times bigger.
- D. The relationship is linear.
- E. The relationship is not linear.

4. For cylinders with radius 2 units, let  $h$  represent the height of the cylinder, in units, and the  $V$  represent the volume of the cylinder, in cubic units.

- a. Complete the table relating the height and volume of cylinders with radius 2 inches. Write each volume as a multiple of  $\pi$ , or round to the nearest cubic unit.
- b. Is there a linear relationship between height and volume? Explain how you know.

$h$	$V$
1	
2	
3	

5. A cone and cylinder have the same height and their bases are congruent circles. If the volume of the cylinder is  $120 \text{ in}^3$ , what is the volume of the cone?

6. Three students each calculated the volume of a sphere with a radius of 6 centimeters.

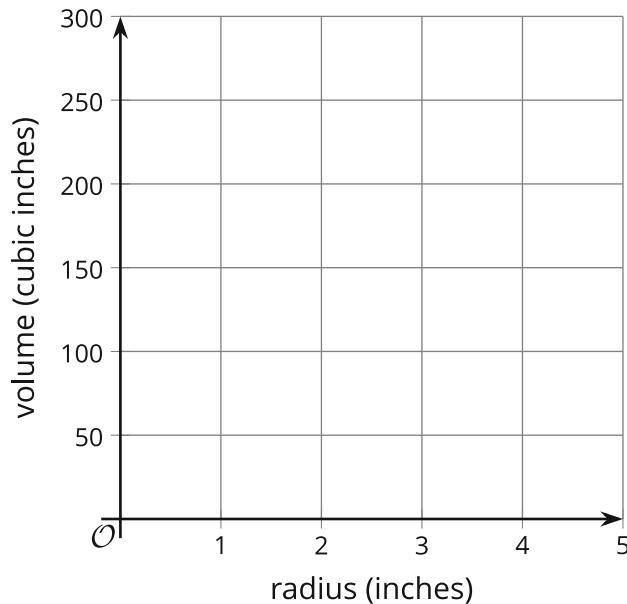
- Diego found the volume to be  $288\pi$  cubic centimeters.
- Andre approximated 904 cubic centimeters.
- Noah calculated 226 cubic centimeters.

Do you agree with any of them? Explain your reasoning.

7. There are many cones with a height of 12 inches. Let  $r$  represent the radius and  $V$  represent the volume of these cones.

a. Write an equation that expresses the relationship between  $V$  and  $r$ . Use 3.14 as an approximation for  $\pi$ .

b. Plot points that show the volume when  $r = 1$ ,  $r = 2$ ,  $r = 3$ , and  $r = 4$ . Show your reasoning.



c. A vendor at a street fair sells popcorn in cones, all of height 9 inches. The sharing-size cone has 3 times the radius of the skinny-size cone. About how many times more popcorn does the sharing cone hold than the skinny cone?



# Assessment Answer Keys

Check Your Readiness A and B  
Mid-Unit Assessment A and B  
End-of-Unit Assessment A and B

# Assessment Answer Keys

## Assessment : Check Your Readiness (A)

### Problem 1

The content assessed in this problem is first encountered in Lesson 3: Equations for Functions.

This problem is designed to expose some common errors that arise when writing equations to represent the relationship between quantities. Choices A and E are correct.

Students who made other choices may have made these errors. Choice B represents the statement "Jada is twice the age of someone three years older than her brother." In choice C, the operations are done in the correct order, but they are the inverse of the operations required. Choice D incorrectly switches the ages of Jada and her brother. Choice F is close, but incorrect. The correct equation  $b = \frac{j-3}{2}$  can be separated into two fractions to obtain  $b = \frac{j}{2} - \frac{3}{2}$ .

If most students struggle with this item, plan to budget class time to discuss the first and third practice problems. This item is also relevant to Activity 3. Prior to this lesson, the third practice problem in Lesson 1 also provides an opportunity to review writing equations in two variables. Plan to offer tape diagrams as a tool from previous work to help students who struggle to connect equations and situations.

### Statement

Jada is three years older than twice her brother's age.

Select **all** the equations that correctly represent the relationship between Jada's age  $j$  and her brother's age  $b$ .

A.  $j = 2b + 3$

B.  $j = 2(b + 3)$

C.  $j = \frac{b}{2} - 3$

D.  $b = 2j + 3$

E.  $b = \frac{j-3}{2}$

F.  $b = \frac{j}{2} - 3$

### Solution

["A", "E"]

## Aligned Standards

7.EE.B.4

### Problem 2

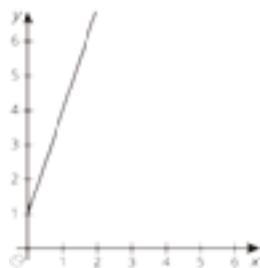
The content assessed in this problem is first encountered in Lesson 8: Linear Functions.

Work with linear functions, which begins in Lesson 8, builds off of students' understanding of proportional relationships. Choice A is worth special attention: it is an example of a linear function that does not represent a proportional relationship.

If most students struggle with this item, plan to revisit the definition of proportional relationships. Students worked with these relationships in the unit "Linear Relationships," so you may want to revisit practice problems from that unit as well. Note that the lesson "Connecting Representations to Functions" has an optional activity that reviews connecting tables, situations, and equations. If students struggle to recall previous work with proportions, find opportunities to highlight this work before this lesson.

### Statement

Select all the proportional relationships.



A.

B. A train is traveling at a constant speed of 60 miles per hour. The number of hours the train has been traveling is  $t$ . The number of miles the train has traveled is  $d$ .

C. The relationship is represented by this table:

x	y
3	6
4	12
5	24

D.  $y = 3x$ , where  $x$  and  $y$  are both positive numbers.

E.  $y = \frac{1}{x}$

## Solution

["B", "D"]

## Aligned Standards

7.RP.A.2.a

### Problem 3

The content assessed in this problem is first encountered in Lesson 8: Linear Functions.

This problem gives a proportional relationship using an equation of the form  $y = kx$ . In solving the problem, students are prompted to remember that this equation can be written in the equivalent form  $x = \frac{1}{k}y$ .

If most students struggle with this item, plan to discuss in Activity 2 the connection between the two equations that are possible for each part of the activity. You may choose to pose this item again as part of the Cool-Down for this lesson. Note that working with equations to think about relationships between say, volume and height for a fixed radius, relies on being able to rewrite simpler proportions for a targeted value. Plan to practice this in the work up to Lessons 14 and 16.

### Statement

There are 16 cups in a gallon. The equation  $c = 16g$  gives the number of cups in terms of the number of gallons. Write another equation for this situation, giving the number of gallons in terms of the number of cups:

$$g =$$

## Solution

$$g = \frac{1}{16}c \text{ (or equivalent)}$$

## Aligned Standards

7.RP.A.2.c

### Problem 4

The content assessed in this problem is first encountered in Lesson 3: Equations for Functions.

When students use equations to find input/output values of functions, they will need to substitute numbers for variables. This problem also assesses signed number arithmetic. Check to make sure students notice that, while parts a and b ask for values of  $y$  given  $x$ , part c asks for the value of  $x$  given  $y$ .

In part a, students might simply add 2.5 to 3 and get an answer of 5.5. They may subtract 2.5 from -3 to get an answer of -5.5. In part b, students who forget that "a negative times a negative is a positive" will get an answer of -2.

If most students struggle with this item, plan to revisit it before Activity 3 to ensure that students understand the intention of substituting a value we know in order to find a value that makes an equation true. If students need additional practice, these and similar problems can be incorporated into an Algebra Talk.

## Statement

Given the equation  $y = -3x + 2.5$ :

1. When  $x$  is 1, what value of  $y$  makes the equation true?
2. When  $x$  is -1.5, what value of  $y$  makes the equation true?
3. When  $y$  is 8.5, what value of  $x$  makes the equation true?

## Solution

1. -0.5
2. 7
3. -2

## Aligned Standards

6.EE.A.2.c, 7.NS.A.1, 7.NS.A.2

## Problem 5

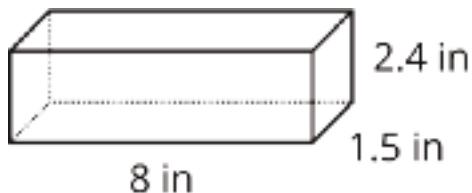
The content assessed in this problem is first encountered in Lesson 12: How Much Will Fit?.

Verify that students perform the right calculation, but also that they use the appropriate units. This is a good problem to remind students about the notation for square and cubic units.

If most students struggle with this item, plan to spend time after Lesson 12 reviewing finding the volume of rectangular prisms using the first and second practice problems.

## Statement

Here is a rectangular prism.



1. What is the surface area of the prism?
2. What is the volume of the prism?

## Solution

1.  $69.6 \text{ in}^2 (2 \cdot (1.5) \cdot (2.4) + 2 \cdot (1.5) \cdot 8 + 2 \cdot (2.4) \cdot 8)$
2.  $28.8 \text{ in}^3 ((1.5) \cdot (2.4) \cdot 8)$

## Aligned Standards

7.G.B.6

### Problem 6

The content assessed in this problem is first encountered in Lesson 13: The Volume of a Cylinder.

This problem is especially helpful for explaining why an approximation for  $\pi$  should only be used when necessary. If students begin by approximating  $\pi$  the work is much harder.

If most students struggle with this item, plan to spend additional time on Activity 1 and revisit this item at the end of Activity 1. Students may not be familiar with using pi in a reported area as opposed to calculating area. Be sure to discuss how this is one way to report measures, and knowing that pi is a little more than 3 gives us an idea of which whole numbers the measure is close to.

## Statement

A circular field has area  $14400\pi$  square feet.

1. What is the radius of the field?
2. What is the diameter of the field?
3. What is the circumference of the field, to the nearest foot?

## Solution

1. 120 feet (the radius  $r$  solves  $\pi r^2 = 14400\pi$ , then  $r^2 = 14400$  and  $r = 120$  since it is positive)
2. 240 feet (twice the radius)
3. 754 feet ( $C = 240\pi$ , then use an approximation for  $\pi$ )

## Aligned Standards

7.G.B.4

### Problem 7

The content assessed in this problem is first encountered in Lesson 3: Equations for Functions.

If students have trouble here, ask them to explain their choices; their difficulties may involve interpreting the variables rather than the geometry or the formulas.

If most students struggle with this item, plan to spend additional time reviewing what students already know about calculating measures of geometric figures. This item assesses both students' knowledge of area and perimeter (useful when working on volume in Lesson 12 and beyond) and reasoning about geometric formulas using variables, which is used to introduce functions in Lesson 3. Plan to revisit this item after Lesson 3.

## Statement

A rectangle has length  $x$  and width  $y$ .



Select **all** the statements that must be true.

- A. The perimeter is  $x + y$ .
- B. The perimeter is  $xy$ .
- C. The perimeter is  $2(x + y)$ .
- D. The perimeter is  $2xy$ .
- E. The perimeter is  $2x + 2y$ .
- F. The area is  $x + y$ .
- G. The area is  $xy$ .
- H. The area is  $2xy$ .

## Solution

["C", "E", "G"]

## Aligned Standards

3.MD.C.7, 3.MD.D.8

# Assessment : Check Your Readiness (B)

## Problem 1

The content assessed in this problem is first encountered in Lesson 3: Equations for Functions.

This problem is designed to expose some common errors that arise when writing equations to represent the relationship between quantities. Choices B and D are correct. Students who made other choices may have made these errors. Students selecting choice A incorrectly wrote the terms of the expression in the order they were given. Choice C represents the statement, "Noah is four times the age of someone three years younger than his brother." Choice E switches the ages of Noah and his brother. Choice F is close, but incorrect. The correct equation can be separated into two fractions to obtain  $b = \frac{n}{4} + \frac{3}{4}$ .

If most students struggle with this item, plan to budget class time to discuss the first and third practice problems. This item is also relevant to Activity 3. Prior to this lesson, the third practice problem in Lesson 1 also provides an opportunity to review writing equations in two variables. Plan to offer tape diagrams as a tool from previous work to help students who struggle to connect equations and situations.

### Statement

Noah is three years younger than four times the age of his brother. Select **all** the equations that correctly represent the relationship between Noah's age  $n$  and his brother's age  $b$ .

