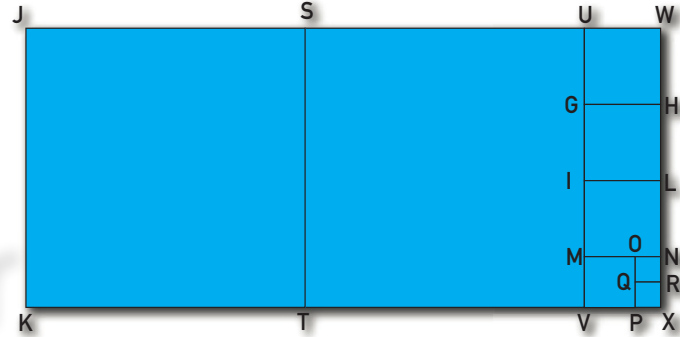


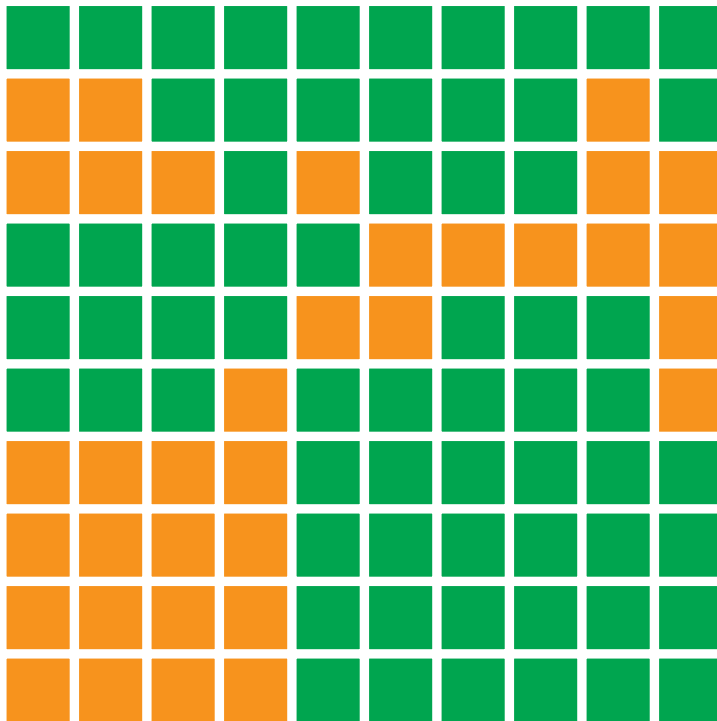
Putting it All Together



Rectangle decomposed into squares.



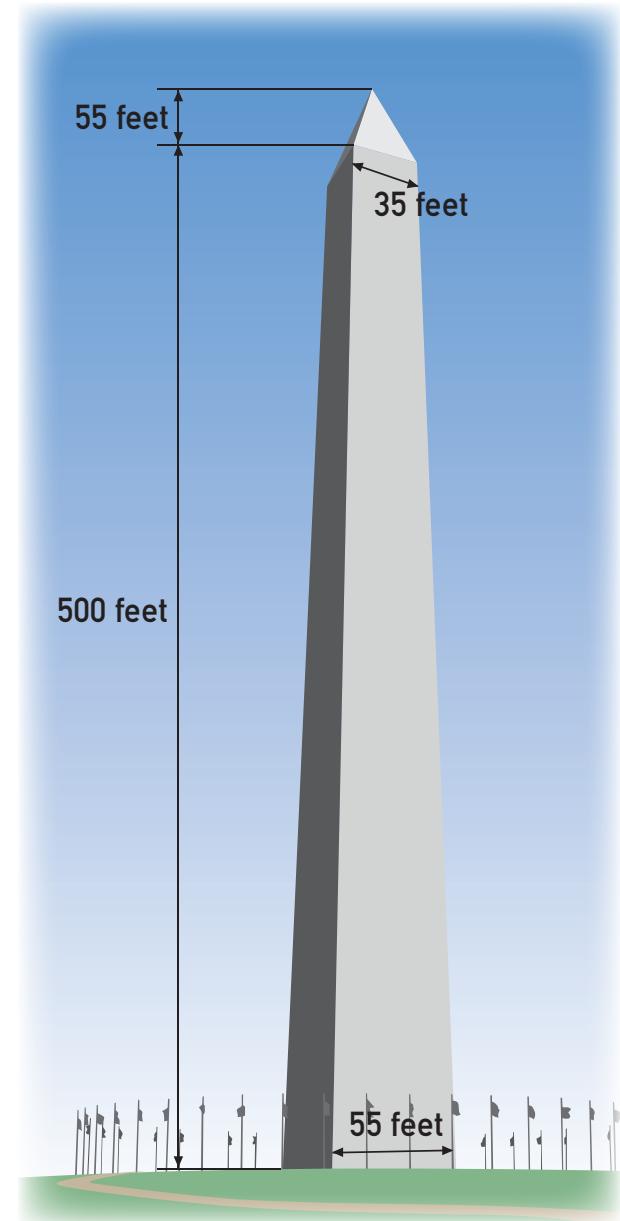
Teacher Guide

Squaretown's map



		
class A	26	14
class B	31	19

Which Was "Yessier"?