- A.  $n = 3 - 4b$
- B.  $n = 4b - 3$
- C.  $n = 4(b - 3)$
- D.  $b = \frac{n+3}{4}$
- E.  $b = 4n - 3$
- F.  $b = \frac{n}{4} + 3$

### Solution

["B", "D"]

### Aligned Standards

7.EE.B.4

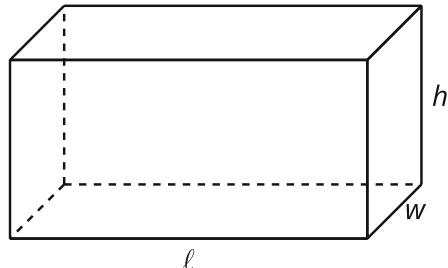
## Problem 2

The content assessed in this problem is first encountered in Lesson 12: How Much Will Fit?.

If students have trouble here, ask them to explain their choices; their difficulties may involve interpreting the variables rather than the geometry or the formulas.

## Statement

A rectangular prism has length  $\ell$ , width  $w$ , and height  $h$ . Select **all** the statements that must be true.



- A. The volume is  $(\ell wh)^2$
- B. The volume is  $2(\ell wh)$
- C. The volume is  $(\ell wh)$
- D. The surface area is  $2(\ell wh)$
- E. The surface area is  $2(\ell w) + 2(wh) + 2(\ell h)$
- F. The surface area is  $2(\ell w) \cdot 2(wh) \cdot 2(\ell h)$

## Solution

["C", "E"]

## Aligned Standards

3.MD.C.7

## Problem 3

The content assessed in this problem is first encountered in Lesson 8: Linear Functions.

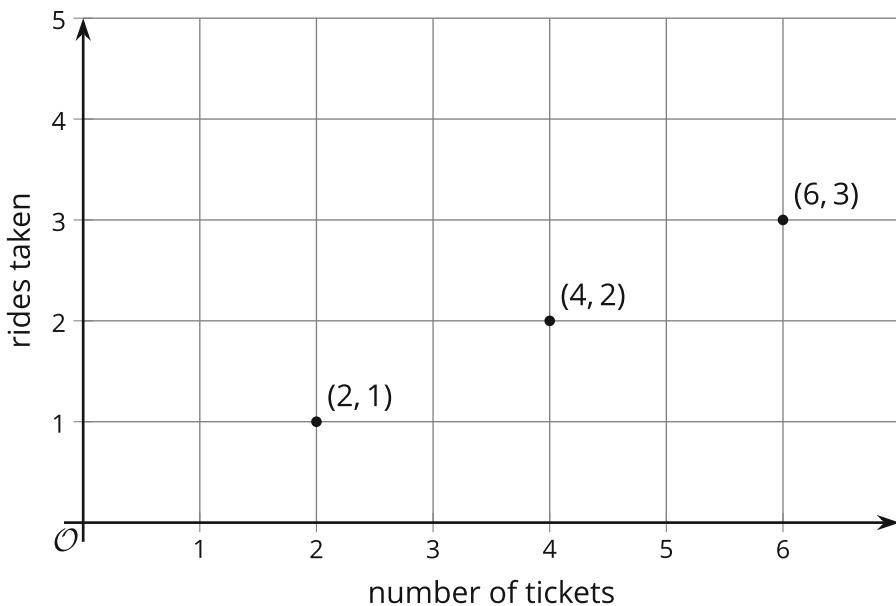
Work with linear functions, which begins in Lesson 9, builds off of students' understanding of proportional relationships. Some students may think that choice A does not represent a proportional relationship, since the point  $(0, 0)$  is not explicitly marked on the graph. Choices B and E are both instructive examples of linear relationships that are not proportional. Students not selecting choice D may be thrown by the fact that the constant of proportionality is a unit fraction: they know that  $y = \frac{1}{x}$  is not a proportional relationship.

If most students struggle with this item, plan to revisit the definition of proportional relationships. Students worked with these relationships in the unit "Linear Relationships," so you may want to revisit practice problems from that unit as well. Note that the lesson "Connecting Representations to Functions" has an optional activity that reviews connecting tables, situations, and equations. If

students struggle to recall previous work with proportions, find opportunities to highlight this work before this lesson.

## Statement

Select all the proportional relationships.



B.

$x$	$y$
0	6
3	9
6	12
9	15
12	18

C.

day	1	2	3	4
number of cell phones sold	5	10	15	20

D.  $y = \frac{1}{3}x$ , where  $x$  and  $y$  are both positive numbers.

E. Clare borrowed \$100 from her father. She pays him back \$10 each week.  $M$  is the amount of money Clare still owes, and  $W$  is the number of weeks:  $100 - 10W = M$

## Solution

["A", "C", "D"]

## Aligned Standards

7.RP.A.2.a

### Problem 4

The content assessed in this problem is first encountered in Lesson 8: Linear Functions.

This problem gives a proportional relationship using an equation of the form  $y = \frac{1}{k}x$ . In solving the problem, students may isolate the variable algebraically, or they may simply recall that an equation of this type can be rewritten in the equivalent form  $x = ky$ .

If most students struggle with this item, plan to discuss in Activity 2 the connection between the two equations that are possible for each part of the activity. You may choose to pose this item again as part of the Cool-Down for this lesson. Note that working with equations to think about relationships between say, volume and height for a fixed radius, relies on being able to rewrite simpler proportions for a targeted value. Plan to practice this in the work up to Lessons 14 and 16.

### Statement

A recipe for salad dressing calls for  $\frac{1}{3}$  tablespoon of vinegar for every tablespoon of oil. The equation  $V = \frac{1}{3}L$  gives the amount of vinegar needed ( $V$ ) in terms of the amount of olive oil used ( $L$ ). Write another equation describing the recipe, this time giving  $L$  in terms of  $V$ .

$$L =$$

## Solution

$$L = 3V \text{ or } L = \frac{V}{\frac{1}{3}}$$

## Aligned Standards

7.RP.A.2.c

### Problem 5

The content assessed in this problem is first encountered in Lesson 3: Equations for Functions.

When students use equations to find input or output values of functions, they will need to substitute numbers for variables. This problem also assesses signed number arithmetic. Check to make sure students notice that, while parts a and b ask for the value of  $y$ , part c asks for the value of  $x$ .

If most students struggle with this item, plan to revisit it before Activity 3 to ensure that students understand the intention of substituting a value we know in order to find a value that makes an

equation true. If students need additional practice, these and similar problems can be incorporated into an Algebra Talk.

## Statement

Given the equation  $y = -5x - (1.5)$ :

1. When  $x$  is 2, what value of  $y$  makes the equation true?
2. When  $x$  is -2.5, what value of  $y$  makes the equation true?
3. When  $y$  is -16.5, what value of  $x$  makes the equation true?

## Solution

1. -11.5
2. 11
3. 3

## Aligned Standards

6.EE.A.2.c, 7.NS.A.1, 7.NS.A.2

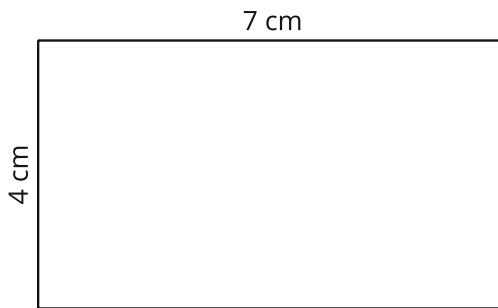
## Problem 6

The content assessed in this problem is first encountered in Lesson 3: Equations for Functions.

Verify that students perform the right calculation, but also that they use the appropriate units. This is a good problem to remind students about the notation for square units.

## Statement

Here is a rectangle:



1. What is the area of the rectangle?
2. What is the perimeter of the rectangle?

## Assessment: Check Your Readiness (B)

## Solution

1.  $28 \text{ cm}^2$
2.  $22 \text{ cm}$

## Aligned Standards

7.G.B.6

### Problem 7

The content assessed in this problem is first encountered in Lesson 13: The Volume of a Cylinder.

This problem is especially helpful for explaining why an approximation for  $\pi$  should only be used when necessary. If students begin by approximating  $\pi$  the work is much harder.

If most students struggle with this item, plan to spend additional time on Activity 1 and revisit this item at the end of Activity 1. Students may not be familiar with using pi in a reported area as opposed to calculating area. Be sure to discuss how this is one way to report measures, and knowing that pi is a little more than 3 gives us an idea of which whole numbers the measure is close to.

### Statement

The area of a circular table is  $225\pi \text{ in}^2$ .

1. What is the radius of the table?
2. What is the circumference of the table, to the nearest inch?
3. What is the diameter of the table?

## Solution

1. 15 inches ( $\pi r^2 = 225\pi$ ,  $r^2 = 225$ ,  $r = 15$ .)
2. 94 square inches ( $2\pi r = 30\pi$ , then use an approximation for  $\pi$ .)
3. 30 inches. (Twice the radius)

## Aligned Standards

7.G.B.4

# Assessment : Mid-Unit Assessment (A)

## Teacher Instructions

Give this assessment after lesson 10. Graphing calculators should not be used. Use of a four-function or scientific calculator is acceptable. Provide access to graph paper.

## Student Instructions

You may use graph paper and a four-function or scientific calculator, but not a graphing calculator.

### Problem 1

A student selecting choice A misunderstands the overall concept relating a function to its graph. A student selecting choices B or C may be mixing variables, or working backwards from the point (16, 4), looking for relationships.

#### Statement

Select all the functions whose graphs include the point (16, 4).

- A.  $y = 2x$
- B.  $y = x^2$
- C.  $y = x + 12$
- D.  $y = x - 12$
- E.  $y = \frac{1}{4}x$

#### Solution

["D", "E"]

#### Aligned Standards

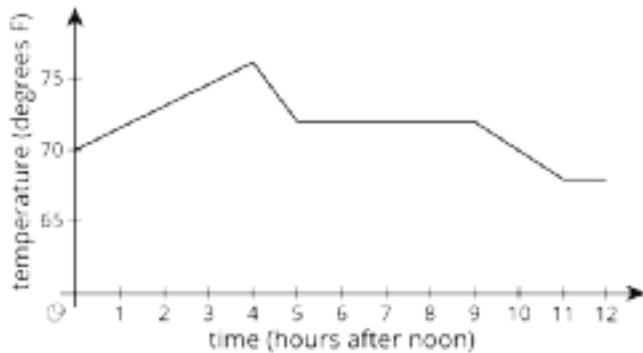
8.F.A.1

### Problem 2

It is easy to read choice A too quickly—time is not a function of temperature, but temperature is a function of time. Students selecting A may either be making this careless error or may be having genuine trouble with the definition of a function. Students selecting B have identified the time period in which the temperature decreases most quickly, but not the period that contains the lowest temperature. Students selecting C are having trouble distinguishing between increasing and decreasing intervals on a graph, or perhaps they are looking at the wrong time interval. Students failing to select D may be having trouble thinking about where to look for multiple occurrences of 74 degrees (visualizing a horizontal line drawn at the 74-degree mark is helpful). Students failing to select E may likewise be unsure how to look for intervals of no change.

## Statement

This graph shows the temperature in Diego's house between noon and midnight one day.



Select **all** the true statements.

- A. Time is a function of temperature.
- B. The lowest temperature occurred between 4:00 and 5:00.
- C. The temperature was increasing between 9:00 and 10:00.
- D. The temperature was 74 degrees twice during the 12-hour period.
- E. There was a four-hour period during which the temperature did not change.

## Solution

["D", "E"]

## Aligned Standards

8.F.B.5

## Problem 3

Students selecting A have calculated the correct rate but are interpreting it incorrectly as an increase instead of a decrease. Students selecting D are correctly interpreting the change as a decrease, but have calculated only the change in gallons, not the rate of change. Students selecting B have made both these errors.

## Statement

This table shows a linear relationship between the amount of water in a tank and time.

time (minutes)	water (gallons)
0	30
5	20
10	10

Which of these statements is true?

- A. The water in the tank is increasing at a rate of 2 gallons per minute.
- B. The water in the tank is increasing at a rate of 10 gallons per minute.
- C. The water in the tank is decreasing at a rate of 2 gallons per minute.
- D. The water in the tank is decreasing at a rate of 10 gallons per minute.

## Solution

C

## Aligned Standards

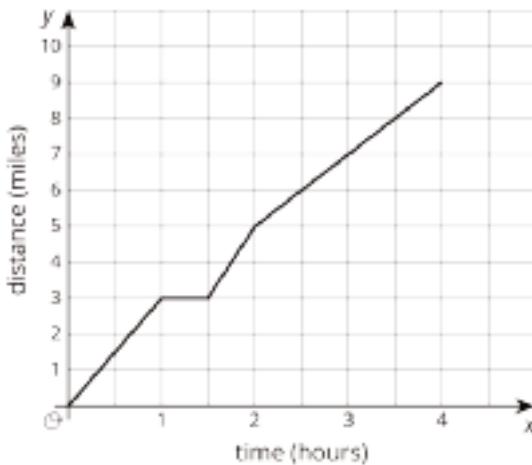
8.F.B.4

## Problem 4

Students answering 2 may be using the longest section of the graph, where Elena walks 4 miles in 2 hours. Students answering 3 may think the first section is the steepest, but it is not. Students answering something other than 2, 3, or 4 may have a deeper misunderstanding about rates of change.

## Statement

Elena goes for a long walk. This graph shows her time and distance traveled throughout the walk.



What was her fastest speed, in miles per hour?

## Solution

4

## Aligned Standards

8.F.B.5

### Problem 5

If necessary, clarify for students that it does not matter that the population of bacteria might be different at different times each day—we are only using the population at each time of day that Lin checks.

## Statement

Lin counts 5 bacteria under a microscope. She counts them again each day for four days, and finds that the number of bacteria doubled each day—from 5 to 10, then from 10 to 20, and so on.

Is the population of bacteria a function of the number of days? If so, is it linear? Explain your reasoning.

## Solution

It is a function, but it is not a linear function. It is a function because there is a single output (the number of bacteria) for each input (the number of days). It is not a linear function because the rate of change does not stay the same throughout. Alternately, the points on the graph of this function clearly do not make a straight line.

Minimal Tier 1 response:

- Work is complete and correct.

- Sample: Yes, because there is one output for every input. (Or: Yes, because each day has only one number of bacteria.) No, because the number of bacteria don't go up by the same amount each day.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: explanation appeals to the fact that the day is the independent variable, but does not get at the “one output for each input” definition of function; one well-explained correct answer along with another answer that is poorly explained but correct, or along with an incorrect answer that shows some understanding.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: an incorrect answer to one or both questions that does not show significant understanding; both responses are flawed in some way.

## Aligned Standards

8.F.A.1, 8.F.A.3

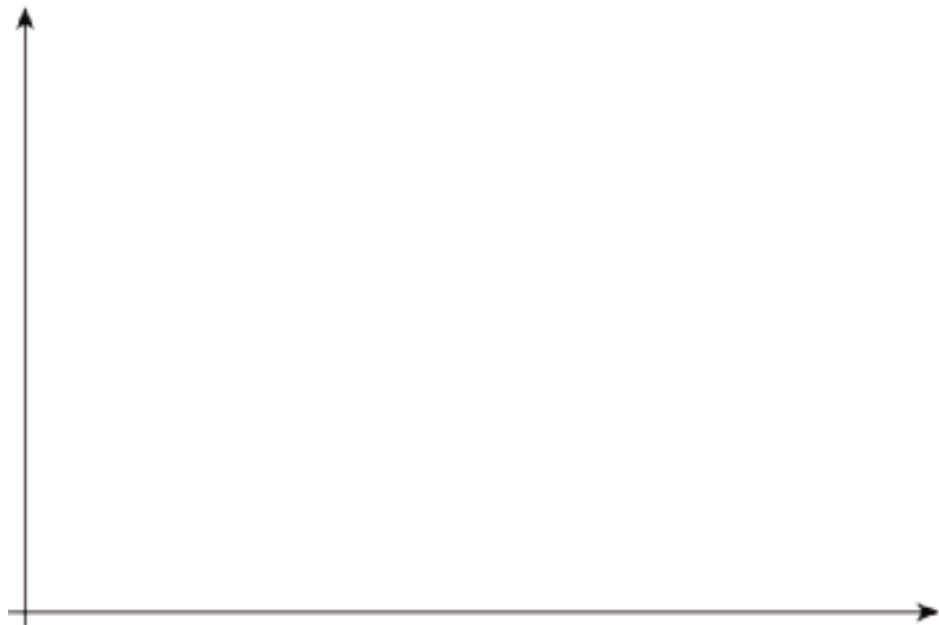
## Problem 6

When judging the quality of a student’s graph, look for a few details: first, the slope of the run forward and run back should be similar, with the stopping point for the run back about  $\frac{1}{4}$  of the way back from the maximum. Second, there should be a clear horizontal line corresponding to standing in place. Third, there should be a less steep line back to the horizontal axis, to indicate the walk back to the start.

### Statement

Draw a graph of Andre’s distance as a function of time for this situation:

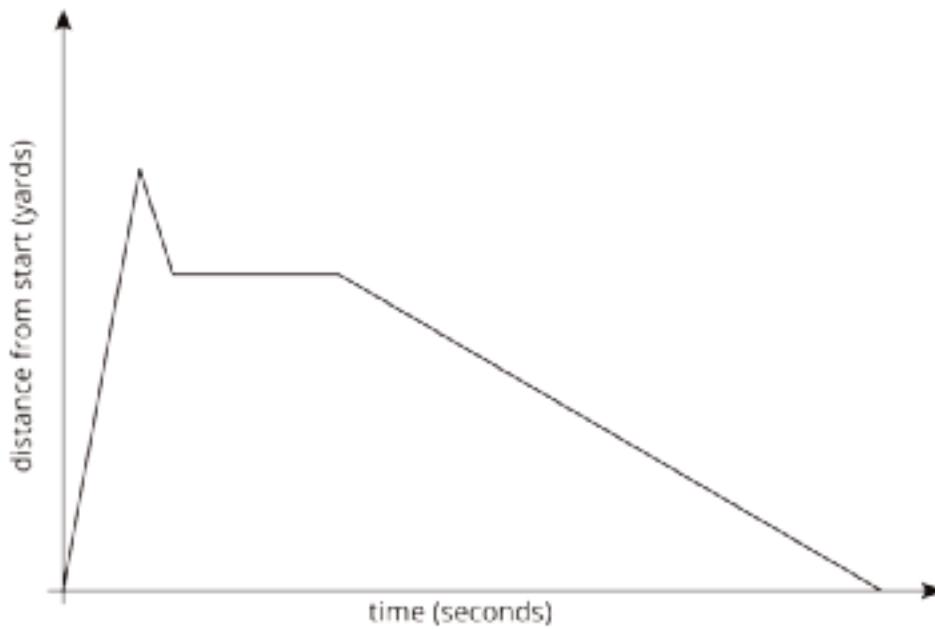
When the football play started, Andre ran forward 20 yards, then turned around and ran 5 yards back. He stood in that spot for 3 seconds, then walked back to where he began.



Label the axes appropriately. You do *not* have to include numbers on the axes or the coordinates of points on your graph.

## Solution

Answers vary. Sample response:



Minimal Tier 1 response:

- Work is complete and correct.
- Sample: See diagram, as well as notes in narrative.

- Acceptable errors: axes are labeled only as “distance/time” or “yards/seconds.”

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: axes unlabeled or labeled incorrectly; graph does not meet one of the criteria mentioned in the narrative.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: graph does not meet two or more criteria mentioned in the narrative; graph fails to meet one of the criteria mentioned in the narrative and the axes are unlabeled/incorrect.

## Aligned Standards

8.F.B.5

### Problem 7

Look for students’ work in part c in particular; it is not necessary to calculate the amount of time. Errors in part a or part b will reveal whether students have difficulty interpreting functions defined by ordered pairs or by graphs. In part c, it is sufficient justification to add a reasonably accurate graph of A Plus Plumbing’s pricing scheme to the existing graph, showing an unlabeled intersection point.

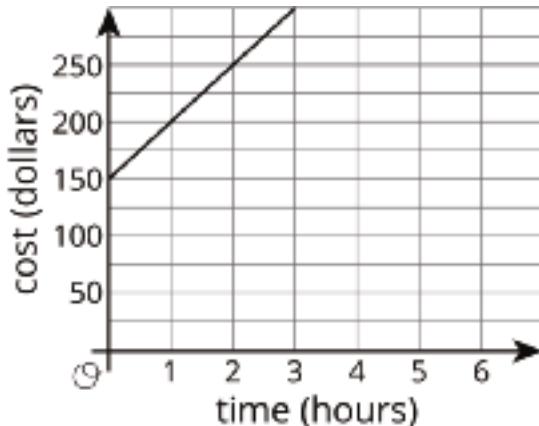
#### Statement

Two plumbing companies charge money for each hour of work, plus a one-time fee.

A Plus Plumbing charges according to this table:

time (hours)	cost (dollars)
1	140
4	320
6	440

Quality Plumbing charges according to this graph:



1. How much does A Plus Plumbing cost for each hour of work, and what is the one-time fee? Explain or show your reasoning.
2. How much does Quality Plumbing charge for each hour of work, and what is the one-time fee? Explain or show your reasoning.
3. Can A Plus Plumbing and Quality Plumbing ever charge the same total for the same amount of time? Explain or show your reasoning.

## Solution

1. The cost for each hour of work is \$60, and the one-time fee is \$80. Determine the cost per hour by finding the rate of change:  $\frac{320-140}{3} = 60$ . Then determine the one-time fee by subtracting \$60 from \$140.
2. The cost for each hour of work is \$50, and the one-time fee is \$150. Determine the cost per hour by finding the slope of the line, which is 50. Determine the one-time fee by using the  $y$ -intercept of the graph.
3. Yes. Explanations vary. Sample response: A Plus Plumbing has a lower one-time fee but costs more per hour, so it will eventually catch up to Quality Plumbing.

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Acceptable errors: omitting units (\$).
- Sample:

1. The cost is \$60, because  $140 + 3 \cdot 60 = 320$ . The fee is \$80 because  $140 - 60 = 80$ .

2. The cost is \$50, because the graph goes up by 50 every hour. The fee is \$150 because for 0 hours, they charge \$150.
3. A Plus starts with a lower price but costs more each hour. This means the graphs must intersect.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: correct work/explanation for part c based on mistakes in parts a and b; approach to part c involves a correct system of equations with arithmetic mistakes in the solution method; work calculating slope/rate of change involves arithmetic mistakes.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: approach to parts a and b shows some understanding that the goal is to find a constant number to add each hour that will result in the numbers in the table, but the work is not systematic and involves errors; badly misinterpreting the table/graph, such as reversing the columns/coordinates, with reasonable work following that; approach to part c involves a correct system of equations but no reasonable approach to solving the system; work for parts a and b is correct but work for part c is conceptually flawed.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: no evidence of understanding the connection between charge per hour and the information in the table and graph; error types from tier 3 response on parts a, b, and c.

## Aligned Standards

8.EE.C, 8.F.A.2, 8.F.B.4

# Assessment : Mid-Unit Assessment (B)

## Teacher Instructions

Give this assessment after lesson 10. Graphing calculators should not be used. Use of a four-function or scientific calculator is acceptable. Provide access to graph paper.

## Student Instructions

You may use graph paper and a four-function or scientific calculator, but not a graphing calculator.

### Problem 1

Students selecting B or C are probably unsure of how to approach the problem, taking the numbers directly from the equation. Students selecting D have switched the input/output pair.

#### Statement

Which point is on the graph of the function  $y = 3x + 2$ ?

- A. (1, 5)
- B. (2, 3)
- C. (3, 2)
- D. (5, 1)

### Solution

A

#### Aligned Standards

8.F.A.1

### Problem 2

Students selecting A looked at the first two rows of the weight column and did not realize that the increase was over two weeks. Students selecting C may not understand how to find the constant rate of change. Students selecting one of D or E, but not both, may not realize that both can be true at once, or may think only D is true because age is the independent variable.

#### Statement

This table shows a linear relationship between the age of a newborn baby in weeks and their weight.

age (weeks)	weight (pounds)
1	6
3	8
10	15
12	17

Select **all** the true statements.

- A. The weight of the baby increases at a constant rate of 2 lbs every week.
- B. The weight of the baby increases at a constant rate of 1 lb every week.
- C. The weight of the baby increases at a constant rate of  $\frac{1}{2}$  lb every week.
- D. Age is a function of weight.
- E. Weight is a function of age.