Covering the Washington Monument

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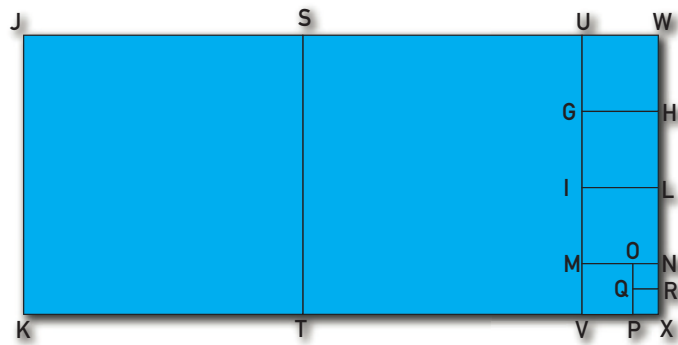
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Putting it All Together

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Putting it All Together
Teacher Guide
 Core Knowledge Mathematics™

Putting it All Together

Unit Narrative

This optional unit consists of six lessons. Each of the first three lessons is independent of the others, requiring only the mathematics of the previous units. The last three lessons build on each other.

The first lesson concerns Fermi problems—problems that require making rough estimates for quantities that are difficult or impossible to measure directly (MP4). The three problems in this lesson involve measurement conversion and calculation of volumes and surface areas of three-dimensional figures or the relationship of distance, rate, and time.

The second lesson involves finding approximately equivalent ratios for groups from two populations, one very large (the population of the world) and one comparatively small (a 30-student class). Students work with percent rates that describe subgroups of the world population, e.g., about 59% of the world population lives in Asia. Using these rates, which include numbers expressed in the form $\frac{a}{b}$ or as decimals, they determine, for example, the number of students who would live in Asia—“if our class were the world” (MP2). Because students choose their own methods to determine these numbers, possibly making strategic use of benchmark percentages or spreadsheets (MP5), there is an opportunity for them to see correspondences between approaches (MP1). Because the size of the world population and its subgroups are estimates, and because pairs of values in ratios may both be whole numbers, considerations of accuracy may arise (MP6).

The third lesson is an exploration of the relationship between the greatest common factor of two numbers, continued fractions, and decomposition of rectangles with whole-number side lengths, providing students an opportunity to perceive this relationship through repeated reasoning (MP8) and to see correspondences between two kinds of numerical relationships, and between numerical and geometric relationships (MP1).

The remaining three lessons explore the mathematics of voting (MP2, MP4). In some activities, students chose how to assign votes and justify their choices (MP3). The first of these lessons focuses on proportions of voters and votes cast in elections in which there are two choices. It requires only the mathematics of the previous units, in particular, equivalent ratios, part–part ratios, percentages, unit rates, and, in the final activity, the concept of area. The second of these lessons focuses on methods for voting when there are more than two choices: plurality, runoff, and instant runoff. They compute percentages, finding that different voting methods have different outcomes. The third of these lessons focuses on representation in the case when voters have two choices. It’s not always possible to have the same number of constituents per representative. How can we fairly share a small number of representatives? Students again compute percentages to find outcomes.

Progression of Disciplinary Language

In this unit, teachers can anticipate students using language for mathematical purposes such as critiquing, justifying, and comparing. Throughout the unit, students will benefit from routines

designed to grow robust disciplinary language, both for their own sense-making and for building shared understanding with peers. Teachers can formatively assess how students are using language in these ways, particularly when students are using language to:

Critique

- reasoning about Fermi problems (Lesson 1)
- claims about percentages (Lesson 4)
- reasoning about the fairness of voting systems (Lesson 6)

Justify

- reasoning about Fermi problems (Lesson 1)
- reasoning about the fairness of voting systems (Lessons 5 and 6)

Compare

- rectangles and fractions (Lesson 3)
- voting systems (Lesson 5)

In addition, students are expected to interpret and represent characteristics of the world population, describe distributions of voters, and generalize about decomposition of area and numbers.

The table shows lessons where new terminology is first introduced, including when students are expected to understand the word or phrase receptively and when students are expected to produce the word or phrase in their own speaking or writing. Terms from the glossary appear bolded. Teachers should continue to support students’ use of a new term in the lessons that follow where it was first introduced.

lesson	new terminology	
	receptive	productive
6.9.3	mixed number	
6.9.4	in favor majority	
6.9.5	plurality runoff	majority
6.9.6	in all fair	

Required Materials

Colored pencils

Four-function calculators

Graph paper

Internet-enabled device

Scissors

Tools for creating a visual display

Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Section: Making Connections

Lesson 1: Fermi Problems

Goals

- Estimate quantities in a real-world situation and explain (orally and in writing) the estimation strategy.
- Justify (orally) why it is unreasonable to have an exact answer for a situation that involves estimation, and critique (orally) different estimates.
- Make simplifying assumptions and determine what information is needed to solve a Fermi problem about distance, volume, or surface area.

Lesson Narrative

This lesson is optional. The activities in this lesson plan are sometimes called “Fermi problems” after the famous physicist Enrico Fermi. A Fermi problem requires students to make a rough estimate for quantities that are difficult or impossible to measure directly. Often, they use rates and require several calculations with fractions and decimals, making them well-aligned to grade 6 work. Fermi problems are examples of mathematical modeling (MP4), because one must make simplifying assumptions, estimates, research, and decisions about which quantities are important and what mathematics to use. They also encourage students to attend to precision (MP6), because one must think carefully about how to appropriately report estimates and choose words carefully to describe the quantities.

Each of these activities can stand on its own. If students do the first before the second, the second will take less time. It is very likely that it would take more than a single day to do all of the activities in this lesson. One option is to let students choose an activity that interests them. If you choose to conduct the lesson in this manner, begin by posing these scenarios, one for each activity in this lesson, to students:

1. “Imagine that an ant ran from Los Angeles to New York City.”
2. “Imagine a warehouse that has a rectangular floor and contains all of the boxes of breakfast cereal bought in the United States every year.”
3. “Imagine that the entire Washington Monument had to be completely retiled.”

Then ask:

1. “Which one interests you the most? Why?”
2. “What questions could we ask about each situation?”
3. “What information would you want to know in order to investigate that particular situation?”

Each student or group can explore the problem that interests them the most and share their findings.

As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

Alignments

Addressing

- 6.G.A: Solve real-world and mathematical problems involving area, surface area, and volume.
- 6.NS.B: Compute fluently with multi-digit numbers and find common factors and multiples.
- 6.RP.A: Understand ratio concepts and use ratio reasoning to solve problems.

Instructional Routines

- Group Presentations
- MLR5: Co-Craft Questions
- MLR8: Discussion Supports
- Poll the Class

Required Materials

Four-function calculators

Internet-enabled device

Tools for creating a visual display

Any way for students to create work that can be easily displayed to the class. Examples: chart

paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Internet-enabled devices are only necessary if students will conduct research to find quantities that they need to know. As an alternative, you can supply the information when they ask for it.

Tools for creating a visual display are only needed if you would like students to present their work in an organized way and have the option of conducting a gallery walk.

Student Learning Goals

Let's make some estimates.

1.1 Ant Trek

Optional: 20 minutes

In this Fermi problem, students estimate the time it would take an ant to run from Los Angeles to New York City. Monitor for different approaches to solving the problem, and select students to share during the discussion. In particular, look for:

- A range of estimates
- Different levels of precision in reporting the final estimate
- Different representations (for example, double number lines and tables)

Addressing

- 6.NS.B
- 6.RP.A

Instructional Routines

- Group Presentations
- MLR5: Co-Craft Questions
- Poll the Class

Launch

Arrange students in groups of 1–4. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

Before starting, ask students to come up with a guess about the answer and poll the class. Record all guesses.

Ask students to brainstorm the information they need to answer the question. Give the information provided when students ask for it, or provide access to internet-enabled devices so that students can find the information they need.

Vital information to have on hand includes:

- The distance between Los Angeles and New York City is about 3,944 km.
- An ant can run about 18 mm per second.

Also of interest is the fact that most ants do not live long enough to complete this trip. Many ants live for only a couple months. If students realize this, ask them how many ant lifetimes it would take for an ant to make this journey.

Access for English Language Learners

Conversing: MLR5 Co-Craft Questions. To use this routine to support student understanding of the context of the problem. Display a map that shows a route between Los Angeles and New York City. Invite students to work with a partner to create a list of possible mathematical questions that could be asked about the situation. Invite students to share their questions with the class and call students' attention to the the different ways students integrate measurements (distance traveled, speed) in their questions.

Design Principle(s): Maximize meta-awareness; Support sense-making

Anticipated Misconceptions

Students may struggle to get started with no given information. Ask these students what they need to know in order to estimate how long it takes the ant to travel across the country. They may respond in a variety of ways, for example:

- How far does the ant travel in a day (or a different period of time)? Give them the information that the ant travels 18 mm per second.
- How far is it from Los Angeles to New York City? (They may ask for this distance in miles.) Give them the information that it is about 3944 km.

Students may struggle to convert mm to km. Ask these students how many mm are in a cm? in a meter? How many meters are in a km?

Students may be distracted by other concerns, like what the ant will eat, whether or not the ant will travel throughout the winter, whether or not the ant can cross mountains. Help these students understand that it is okay to make simplifying assumptions so that the calculations are reasonable.

Student Task Statement

How long would it take an ant to run from Los Angeles to New York City?

Student Response

Answers vary. Sample response:

It will take the ant about 7 years to run from Los Angeles to New York City without stopping. The ant can run 18 mm per second. There are 60 seconds per minute and 60 minutes per hour, so this is $18 \cdot 60 \cdot 60 = 64,800$, or 64,800 mm per hour. 64,800 mm per hour is 64.8 meters per hour, or 0.0648 km per hour. A speed of 0.0648 km per hour translates to a pace of $1 \div 0.0648$ hours per km. So to travel 3944 km, it would take $3944 \div 0.0648$ or approximately 60,864 hours. There are 24 hours per day and 365 days per year, so dividing by 24 and then 365 tells us that it will take the ant about 7 years to make this trip without stopping.

Activity Synthesis

Invite students or groups to share different solution approaches. Consider asking students to create a visual display and conducting a gallery walk. Highlight strategies that include keeping careful track of the information used including:

- How far the ant has to travel
- How fast the ant travels
- A step-by-step analysis changing mm per second to km per day and eventually km per year

Make sure to discuss why student estimates are not all identical. For these open-ended problems that require many assumptions, there is not a single valid approach or answer. Some reasons for a variety of answers in this case include:

- Students may take into account that the ant will need to rest or sleep.
- Students may choose different speeds for the ant.
- Students may round their answers differently.