## Solution

["B", "D", "E"]

## Aligned Standards

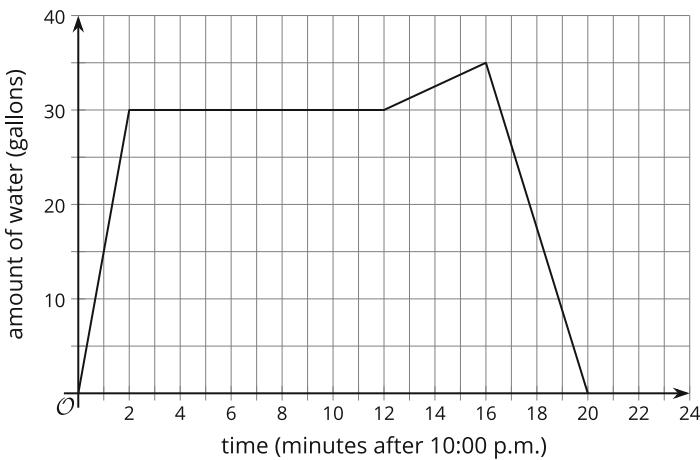
8.F.B.4

## Problem 3

Students selecting A may think that because the point with  $x$ -coordinate is higher on the graph, this corresponds to a faster rate of change. Students selecting B may be confusing the constant rate of change with a constant amount of water in the tub, or may have a deeper misunderstanding about interpreting graphs.

## Statement

The graph shows the amount of water in a bathtub starting at 10:00. Select **all** true statements.



- A. The tub was filling faster at 10:14 than at 10:01.
- B. At 10:06, the tub was neither filling nor draining.
- C. The maximum amount of water in the tub was about 35 gallons.
- D. The amount of water in the tub stayed the same from 10:16 to 10:20.
- E. It took 4 minutes for the tub to drain.

## Solution

["B", "C", "E"]

## Aligned Standards

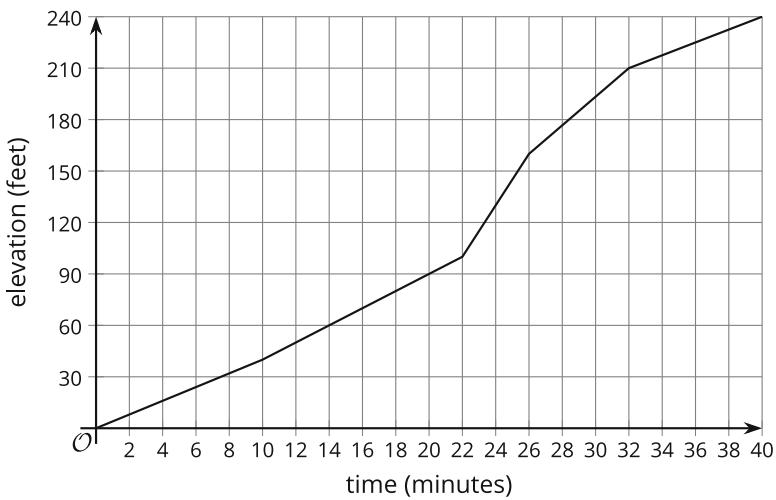
8.F.B.5

## Problem 4

Check to see if students understand what “time period” means in this context. Some students may answer 10–22 minutes, looking at the longest section of the graph. Students answering 32–40 minutes may simply be choosing the section of the graph corresponding to the greatest height.

## Statement

Mai hiked up a trail for 40 minutes. The graph shows the elevation in feet that she reached throughout her hike. Name the time period where Mai gained elevation at the fastest rate.



## Solution

22 to 26 minutes

## Aligned Standards

8.F.B.5

### Problem 5

Students may want to include a table or a graph to get a visual of how tall the trees get each month.

#### Statement

A tree planted today has a height of 5 feet and grows one foot each month.

Is the height of the tree a function of the number of months? If so, is it a linear or a nonlinear function? Explain your reasoning.

## Solution

It is a linear function. It is a function because there is a single output (the height of the tree) for each input (the number of months). It is linear because the growth rate of 1 foot per month is a constant rate of change. Alternately, we know this is a linear function because it has equation  $h = M + 5$ .

Minimal Tier 1 response:

- Work is complete and correct.
- Sample: Yes, because there is one output for every input. (Or: Yes, because each month the tree has a constant rate of growth.)

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.

- Sample errors: explanation appeals to the fact that the month is the independent variable, but does not get at the "one output for each input" definition of function; one well-explained correct answer along with another answer that is poorly explained but correct, or along with an incorrect answer that shows some understanding.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: an incorrect answer to one or both questions that does not show significant understanding; both responses are flawed in some way.

## Aligned Standards

8.F.A.1, 8.F.A.3

### Problem 6

When judging the quality of a student's graph, look for a few details: the slopes of the walking segments should be similar (with opposite sign). A horizontal line should represent the 5 minute wait time. The slope of the sprint home should be steeper than the walking slopes. The change from sprinting from walking should occur "halfway home" - that is, at half of the maximum distance.

#### Statement

Draw a graph of Lin's distance as a function of time for this situation:

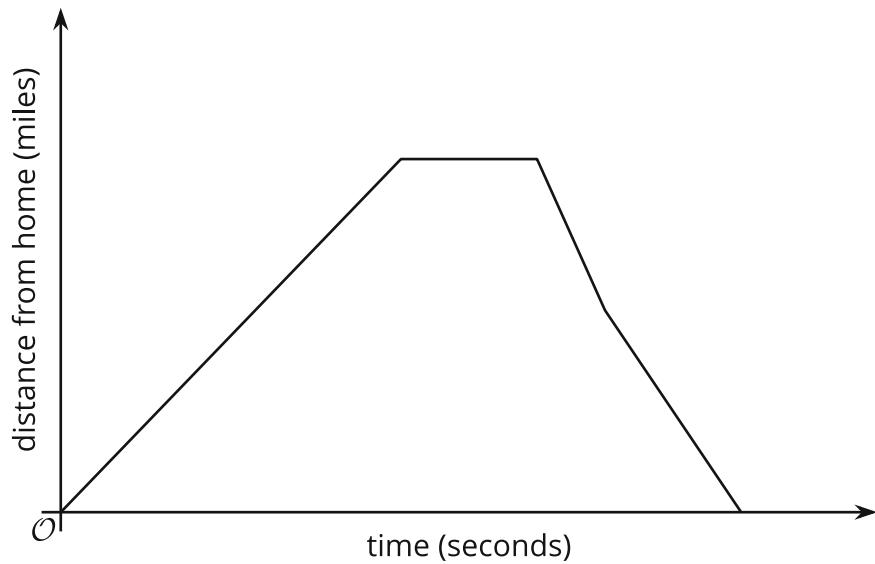
Lin walked a half-mile to school at a constant rate. Five minutes after arriving at school, Lin realized that she had left her permission slip at home. Lin began sprinting home, but when she was halfway there she got tired and walked the rest of the way.



Label the axes appropriately. You do *not* have to include numbers on the axes or the coordinates of points on your graph.

#### Solution

Answers vary. Sample response:



Minimal Tier 1 response:

- Work is complete and correct.
- Sample: See diagram, as well as notes in narrative.
- Acceptable errors: axes are labeled only as “distance/time” .

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: axes unlabeled or labeled incorrectly; graph does not meet one of the criteria mentioned in the narrative.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: graph does not meet two or more criteria mentioned in the narrative; graph fails to meet one of the criteria mentioned in the narrative and the axes are unlabeled/incorrect.

## Aligned Standards

8.F.B.5

### Problem 7

Students should be able to use a graph, table, and equation interchangeably. Look for students who write equations for the two companies' pricing schemes and use the equations throughout the problem. Other methods involve finding the hourly rate for each company and using this to find data points not visible in the table or graph, and plotting the points from the table directly onto the graph for comparison.

**Assessment: Mid-Unit Assessment (B)**

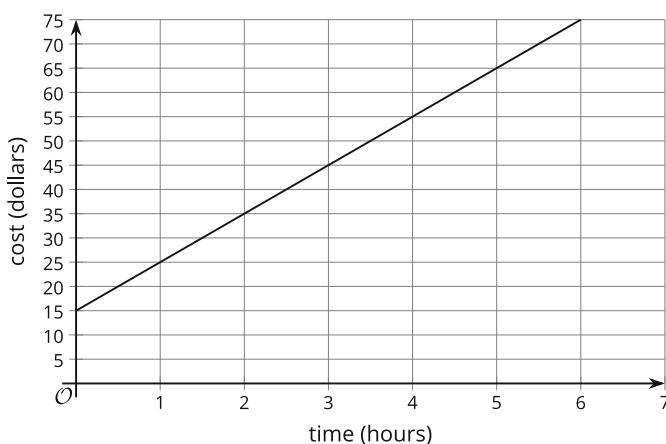
## Statement

Two cleaning services charge money for each hour of work, plus a one-time fee.

Sparkle Team Cleaners charges according to this table:

time (hours)	cost (dollars)
1	28
2	36
4	52
5	60
8	84

So Fresh & So Clean charges according to this graph:



1. A customer would like a company to visit their home and give a quote for their services, without doing further work. Which company is the most cost effective for this customer? Explain your answer.
2. Another customer would like the company to perform seven hours of work. Which company should this customer choose?
3. What is the smallest number of hours of work for which Sparkle Team Cleaners is cheapest?

## Solution

1. So Fresh and So Clean. To find the flat fee for Sparkle Team, notice that they charge 8 dollars per hour. Subtract \$8 from \$28, the cost for 1 hour of cleaning, to find a flat fee of \$20. The flat fee for So Fresh and So Clean is the  $y$ -intercept, 15 dollars.
2. Sparkle Team. Since Sparkle Team charges \$8 per hour, subtract \$8 from \$84, the cost for 8 hours of cleaning, to find that seven hours of cleaning costs \$76. From the graph, we can see that So Fresh and So Clean charges \$10 per hour. The point (6, 75) is on the graph, so 7 hours of cleaning must cost \$85.
3. 3 hours. Sparkle Team Cleaners charges \$44 for 3 hours of work, while So Fresh and So Clean charges \$45. (2.5 hours is also an acceptable solution: it's the point when both companies charge the same amount and the answers students will find if they solve a system of equations.)

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Acceptable errors: omitting units (\$).
- Sample:
  1. It's \$15 vs. \$20, so. So Fresh and Clean costs less.
  2. Sparkle Team is \$76 and Fresh and Clean is \$85, so Sparkle Team.
  3. 3 hours, because Sparkle Team just barely passes Fresh and Clean with \$44 vs. \$45.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: correct work/explanation for part c based on mistakes in parts a and b; approach to part c involves a correct system of equations with arithmetic mistakes in the solution method; work calculating slope/rate of change involves arithmetic mistakes.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: approach to parts a and b shows some understanding that finding a starting point and a constant rate of change is needed, but the work is not systematic and involves errors; badly misinterpreting the table/graph, such as reversing the columns/coordinates, with reasonable work following that; approach to part c involves a correct system of equations but no reasonable approach to solving the system; work for parts a and b is correct but work for part c is conceptually flawed.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: no evidence of understanding the connection between charge per hour and the information in the table and graph; error types from tier 3 response on parts a, b, and c.

## Aligned Standards

8.EE.C, 8.F.A.2, 8.F.B.4

# Assessment : End-of-Unit Assessment (A)

## Teacher Instructions

Students will need a calculator for approximating  $\pi$ , and may find it helpful in several problems. Note that some calculators have the ability to store formulas in memory. If you want to test whether students know formulas, you might take steps to clear the calculator memory or provide scientific calculators that don't allow for anything to be stored in memory.

## Student Instructions

You may use any type of calculator.

### Problem 1

Students selecting B or C might have a strong visual misconception about the volume of the two shapes, or might accidentally think they can apply the rule for a triangle's area. Students selecting D have the relationship backwards.

#### Statement

A cylinder has volume  $78 \text{ cm}^3$ . What is the volume of a cone with the same radius and height?

- A.  $26 \text{ cm}^3$
- B.  $39 \text{ cm}^3$
- C.  $156 \text{ cm}^3$
- D.  $234 \text{ cm}^3$

#### Solution

A

#### Aligned Standards

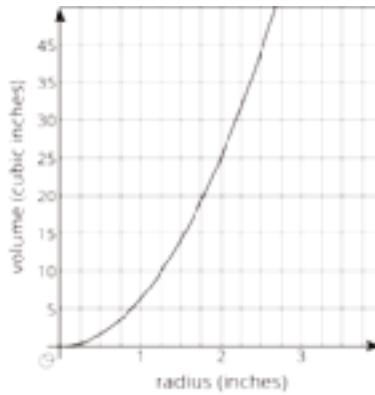
8.G.C.9

### Problem 2

Students selecting A instead of B have a misunderstanding about linear functions, and may have used some information about cones to come to this conclusion. Students selecting C instead of D may have thought proportional reasoning applies, but it does not here. Students failing to select E may need a reminder about the relationship between a function and its graph, or may have made an error in calculating the volume of the cone directly.