If students report their answers in a variety of ways (to the nearest year, nearest month, nearest day, nearest hour, etc.), ask them what is most appropriate. Reporting to the nearest year is perhaps most appropriate, however, because there are so many hidden variables that influence the answer, including:

- The speed is only approximate and would not remain constant for years.
- The distance is also only approximate as the ant will need to circumnavigate many obstacles.
- An answer to the nearest year communicates the general idea of how long it will take very effectively.

1.2 Stacks and Stacks of Cereal Boxes

Optional: 20 minutes

In this Fermi problem, students estimate the total volume occupied by all of the breakfast cereal purchased in a year in the United States.

Monitor for different approaches to solving the problem, and select students to share during the discussion. In particular, look for:

- A range of estimates
- Different levels of precision in reporting the final estimate
- Different arrangements of boxes

Addressing

- 6.G.A
- 6.NS.B
- 6.RP.A

Instructional Routines

- Group Presentations

Launch

Arrange students in groups of 1–4. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

Before starting, ask students to come up with a guess about the answer and poll the class. Record all guesses.

Ask students to brainstorm the information they need to answer the question. Give the information provided when students ask for it, or provide access to internet-enabled devices so that students can find the information they need.

Vital information to have on hand includes:

- Every year, people in the U.S. buy 2.7 billion boxes of breakfast cereal.
- A “typical” cereal box has dimensions of 2.5 inches by 7.75 inches by 11.75 inches.

Access for Students with Disabilities

Representation: Internalize Comprehension. Represent the same information through different modalities by using physical objects to represent abstract concepts. Some students may benefit by starting with an image, video, or a typical size cereal box to support visualization.

Supports accessibility for: Conceptual processing; Visual-spatial processing

Anticipated Misconceptions

- Students may not think about *how* the boxes are packed. That is, they may make a volume calculation and then perform division. (Ask these students to think about how the boxes are placed in the warehouse.)
- Students may struggle with unit conversions. (Encourage them to go step by step: How many inches are in a foot? How many square inches are in a square foot? How many cubic inches are in a cubic foot?)

Student Task Statement

Imagine a warehouse that has a rectangular floor and that contains all of the boxes of breakfast cereal bought in the United States in one year.

If the warehouse is 10 feet tall, what could the side lengths of the floor be?

Student Response

Answers vary. Sample response:

The warehouse would be a little more than one mile wide on each side. Boxes of cereal come in many different sizes, but for our purpose, let's assume that the dimensions are 2.5 inches by 7.75 inches by 11.75 inches. So the volume of a single box is about 228 cubic inches. There are 12^3 or 1728 cubic inches per cubic foot, so a box of cereal is about 0.13 cubic feet. Since there are 2.7 billion boxes, the total volume is about 360,000,000 cubic feet. If the warehouse is 10 feet tall, then the area of the floor needs to be at least $360,000,000 \div 10$ or 36,000,000 square feet. So if the warehouse floor were a square with side length 6,000 feet, it could hold all of the boxes. Note that there are 5,280 feet in a mile, so the warehouse would be a little more than one mile wide on each side.

Activity Synthesis

Invite students or groups to share different solution approaches. Consider asking students to create a visual display and conducting a gallery walk. Highlight strategies that include keeping careful track of the information used including:

- What is the volume of each box?
- How are the boxes arranged for storage?
- How are the dimensions of the warehouse floor determined?

Make sure to discuss why different people have different estimates.

- Some students may just calculate volume and not think of how the boxes are packed. (This is OK as the amount of empty space will not be large unless the boxes are placed haphazardly.)
- Students may place the boxes in a different way.
- Students may or may not take account of the small amount of space in the warehouse that is not filled by the boxes.

There are an infinite number of rectangles with suitable area to serve as the floor of the warehouse. Note, however, that having an appropriate area (e.g., 36,000,000 square feet) is *not* sufficient. For example, we could not have a warehouse that is 1 inch wide and 432,000,000 feet long. In general, if one of the dimensions is too small, it increases the amount of wasted space.

1.3 Covering the Washington Monument

Optional: 20 minutes

In this Fermi problem, students estimate the total number of tiles needed to cover the Washington Monument.

Monitor for different approaches to solving the problem, and select students to share during the discussion. In particular, look for:

- A range of estimates
- Different levels of precision in reporting the final estimate
- Different choices made to simplify the situation

Addressing

- 6.G.A
- 6.NS.B
- 6.RP.A

Instructional Routines

- Group Presentations
- MLR8: Discussion Supports

Launch

Arrange students in groups of 1–4. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

Show a picture of the Washington Monument.



Before starting, ask students to come up with a guess how many tiles it would take to cover the Washington Monument and poll the class. Record all guesses.

Ask students to brainstorm the information they need to answer the question. Give the information provided when students ask for it, or provide access to internet-enabled devices so that students can find the information they need.

Vital information to have on hand includes:

- Dimensions of the monument:



- Standard sizes for square tiles: side lengths of 1 inch, 6 inches, 8 inches, 1 foot, and $1\frac{1}{2}$ feet.

Note that 55 feet is the height of the pyramid at the top, not the height of each triangular face. Students would need the Pythagorean Theorem to find the height of these faces, so either tell them it is close to 57.5 feet or they can use the 55 feet for their estimate. This does not have a significant impact on the area calculation, as the pyramid at the top does not account for most of the surface area.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer to capture information about the different assumptions they made when calculating the areas of the trapezoidal faces of the Washington Monument building.

Supports accessibility for: Language; Organization

Anticipated Misconceptions

Students may not account for the faces of the monument that you cannot see in the picture. Remind these students that some faces are hidden. If three-dimensional building blocks are available, consider having students who have difficulty visualizing the monument build a small model. It is difficult to model the exact shape of the Washington Monument, but the physical model may help students make the needed calculations.

Students may have difficulty converting units, especially if their tiles are measured in inches. Ask these students how many inches are in a foot and how many square inches are in a square foot.

Student Task Statement

How many tiles would it take to cover the Washington Monument?



Student Response

Answers vary. Sample response:

First, decide how big the tiles are. Let's assume they are square tiles with side length 1 inch. If two of the trapezoidal sides of the Washington Monument are put together (one right side up, the other upside down), the result is a parallelogram with base 90 feet and height 500 feet. So the area of those two sides would be 45,000 square feet, as would the area of the other two sides. Using 57.5 feet for the height of the triangular faces at the top, their total area is $2 \cdot 35 \cdot 57.5$ or 4,025 square feet. So the total surface area of the monument is about 94,025 square feet. Since there are 12 inches per foot, there are 144 square inches per square foot. So the surface area of the monument is about $94,025 \cdot 144$ or 13,539,600 square inches. So it would require about 13,540,000 square tiles of side length 1 inch to cover the Washington Monument.

Activity Synthesis

Invite students or groups to share different solution approaches. Consider asking students to create a visual display and conducting a gallery walk. Highlight different methods of calculating the area of the trapezoidal faces of the Washington Monument including:

- Putting two trapezoid faces (or all 4 faces) together to make a parallelogram
- Decomposing each trapezoid face into a rectangle and two triangles and then rearranging

Discuss why different people have different estimates. Some reasons to highlight include:

- Using different-sized tiles to cover the Washington Monument
- Thinking about how some tile is wasted, namely the tiles that go on the edge of each face
- Rounding

This problem has a different flavor than the two preceding ones. The only place estimation comes into play is with how to cover the top triangles and perhaps thinking about what happens to tiles at the edge of a face. What all the problems share, however, is that we cannot give an *exact* answer. To provide a reasoned estimate, keeping track of all assumptions being made is vital. This tells us not only how accurately we should report our answer but also helps us communicate clearly to others who may have made different assumptions.

Access for English Language Learners

Speaking: MLR8 Discussion Supports. Give students additional time to make sure that everyone in their group can explain their strategy for estimating the number of tiles using appropriate mathematical language (e.g., face, height, width, surface area, trapezoid, etc.). Be sure to vary who is called on to represent the ideas of each group. This routine will prepare students for the role of group representative and to support each other to take on that role.

Design Principle(s): Optimize output (for explanation)

Lesson 2: If Our Class Were the World

Goals

- Apply reasoning about percentages and equivalent ratios to analyze and approximate characteristics of the world's population.
- Generate (orally and in writing) mathematical questions about the world's population, e.g., "How many people . . . ?"
- Present (using words and other representations) a comparison that uses the number of students in the class to represent the proportion of the world's population with a particular characteristic.

Lesson Narrative

This lesson is optional. In this lesson, students look at ratios of different populations in the world and determine what their class would be like if its ratios were equivalent (MP1, MP2). In the process, they again work with percentages that are not whole numbers, using knowledge gained in a previous unit. Moreover, the ratios will be "close" to being equivalent because the exact world population is not known and all populations need to be whole numbers (MP6). The activities in this lesson could take anywhere from one to four days, depending on how much time is available and how far the class takes it. Earlier activities are needed for later ones in this lesson. A variant on this activity involves developing, administering, and analyzing a survey: If the *school* were our class. Students brainstorm some questions they would like to know about the students in their school. Questions might include:

- How many people in the school play an instrument?
- How many people in the school eat school lunches?
- How many people in the school ride the bus to school?
- How many people in the school have a cell phone?
- How many people in the school plan to attend a four-year college or university?
- How many people in the school were born outside of this state?
- How many people in the school have traveled outside of this country?

As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

Alignments

Addressing

- 6.NS.B: Compute fluently with multi-digit numbers and find common factors and multiples.
- 6.RP.A: Understand ratio concepts and use ratio reasoning to solve problems.

Instructional Routines

- MLR8: Discussion Supports

Required Materials

Four-function calculators

Internet-enabled device

Tools for creating a visual display

Any way for students to create work that can be easily displayed to the class. Examples: chart

paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Internet-enabled devices are only necessary if students will conduct research to find quantities that they need to know. As an alternative, you can supply the information when they ask for it.

Tools for creating a visual display are only needed if you would like students to present their work in an organized way and have the option of conducting a gallery walk.

Student Learning Goals

Let's use math to better understand our world.

2.1 All 7.4 Billion of Us

Optional: 10 minutes

In this activity, students review work with percentages and arithmetic from earlier units. It also gives them a framework for thinking about the next activity. Students have to decide how to treat a situation in which ratios are approximately equivalent (MP6). People can only be reported in whole number quantities, and students need to decide how to round numbers appropriately as a result.