#### Statement

The graph shows the relationship between the radius and volume for many cones whose height is 6 inches.



Select **all** the true statements about such cones.

- A. The relationship between radius and volume is linear.
- B. The relationship between radius and volume is not linear.
- C. If the radius of the cone doubles, the volume of the cone doubles.
- D. If the radius of the cone doubles, the volume of the cone is multiplied by 4.
- E. If the radius of the cone is 2 inches, the volume of the cone is about 25 cubic inches.

## Solution

["B", "D", "E"]

## Aligned Standards

8.F.A.1, 8.F.A.3, 8.G.C.9

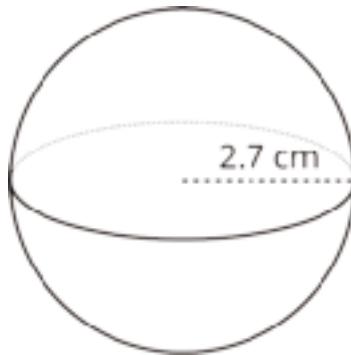
## Problem 3

Students selecting A used the formula  $\pi r^2$  instead. Students selecting B left out the  $\pi$  in their calculation (the volume is close to  $26\pi$ ). Students selecting C left the  $\frac{4}{3}$  out of their calculation or misapplied the formula for the volume of a cylinder.

Students should arrive at the same correct answer for any reasonable choice of the approximation of  $\pi$ .

## Statement

A sphere has radius 2.7 centimeters.



What is its volume, to the nearest cubic centimeter?

- A. 23
- B. 26
- C. 62
- D. 82

## Solution

D

## Aligned Standards

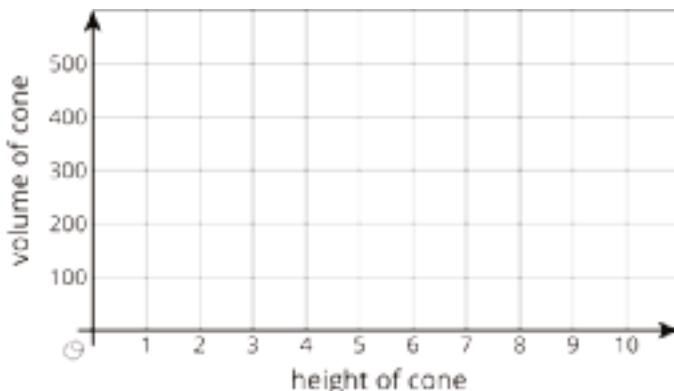
8.G.C.9

## Problem 4

Students should arrive at similar graphs for any reasonable choice of the approximation of  $\pi$ . Teachers who prefer to assign specific approximations are fine to do so.

## Statement

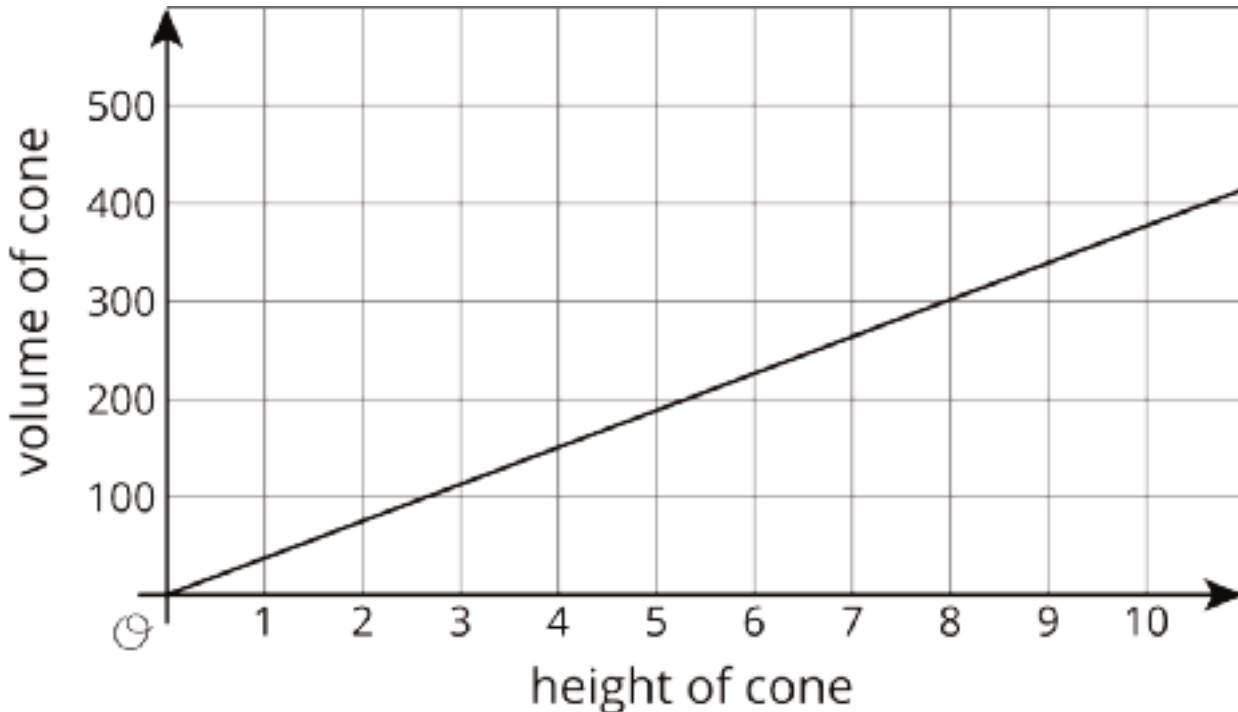
For cones with radius 6 units, the equation  $V = 12\pi h$  relates the height  $h$  of the cone, in units, and the volume  $V$  of the cone, in cubic units.



1. Sketch the graph of this equation on the axes.

2. Is there a linear relationship between height and volume? Explain how you know.

### Solution



1. See graph.
2. Yes. Explanations vary. Sample explanation: there is a linear relationship because the equation relating height and volume is in the same form as  $y = mx + b$ . Alternate explanation: there is a linear relationship because there is a proportional relationship, and all proportional relationships are linear.

Minimal Tier 1 response:

- Work is complete and correct.
- Sample:

1. See graph.
2. Yes, because the volume is  $12\pi$  times the height.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: some points in part a are incorrectly plotted, but the explanation for part b is correct based on independent justification; answer for part b is something like “the graph looks like a line” without further justification that the graph is linear.

Tier 3 response:

### Assessment: End-of-Unit Assessment (A)

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: any explanation for part b that does not appeal to slope, proportionality, the form of the equation for a line, or other concepts related to linearity; an incorrect graph in part a, with an incorrect answer to part b (including answers based on the non-linearity of the graph).

## Aligned Standards

8.F.A.1, 8.F.B.4

### Problem 5

Watch for students ignoring the  $\pi$  in the calculation, acting as though the volume is  $95\pi$ . Students who answer 37.1 meters have made this error.

Students should arrive at similar graphs for any reasonable choice of the approximation of  $\pi$ . Teachers who prefer to assign specific approximations are fine to do so.

#### Statement

A cylinder has radius 1.6 meters. Its volume is 95 cubic meters. Find its height to the nearest tenth of a meter.

### Solution

11.8 meters. If the height in meters is  $h$ , then the equation  $\pi \cdot (1.6)^2 \cdot h = 95$  is true. Using 3.14 as an approximation for  $\pi$  gives the equation  $8.04h = 95$ , and the solution to this equation is  $h \approx 11.8$ .

## Aligned Standards

8.G.C.9

### Problem 6

To check students' answers, see that their choices for  $r$  and  $h$  make the equation  $r^2h = 144$  true. Some students will not use the information about Cone A, and that's fine.

#### Statement

Cones A and B both have volume  $48\pi$  cubic units, but have different dimensions. Cone A has radius 6 units and height 4 units. Find one possible radius and height for Cone B. Explain how you know Cone B has the same volume as Cone A.

### Solution

Answers and explanations vary. Sample explanations:

- Cone B has radius 3 units and height 16 units. When the radius is halved, the height must be multiplied by 4 to keep the volume the same.

- Cone B could have any radius  $r$  and height  $h$  as long as  $\frac{1}{3}r^2h = 48$ , or  $r^2h = 144$ . One possible solution is radius 2 units and height 36 units, but there are others.

Minimal Tier 1 response:

- Work is complete and correct.
- Acceptable errors: omission of units; work demonstrates that the volume of Cone B is  $48\pi$  but does not directly compare to Cone A.
- Sample: Radius 4 and height 9, which has volume  $\frac{1}{3} \cdot \pi \cdot 4^2 \cdot 9$ . This is  $48\pi$  cubic units, the same as Cone A.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: arithmetic errors involving scaling; carefully written work reveals arithmetic errors involving the volume formula of a cone; scaling arguments involving a similar but incorrect volume formula, such as the formula for the volume of a cylinder.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: conceptual errors involving scaling; calculations are done using an incorrect volume formula; incorrect answer with no or little work shown; correct answer with no work shown and no comparison to Cone A.

## Aligned Standards

8.G.C.9

### Problem 7

While the problem does not require students to write an equation for the volume in terms of the radius, it is likely students will do this—the equation is  $V = 9\pi r^2$ .

To determine whether there is a linear relationship, students may plot points or calculate the slope (or rate of change) between two pairs of points. At this stage in their learning, it is not expected that students will reason that a relationship is linear or quadratic based on the form of an equation.

## Statement

There are many cylinders with a height of 9 inches. Let  $r$  represent the radius in inches and  $V$  represent the volume in cubic inches.

$r$	$V$
1	
2	
3	

1. Complete the table relating the radius and volume of cylinders with height 9 inches. Write each volume as a multiple of  $\pi$ , or round to the nearest cubic inch.
2. Is there a linear relationship between the radius and the volume of these cylinders? Explain how you know.
3. If a cylinder with height 9 inches and radius  $r$  is filled with water, it can fill a certain pitcher. How many of these pitchers can a cylinder with height 9 inches and radius  $2r$  fill? Explain how you know.

## Solution

1.

$r$	$V$
1	$9\pi$ or 28
2	$36\pi$ or 113
3	$81\pi$ or 254

2. No. The three points in the table are not on the same line, because the slope between the pairs of points is not the same: the slope between  $r = 1$  and  $r = 2$  is about 85 cubic inches per inch, but the slope between  $r = 2$  and  $r = 3$  is about 141 cubic inches per inch.
3. 4 pitchers, because scaling up the radius is scaling up in two dimensions. When you scale up in two dimensions, the scale factor gets squared.  $2^2 = 4$ .

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Acceptable errors: minor rounding mistakes, omission of units.
- Sample:
  1. See table.

2. No. From a 1-inch to a 2-inch radius, the volume goes up by 85, but from a 2-inch to a 3-inch radius, the volume goes up by 141.
3. 4 pitchers, because if you double the radius of a cone, the volume gets quadrupled.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: good work on parts b and c with errors in part a; work in part b states that points do not lie along the same line or do not fit a consistent slope, but does not provide numerical evidence; correct answer to part c without justification.
- Acceptable errors: a good argument about linearity in part b is based on incorrect table entries in part a.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: argument in part b conflates “linear” with “proportional”; argument in part b does not involve comparing rates in some way; answer to part c is “2 pitchers,” regardless of justification.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: errors in applying the formula for the volume of a cone prevent meaningful work on any of the problem parts; two or more error types under Tier 3 response.

## Aligned Standards

8.F.A.3, 8.G.C.9

# Assessment : End-of-Unit Assessment (B)

## Teacher Instructions

Students will need a calculator for approximating  $\pi$ , and may find it helpful in several problems. Note that some calculators have the ability to store formulas in memory. If you want to test whether students know formulas, you might take steps to clear the calculator memory or provide scientific calculators that don't allow for anything to be stored in memory.

## Student Instructions

You may use any type of calculator.

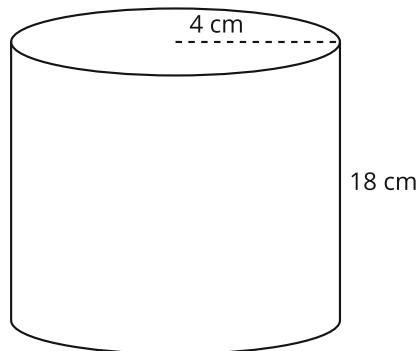
### Problem 1

Students selecting A may have squared both the radius and the height before determining the difference. Students selecting B may have squared  $\pi$  as well as the radius. Students selecting D may have calculated the volume of Cylinder B with having a radius of 5 instead of 2.5.

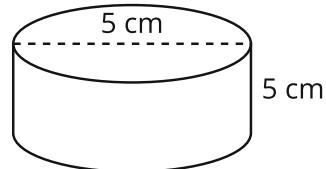
#### Statement

Which is closest to the difference in the volume of the two cylinders?

A



B



A.  $15,795 \text{ cm}^3$

B.  $2,534 \text{ cm}^3$

C.  $806 \text{ cm}^3$

D.  $512 \text{ cm}^3$

## Solution

C

## Aligned Standards

8.G.C.9

### Problem 2

Students selecting B used the formula  $V = \frac{4}{3}\pi r^2$  instead of cubing  $r$ . Students selecting C may have failed to multiply by the reciprocal of  $\frac{4}{3}$  while solving for  $r$ . Students selecting A may have made both errors mentioned above. Students should arrive at the same correct answer for any reasonable choice of the approximation of  $r$ .

#### Statement

A sphere has a volume of  $972\pi \text{ cm}^3$ . What is its radius in centimeters?

- A. 4.5
- B. 9
- C. 11
- D. 18

#### Solution

B

## Aligned Standards

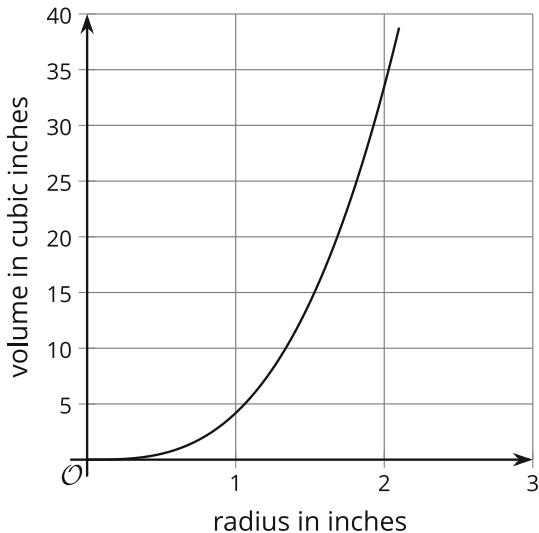
8.G.C.9

### Problem 3

Students selecting A instead of C may have thought proportional reasoning applies, but it does not here. Students selecting B instead of C may be remembering examples involving cones and cylinders of a given height, in which the volume does become 4 times bigger when doubling the radius. The difference here is that a sphere's volume is a cubic function of its radius. Students selecting D instead of E have a misunderstanding about linear functions.

## Statement

The graph shows the relationship between a sphere's radius and volume.



Select **all** the statements that are true about this relationship.

- A. If the radius doubles, the volume doubles.
- B. If the radius doubles the volume becomes about 4 times bigger.
- C. If the radius doubles, the volume becomes about 8 times bigger.
- D. The relationship is linear.
- E. The relationship is not linear.

## Solution

["C", "E"]

## Aligned Standards

8.F.A.1, 8.F.A.3, 8.G.C.9

## Problem 4

While most students will argue that the relationship is linear based on the constant differences in the table, they may also find a constant of proportionality or appeal to the equation relating height and volume,  $V = 4\pi h$ .

## Statement

For cylinders with radius 2 units, let  $h$  represent the height of the cylinder, in units, and the  $V$  represent the volume of the cylinder, in cubic units.

1. Complete the table relating the height and volume of cylinders with radius 2 inches. Write each volume as a multiple of  $\pi$ , or round to the nearest cubic unit.

2. Is there a linear relationship between height and volume?  
Explain how you know.

$h$	$V$
1	
2	
3	

## Solution

1.

$h$	$V$
1	$4\pi$
2	$8\pi$
3	$12\pi$

2. Yes. Explanations vary. Sample explanation: there is a linear relationship because the points would lie on the same line, the rate of change is the same. Alternate explanation: there is a linear relationship because there is a proportional relationship, and all proportional relationships are linear.

Minimal Tier 1 response:

- Work is complete and correct.
- Sample:

1. See table.

2. Yes, because the points would lie on the same line. The rate of change is the same.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: one volume in part a is incorrect, but the explanation for part b is correct based on independent justification; answer for part b is something like “because 8 is two times 4”, but does not extend the proportionality argument to address the entire table.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: any explanation for part b that does not appeal to slope, proportionality, the form of the equation for a line, or other concepts related to linearity; the table completed

## Assessment: End-of-Unit Assessment (B)

incorrectly for part a, with an incorrect answer to part b (including answers based on an incorrect table).

## Aligned Standards

8.F.A.1, 8.F.B.4, 8.G.C.9

### Problem 5

For this question, students can use the fact that the volume of a cone is  $\frac{1}{3}$  of the volume of a cylinder with the same height and same base. Students may also use the volume of the cylinder to solve for the radius and in turn use the radius to determine the volume of the cone.

#### Statement

A cone and cylinder have the same height and their bases are congruent circles. If the volume of the cylinder is 120 in<sup>3</sup>, what is the volume of the cone?

#### Solution

40 in<sup>3</sup>

## Aligned Standards

8.G.C.9

### Problem 6

There are two correct answers for this question. Both are approximations of the sphere's volume, with one expressed in terms of  $\pi$ . Students selecting Noah have applied the wrong formula for volume of a sphere.

#### Statement

Three students each calculated the volume of a sphere with a radius of 6 centimeters.

- Diego found the volume to be  $288\pi$  cubic centimeters.
- Andre approximated 904 cubic centimeters.
- Noah calculated 226 cubic centimeters.

Do you agree with any of them? Explain your reasoning.

#### Solution

Diego and Andre are both correct. Sample explanation:

$$\text{Diego: } V = \frac{4}{3}\pi r^3, V = \frac{4}{3}\pi \cdot 6^3, V = \frac{4}{3} \cdot 216\pi, V = 288\pi.$$

Andre also got  $288\pi$ , but multiplied 288 by 3.14 and rounded his answer to 904.

Diego expressed his answer in terms of  $\pi$ , whereas Andre's answer is the volume to the nearest cubic centimeter.

Minimal Tier 1 response:

- Work is complete and correct.
- Acceptable errors: saying that Andre is incorrect with the rationale that using a multi-digit approximation of  $\pi$  yields an answer closer to 905 cm.
- Sample: Diego and Andre are both correct.  $V = \frac{4}{3}\pi \cdot 6^3$ , so Diego got  $288\pi$  and Andre multiplied 288 by 3.14 instead of  $\pi$  to get 904.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: identifying only Andre or only Diego as being correct, not explaining that Andre's answer involves an approximation for  $\pi$ ; correctly identifying Andre and Diego as correct but with no or very little explanation.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: calculations are done using an incorrect volume formula; incorrect answer with no or little work shown; correct answer with no work shown.

## Aligned Standards

8.G.C.9

### Problem 7

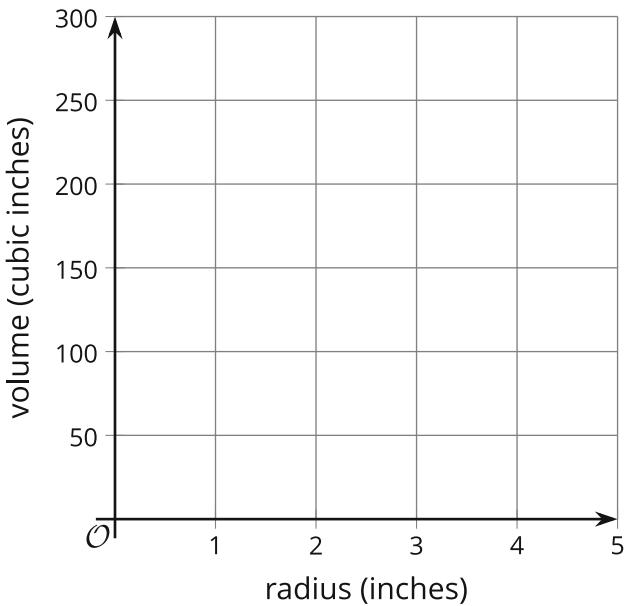
Students will have the easiest time plotting the points if they approximate  $\pi$  in their equation.

#### Statement

There are many cones with a height of 12 inches. Let  $r$  represent the radius and  $V$  represent the volume of these cones.

1. Write an equation that expresses the relationship between  $V$  and  $r$ . Use 3.14 as an approximation for  $\pi$ .

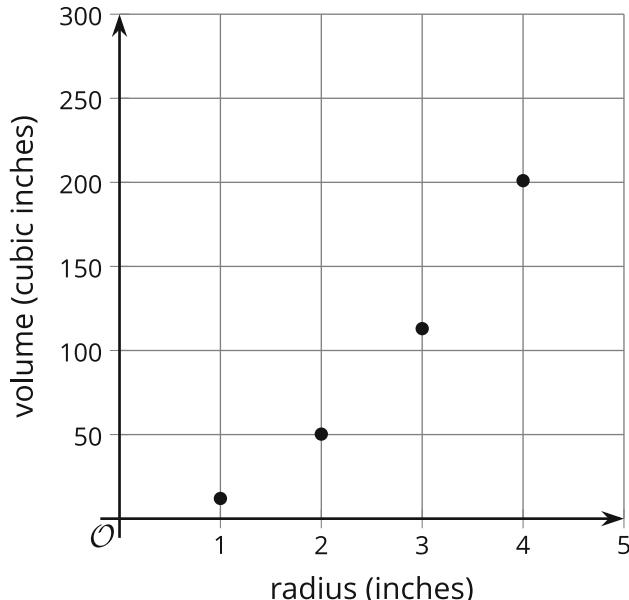
2. Plot points that show the volume when  $r = 1$ ,  $r = 2$ ,  $r = 3$ , and  $r = 4$ . Show your reasoning.



3. A vendor at a street fair sells popcorn in cones, all of height 9 inches. The sharing-size cone has 3 times the radius of the skinny-size cone. About how many times more popcorn does the sharing cone hold than the skinny cone?

## Solution

1.  $V = 12.56r^2$  or an equivalent equation.



2.

3. 9 times more popcorn. Since the sharing size cone is 3 times wider in two different dimensions, square the scale factor. ( $3^2 = 9$ .)

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Acceptable errors: The equation in part a uses  $\pi$  instead of 3.14.
- Sample:

1.  $V = \frac{1}{3} \cdot 3.14 \cdot 12 \cdot r^2, V = 12.56r^2$

2. See graph.

3. 9 times more, because  $3^2 = 9$ .

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: one or two points on the graph are plotted incorrectly; a small mistake in the equation results in incorrect but consistent answers in parts a and b; a correct answer to part c without justification.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: little progress made on parts a and b but a good explanation for part c; points in part b are consistently off, perhaps because of an error applying the equation such as squaring the coefficient of  $r$ ; answer to part c is "3 times bigger," regardless of justification.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: errors in applying the formula for the volume of a cone prevent meaningful work on any of the problem parts; multiple Tier 3 error types.

## Aligned Standards

8.F.A.3, 8.G.C.9

**Assessment: End-of-Unit Assessment (B)**

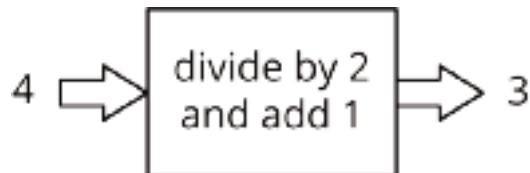


# Lesson Cool Downs

# Lesson 1: Inputs and Outputs

## Cool Down: What's the Rule?

Fill in the table for this input-output rule:



input	output
0	
2	
-8	
100	

# Lesson 2: Introduction to Functions

## Cool Down: Wait Time

You are told that you will have to wait for 5 hours in a line with a group of other people. Determine whether:

1. You know the number of minutes you have to wait.

2. You know how many people have to wait.

For each statement, if you answer yes draw an input-output diagram and write a statement that describes the way one quantity depends on another.