Addressing

- 6.NS.B
- 6.RP.A

Instructional Routines

- MLR8: Discussion Supports

Launch

Give students 5 minutes of quiet think time followed by a whole-class discussion. Provide access to four-function calculators.

Anticipated Misconceptions

For the third question, students may not recognize that the ratio of people from Africa to the world population is only approximately equal to the ratio of students in the class from Africa to the total number of students. Ask these students *about* how many people they think live in Africa.

Student Task Statement

There are 7.4 billion people in the world. If the whole world were represented by a 30-person class:

- 14 people would eat rice as their main food.
- 12 people would be under the age of 20.
- 5 people would be from Africa.

1. How many people in the class would *not* eat rice as their main food?
2. What percentage of the people in the class would be under the age of 20?
3. Based on the number of people in the class representing people from Africa, how many people live in Africa?

Student Response

1. 16. There are 30 students and 14 eat rice as their main food, so the other 16 do not eat rice as their main food.
2. 40% since $12 \div 30 = 0.4$
3. About 1.2 billion. About 16.7% of the world's population live in Africa since $5 \div 30 = 0.167$, and 16.7% of 7.4 billion is about 1.2 billion.

Activity Synthesis

Make sure that students recognize 12 students out of 30 as a benchmark percentage. Point out to them that the fraction $\frac{12}{30}$ is equal to $\frac{2}{5}$ (a benchmark fraction) and also to $\frac{40}{100}$ (showing 40% explicitly).

For the third question, ask students if the ratio for the world population is *equivalent* to the population for people in the class. The answer is no. But 7.4 billion is only an approximate value of the population, so what we are looking for here is a value of people from Africa that gives a ratio close to 5 out of 30 (or equivalent).

Access for English Language Learners

Speaking: MLR8 Discussion Supports. As students describe their strategies for calculating the number of people that live in Africa, revoice student ideas to demonstrate mathematical language use. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example. This will help students to produce and make sense of the language needed to communicate their own ideas.

Design Principle(s): Support sense-making; Optimize output (for explanation)

2.2 About the People in the World

Optional: 10 minutes

Students brainstorm quantitative questions about the world and submit their best ideas to the class. For the next classroom activity, compile all of the questions into one document, removing inappropriate questions. Either do the necessary research, provide resources that students can use to find the needed information, or let students do the research themselves, depending on time and internet access.

Note that some student ideas may be challenging to research (for example, how many people in the world have a car?). If there is no chance to gather data beforehand, encourage students to make some simplifying assumptions with the data that they do find.

Addressing

- 6.NS.B
- 6.RP.A

Launch

Arrange students in groups of 2–4. “We are going to investigate the world population. Think about what you would like to know about all the people of the world!”

Student Task Statement

With the members of your group, write a list of questions about the people in the world. Your questions should begin with “How many people in the world. . .” Then, choose several questions on the list that you find most interesting.

Student Response

Answers vary. Sample responses:

- How many people in the world live on each continent?
- How many people in the world are adults?
- How many people in the world own a car?
- How many people in the world can read?
- How many people in the world speak more than one language?
- How many people in the world have a mobile phone?
- How many people in the world play or have played soccer?

Activity Synthesis

Invite students to share which questions they are *most* interested in investigating. Ask them how they might learn more about a given topic and how easy or hard it might be to find the information.

The most likely resource is the Internet and some populations will be easier to investigate than others. For example, it is not difficult to find information about the number of cars in the world (more than 1 billion but well less than 2 billion). But this does not answer how many people own a car, and that data can be very difficult to find. Encourage students, in preparation for investigating these questions, to make some reasoned assumptions with the data that they can find to make their best estimate. For the car example, they might assume, for example, that 1 billion people in the world own a car.

2.3 If Our Class Were the World

Optional: 30 minutes

If students investigated the answers to their questions in the previous activity, they need access to their findings. They can either work with the questions generated by their group, or everyone can work from a master list of questions compiled by the teacher.

If students have access to a spreadsheet program, they can use it to find the results (MP5). Students need to communicate their findings clearly in graphical displays (MP6). They can choose to illustrate all or some of the questions in their graphical displays. Encourage the students to be creative with their displays. For example, if students are examining the question “How many people would live on each continent?” they might sketch the continents and place stick figures on the continents to represent the class. They might also draw a bar graph showing the same information in a different way.

Addressing

- 6.NS.B
- 6.RP.A

Launch

Students work in same groups of 2–4 from previous activity. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

To give students a sense of the magnitude of the world versus the classroom, consider zooming in from Earth to North America, the United States, your state, your city, and finally your school.

Invite students to work on the question or questions that are of most interest to them either from their own group work or from the compiled list.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills, check in with students within the first 2–3 minutes of work time. Check to make sure students are beginning to choose characteristics about the world’s population that are available to find, and that they can calculate percentages for the class for those characteristics.
Supports accessibility for: Memory; Organization

Anticipated Misconceptions

Students may struggle with the difficult numbers for the percent of the world population with different characteristics. For example, there are about 1.2 billion people living in Africa. Ask them to calculate $1.2 \div 7.4 \approx 0.162$. "What percent of the world population this is?" (16%). "If our class were the world, then about 16% of the class would live in Africa. So how many students in our class would live in Africa?" (Since 16% of 25 is 4, that means that 4 students in our class would live in Africa.)