If you answer no give an example of 2 outputs that are possible for the same input.

# Lesson 3: Equations for Functions

## Cool Down: The Value of Some Quarters

The value  $v$  of your quarters (in cents) is a function of  $n$ , the number of quarters you have.

1. Draw an input-output diagram to represent this function.

2. Write an equation that represents this function.

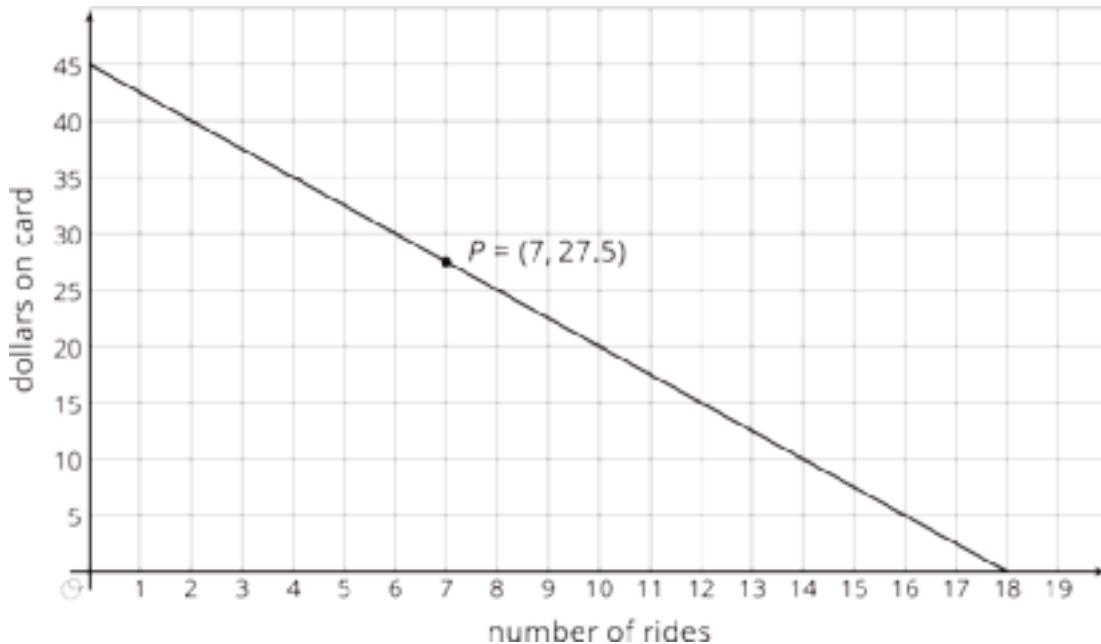
3. Find the output when the input is 10.

4. Identify the independent and dependent variables.

# Lesson 4: Tables, Equations, and Graphs of Functions

## Cool Down: Subway Fare Card

Here is the graph of a function showing the amount of money remaining on a subway fare card as a function of the number of rides taken.

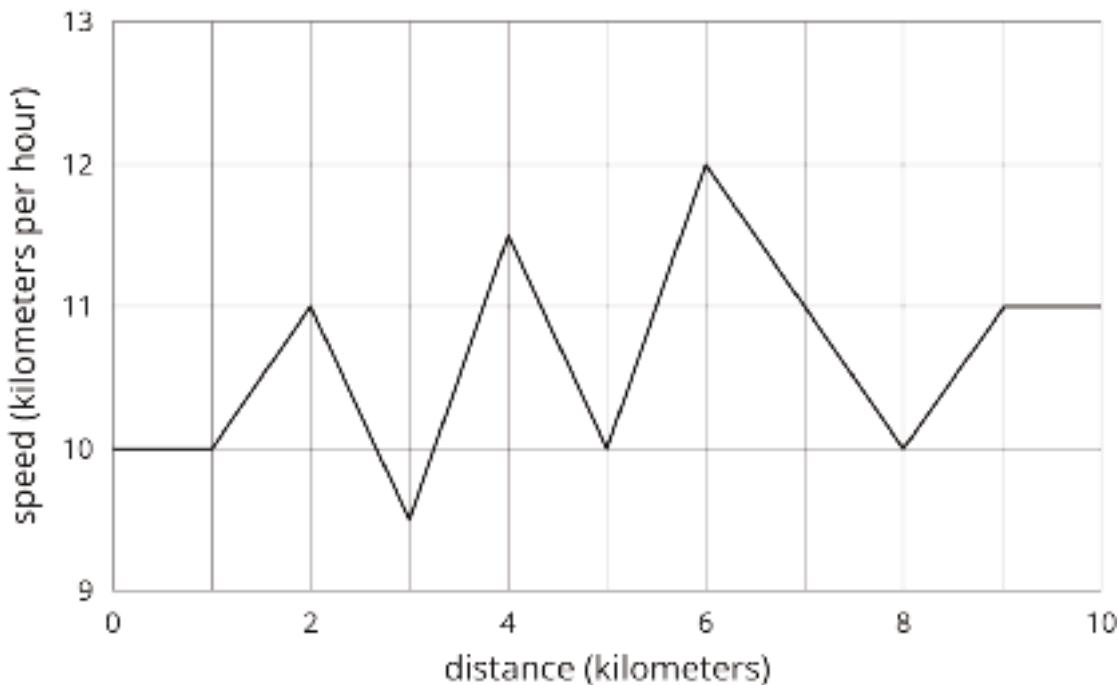


1. What is the output of the function when the input is 10? On the graph, plot this point and label its coordinates.
2. What is the input to the function when the output is 5? On the graph, plot this point and label its coordinates.
3. What does point  $P$  tell you about the situation?

# Lesson 5: More Graphs of Functions

## Cool Down: Diego's 10K Race

Diego runs a 10 kilometer race and keeps track of his speed.



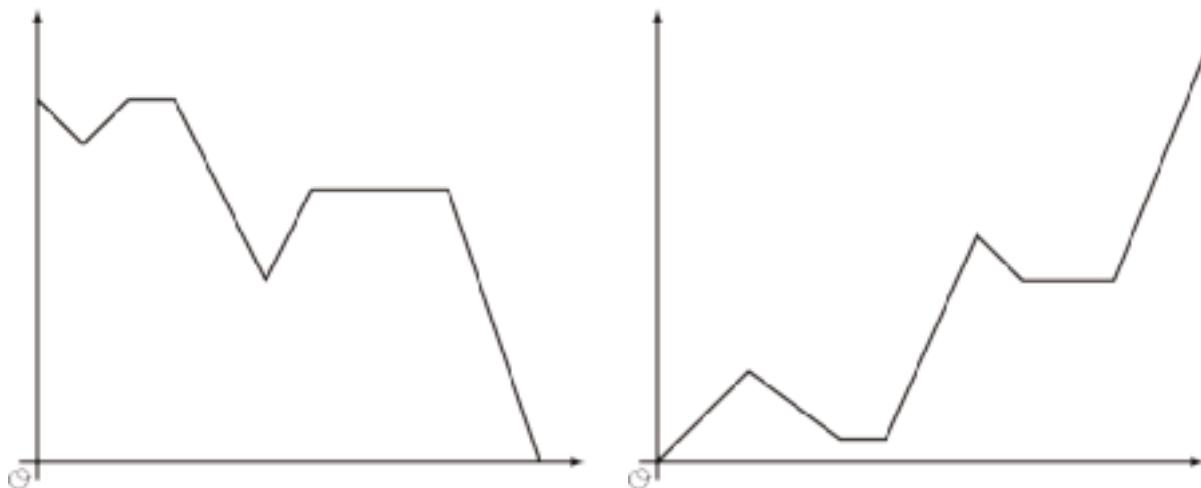
1. What was Diego's speed at the 5 kilometer mark in the race?
2. According to the graph, where was Diego when he was going the slowest during the race?
3. Describe what happened to Diego's speed in the second half of the race (from 5 km to 10 km).

# Lesson 6: Even More Graphs of Functions

## Cool Down: Walking Home From School

Elena starts to walk home from school, but has to turn around and go back because she left something in her locker. On her way back home (the second time), she runs into her friend who invites her to the library to do homework with her. She stays at the library and then heads home to do her chores. Determine:

- Which graph fits Elena's story.
- What the two quantities are.
- Which quantity is a function of which.



# Lesson 7: Connecting Representations of Functions

## Cool Down: Comparing Different Areas

The table shows the area of a square for specific side lengths.

side length (inches)	0.5	1	2	3
area (square inches)	0.25	1	4	9

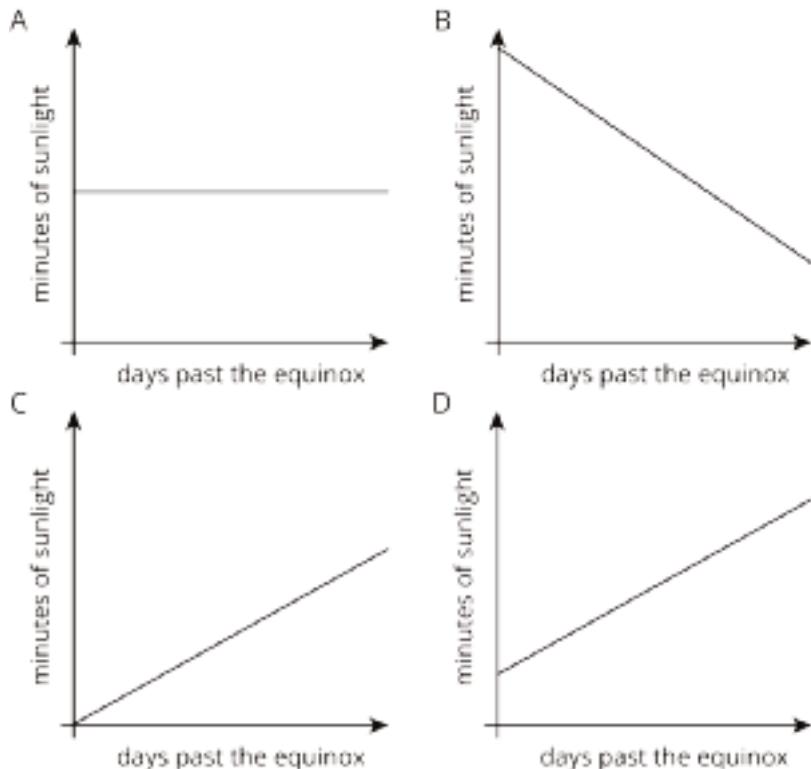
The area  $A$  of a circle with radius  $r$  is given by the equation  $A = \pi \cdot r^2$ .

Is the area of a square with side length 2 inches greater than or less than the area of a circle with radius 1.2 inches?

# Lesson 8: Linear Functions

## Cool Down: Beginning to See Daylight

In a certain city in France, they gain 2 minutes of daylight each day after the spring equinox (usually in March), but after the autumnal equinox (usually in September) they lose 2 minutes of daylight each day.



1. Which of the graphs is most likely to represent the graph of daylight for the month after the spring equinox?
2. Which of the graphs is most likely to represent the graph of daylight for the month after the autumnal equinox?
3. Why are the other graphs not likely to represent either month?

# Lesson 9: Linear Models

## Cool Down: Board Game Sales

A small company is selling a new board game, and they need to know how many to produce in the future.

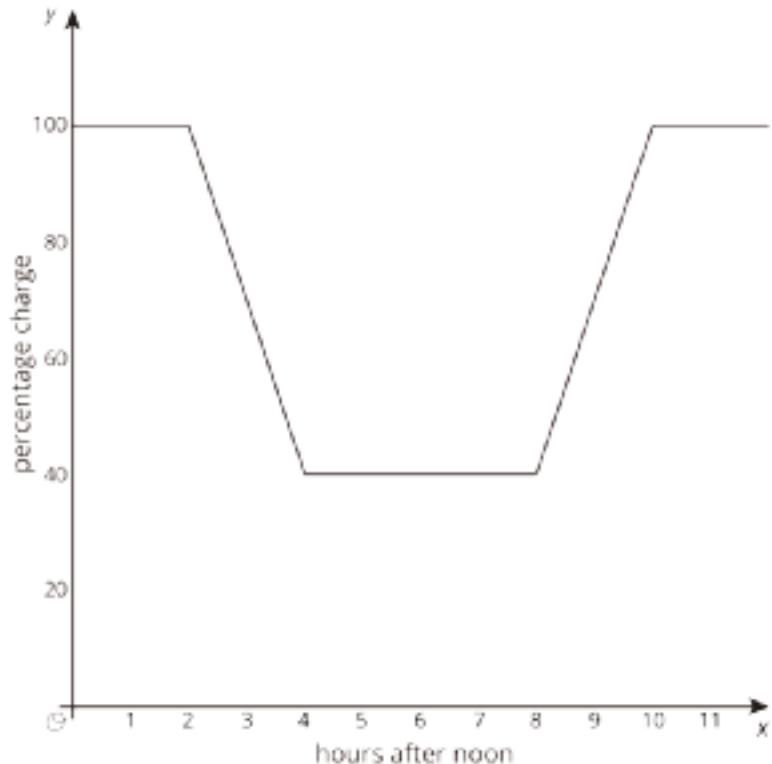
After 12 months, they sold 4 thousand games; after 18 months, they sold 7 thousand games; and after 36 months, they sold 15 thousand games.