In some cases, there might be two acceptable answers to how many students in our class have a certain characteristic. About 10% of the world population is in Europe. For our class, we could have 2 students (8%) or 3 students (12%) that are in Europe. Either one is an acceptable answer. Make sure that students understand that 2.5 (10% of 25) is not a reasonable answer.

Student Task Statement

Suppose your class represents all the people in the world.

Choose several characteristics about the world’s population that you have investigated. Find the number of students in *your* class that would have the same characteristics.

Create a visual display that includes a diagram that represents this information. Give your display the title "If Our Class Were the World."

Student Response

Answers vary. Sample response (based on a class of 25 students):

- 800 million people do not have access to clean water. That is about 11% of the world population. If our class were the world, that would mean that about 3 of us (12%) would not have access to clean water.
- 300 million people play soccer. That’s about 4% of the world population. If our class were the world, that would mean that 1 of us (4%) would play soccer.
- 2.2 billion people practice Christianity. That’s about 30% of the world population. If our class were the world, 7 (28%) or 8 (32%) of us would practice Christianity.

- 4.4 billion people live in Asia. That's about 59% of the world population. If our class were the world, 15 (60%) of us would live in Asia.
- 6.6 billion people can read and write. That's about 89% of the world population. If our class were the world, 22 (88%) of us would be able to read and write.
- 1.9 billion people are children (under the age of 15). That's about 26% of the world population. If our class were the world, 6 (24%) or 7 (28%) of us would be children.

Activity Synthesis

Invite groups to share their displays. Consider doing a gallery walk and asking:

- Were you surprised by any of your findings?
- In what ways is the class actually representative of the world? (possibly gender distribution, religious adherence, etc.)
- In what ways is it not? (age, place of residence, access to clean water, etc.)

Make sure to highlight the fact that having a small class limits the percentages that are possible. This can make deciding how many students in the class have a particular characteristic difficult. Consider asking:

- What percentages are possible for students in the class? (0, 4, 8, . . . multiples of 4 up to 100)
- There are 1 billion people living in the Americas. That's a little more than 13.5% of the world population. If our class was the world, how many students in our class would be from the Americas? (3 students would be 12%, and 4 would be 16%. So 3 students is as close as we can get.)

Lesson 3: Rectangle Madness

Goals

- Coordinate diagrams and expressions involving equivalent fractions.
- Interpret and create diagrams involving a rectangle decomposed into squares.
- Recognize that decomposing rectangles into squares is a geometric way to determine the greatest common factor of two numbers.

Lesson Narrative

This lesson is optional. In this exploration in pure mathematics, students tackle a series of activities that explore the relationship between the greatest common factor of two numbers and related fractions using a geometric representation. The activities in this lesson build on each other, providing students an opportunity to express the relationship between the greatest common factor of two numbers and related fractions through repeated reasoning (MP8). Thus, the activities should be done in order. Doing all of the activities would take more than a single class period—possibly as many as four. It is up to the teacher how much time to spend on this topic. It is not necessary to do the entire set of problems to get some benefit from the activities in this lesson, although more connections are made the farther one gets. As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

Alignments

Building On

- 4.MD.A.3: Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.
- 5.NF.B.3: Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret $3/4$ as the result of dividing 3 by 4, noting that $3/4$ multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size $3/4$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?
- 6.NS.B.4: Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ as $4(9 + 2)$.

Addressing

- 6.NS.A: Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Instructional Routines

- MLR7: Compare and Connect
- MLR8: Discussion Supports

Required Materials

Graph paper

Student Learning Goals

Let's cut up rectangles.

3.1 Squares in Rectangles

Optional: 15 minutes

This first activity helps students understand the geometric process that they use in later activities to connect the greatest common factor with related fractions. The first question helps students focus on the impact on side lengths of decomposing a rectangle into smaller rectangles. The second question has students analyze a rectangle that has been decomposed into squares (MP7). The third question has students decompose a rectangle into squares themselves. In the next activity, students relate this process to greatest common factors and fractions.

Building On

- 4.MD.A.3

Instructional Routines

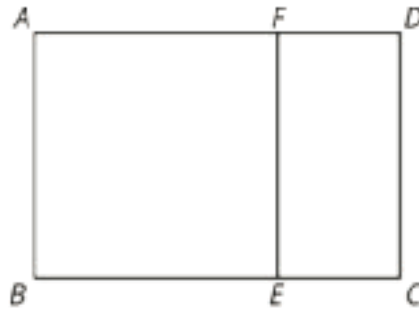
- MLR7: Compare and Connect

Launch

Give 5 minutes of quiet think time followed by whole-class discussion. Consider doing a notice and wonder with the first diagram to help students make sense of the way the vertices are named.