1. Could this information be reasonably estimated using a single linear model?
2. If so, use the model to estimate the number of games sold after 48 months. If not, explain your reasoning.

# Lesson 10: Piecewise Linear Functions

## Cool Down: Lin's Phone Charge

Lin uses an app to graph the charge on her phone.

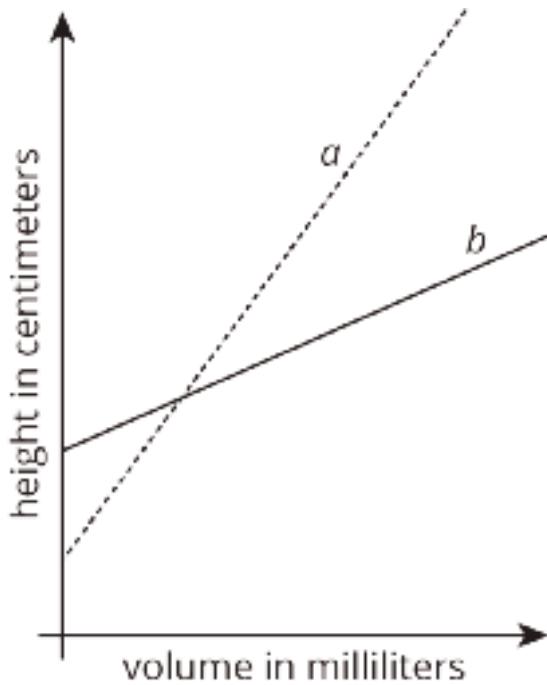


1. When did she start using her phone?
2. When did she start charging her phone?
3. While she was using her phone, at what rate was Lin's phone battery dying?

# Lesson 11: Filling Containers

## Cool Down: Which Cylinder?

Two cylinders,  $a$  and  $b$ , each started with different amounts of water. The graph shows how the height of the water changed as the volume of water increased in each cylinder. Which cylinder has the larger radius? Explain how you know.



# Lesson 12: How Much Will Fit?

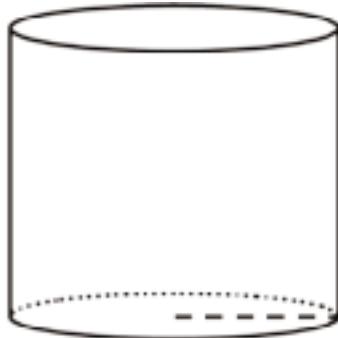
## Cool Down: Rectangle to Round



Here is a box of pasta and a cylindrical container. The two objects are the same height, and the cylinder is just wide enough for the box to fit inside with all 4 vertical edges of the box touching the inside of the cylinder. If the box of pasta fits 8 cups of rice, estimate how many cups of rice will fit inside the cylinder. Explain or show your reasoning.

# Lesson 13: The Volume of a Cylinder

## Cool Down: Liquid Volume

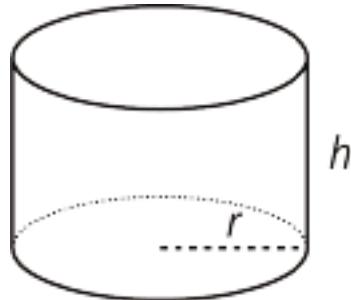


The cylinder shown here has a height of 7 centimeters and a radius of 4 centimeters.

1. What is the area of the base of the cylinder? Express your answer in terms of  $\pi$ .
2. How many cubic centimeters of fluid can fill this cylinder? Express your answer in terms of  $\pi$ .
3. Give a decimal approximation of your answer to the second question using 3.14 to approximate  $\pi$ .

# Lesson 14: Finding Cylinder Dimensions

## Cool Down: Find the Height

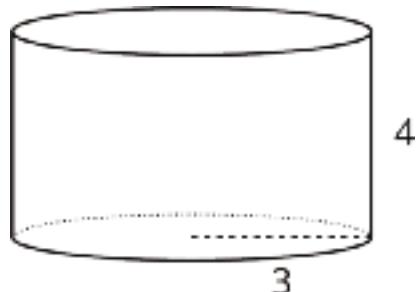


This cylinder has a volume of  $12\pi$  cubic inches and a diameter of 4 inches. Find the cylinder's radius and height.

# Lesson 15: The Volume of a Cone

## Cool Down: Calculate Volumes of Two Figures

A cone with the same base but a height 3 times taller than the given cylinder exists. What is the volume of each figure? Express your answers in terms of  $\pi$ .



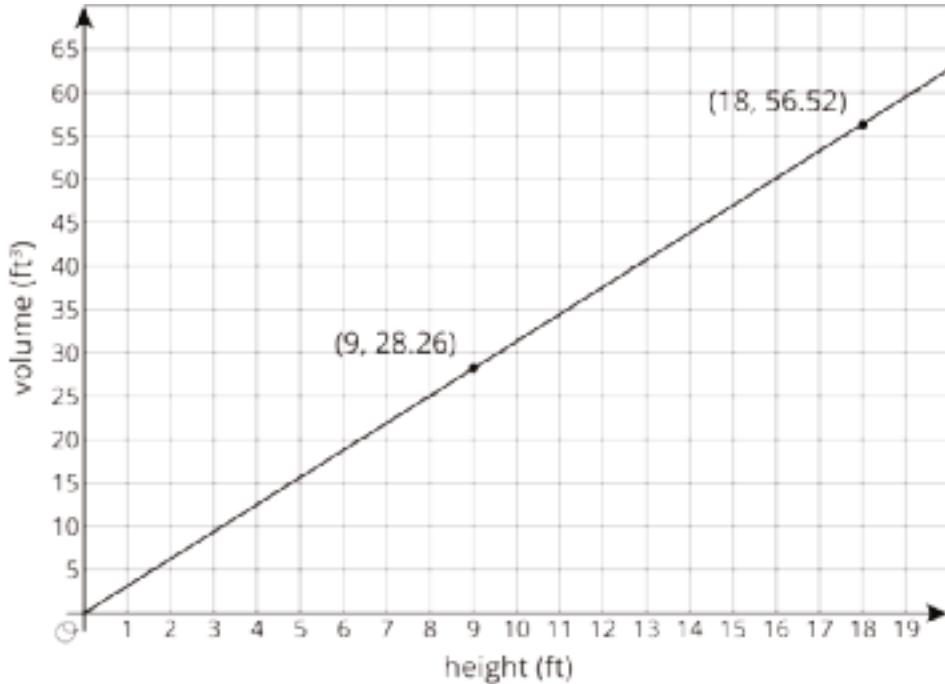
# Lesson 16: Finding Cone Dimensions

## Cool Down: A Square Radius

Noah and Lin are making paper cones to hold popcorn to hand out at parent math night. They want the cones to hold  $9\pi$  cubic inches of popcorn. What are two different possible values for height  $h$  and radius  $r$  for the cones?

# Lesson 17: Scaling One Dimension

## Cool Down: A Missing Radius



Here is a graph of the relationship between the height and volume of some cylinders that all have the same radius,  $R$ . An equation that represents this relationship is  $V = \pi R^2 h$  (use 3.14 as an approximation for  $\pi$ ).

What is the radius of these cylinders?

# Lesson 18: Scaling Two Dimensions

## Cool Down: Halving Dimensions

There are many cylinders for which the height and radius are the same value. Let  $c$  represent the height and radius of a cylinder and  $V$  represent the volume of the cylinder.

1. Write an equation that expresses the relationship between the volume, height, and radius of this cylinder using  $c$  and  $V$ .
2. If the value of  $c$  is halved, what must happen to the value of the volume  $V$ ?

# Lesson 19: Estimating a Hemisphere

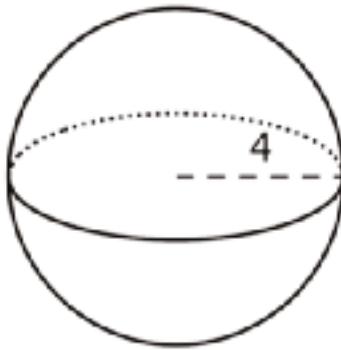
## Cool Down: A Mirror Box

A hemisphere-shaped security mirror fits exactly inside a rectangular prism box with a square base that has edge length 10 inches. What is a reasonable estimate for the volume of this mirror?

# Lesson 20: The Volume of a Sphere

## Cool Down: Volumes of Spheres

Recall that the volume of a sphere is given by the formula  $V = \frac{4}{3}\pi r^3$ .



1. Here is a sphere with radius 4 feet. What is the volume of the sphere? Express your answer in terms of  $\pi$ .
2. A spherical balloon has a diameter of 4 feet. Approximate how many cubic feet of air this balloon holds. Use 3.14 as an approximation for  $\pi$ , and give a numerical answer.

# Lesson 21: Cylinders, Cones, and Spheres

## Cool Down: New Four Spheres

Some information is given about each sphere. Order them from least volume to greatest volume. You may sketch a sphere to help you visualize if you prefer.

Sphere A: Has a radius of 4

Sphere B: Has a diameter of 6

Sphere C: Has a volume of  $64\pi$

Sphere D: Has a radius double that of sphere B.



# Instructional Masters

# Instructional Masters for Functions and Volume

address	title	students per copy	written on?	requires cutting?	card stock recommended?	color paper recommended?
Activity Grade8.5.21.3	Info Gap: Unknown Dimensions	1	no	yes	no	no
Activity Grade8.5.1.2	Guess My Rule	1	no	yes	no	no

8.5.1.2 Guess My Rule.

Guess my Rule

A

Add 7

Guess my Rule

B

Multiply by 3

Guess my Rule

C

Double, then subtract 5

Guess my Rule

D

- If even, divide by 2
- If odd, multiply by 3 and add 1

Guess my Rule

A

Add 7

Guess my Rule

B

Multiply by 3

Guess my Rule

C

Double, then subtract 5

Guess my Rule

D

- If even, divide by 2
- If odd, multiply by 3 and add 1

## Info Gap: Unknown Dimensions

**Problem Card 1**

A cone and a sphere have the same dimensions. What is the volume of the sphere?

## Info Gap: Unknown Dimensions

**Problem Card 2**

A cone and a sphere have the same height. What is the volume of the sphere?

## Info Gap: Unknown Dimensions

**Problem Card 1**

A cone and a sphere have the same dimensions. What is the volume of the sphere?

## Info Gap: Unknown Dimensions

**Problem Card 2**

A cone and a sphere have the same height. What is the volume of the sphere?

## Info Gap: Unknown Dimensions

**Data Card 1**

- The volume of the cone is  $V = 144\pi$  cm<sup>3</sup>.
- The radius of the cone is the same as the radius of the sphere.
- $4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$

## Info Gap: Unknown Dimensions

**Data Card 2**

- The volume of the cone is  $V = 18\pi$  cm<sup>3</sup>.
- The radius of the sphere is half the height of the cone.
- The height of the cone is twice the value of the radius of the cone.
- $4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$

## Info Gap: Unknown Dimensions

**Data Card 1**

- The volume of the cone is  $V = 144\pi$  cm<sup>3</sup>.
- The radius of the cone is the same as the radius of the sphere.
- $4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$

## Info Gap: Unknown Dimensions

**Data Card 2**

- The volume of the cone is  $V = 18\pi$  cm<sup>3</sup>.
- The radius of the sphere is half the height of the cone.
- The height of the cone is twice the value of the radius of the cone.
- $4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$



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