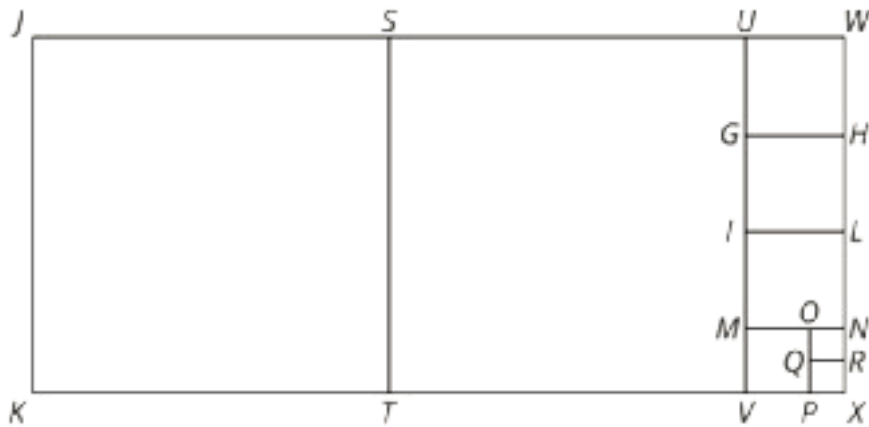
Student Task Statement

1. Rectangle $ABCD$ is not a square. Rectangle $ABEF$ is a square.



- Suppose segment AF were 5 units long and segment FD were 2 units long. How long would segment AD be?
- Suppose segment BC were 10 units long and segment BE were 6 units long. How long would segment EC be?
- Suppose segment AF were 12 units long and segment FD were 5 units long. How long would segment FE be?
- Suppose segment AD were 9 units long and segment AB were 5 units long. How long would segment FD be?

2. Rectangle $JKXW$ has been decomposed into squares.



Segment JK is 33 units long and segment JW is 75 units long. Find the areas of all of the squares in the diagram.

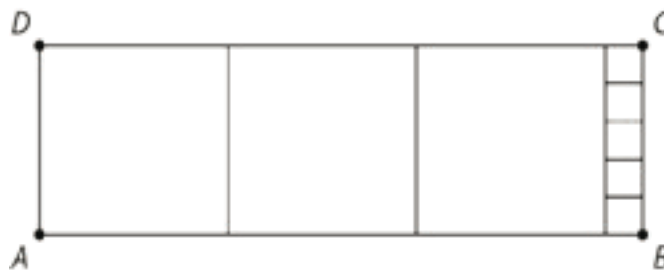
3. Rectangle $ABCD$ is 16 units by 5 units.



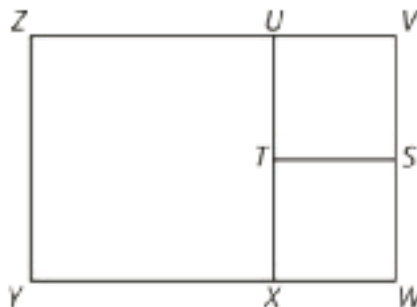
- In the diagram, draw a line segment that decomposes $ABCD$ into two regions: a square that is the largest possible and a new rectangle.
- Draw another line segment that decomposes the *new* rectangle into two regions: a square that is the largest possible and another new rectangle.
- Keep going until rectangle $ABCD$ is entirely decomposed into squares.
- List the side lengths of all the squares in your diagram.

Student Response

- In rectangle $ABCD$:
 - 7
 - 4
 - 12
 - 4
- 1,089, 81, 36, and 9. The area of both $JSTK$ and $SUVT$ is 33^2 , or 1,089 square units. The sides of $UWHG$, $GHLI$, and $ILNM$ are 9 units, so each of their areas is 81 square units. The side of $MOPV$ is 6 units, so its area is 36 square units. The sides of $ONRQ$ and $QRXP$ are 3 units, so their areas are 9 square units.
- Here is the rectangle decomposed into squares. (There is more than one way to do it, but any approach will result in the same number of squares of the same size.) There are three 5-by-5 squares and five 1-by-1 squares.



Are You Ready for More?



1. The diagram shows that rectangle $VWYZ$ has been decomposed into three squares. What could the side lengths of this rectangle be?
2. How many different side lengths can you find for rectangle $VWYZ$?
3. What are some rules for possible side lengths of rectangle $VWYZ$?

Student Response

Answers vary. Sample responses:

1. Height is 2 units, width is 3 units.
2. Any number of distinct Rectangles $VWYZ$ could be created.
3. The length of line segment XW can be any number. Then the width of $VWYZ$ is 3 times the length of line segment XW , and the height of $VWYZ$ is 2 times the length of line segment XW .

Activity Synthesis

Ask students what difficulties they had and how they resolved them. Two insights are helpful when working with rectangles that have been partitioned into squares:

- Squares have four sides of the same length. If you know the length of one side of a square, then you know the lengths of the other three sides of the square.
- If a line segment of length x units is decomposed into two segments of length y units and z units, then $x = y + z$. This relationship can be expressed in other ways, like $x - z = y$.

Access for English Language Learners

Speaking: MLR7 Compare and Connect. Use this routine when students share their process for decomposing the rectangle for the final question. Ask students, to compare the different ways the rectangle was decomposed. For example, ask: "How are the decompositions of the rectangle the same? How are they different?" In this discussion, emphasize that the number of squares created by the different decompositions is the same, regardless of the process. These exchanges can strengthen students' mathematical language use and their ability to recognize patterns across geometric processes.

Design Principle(s): Maximize meta-awareness

3.2 More Rectangles, More Squares

Optional: 30 minutes

In this activity, students apply the geometric process that they saw in the last activity and are asked to make connections between the result and the greatest common factor of the side lengths of the original rectangle. They work on a sequence of similar problems, allowing them to see and begin to

articulate a pattern (MP8). Then they make connections between this pattern and fractions that are equivalent to the fraction made up of the side lengths of the original rectangle.

Building On

- 5.NF.B.3
- 6.NS.B.4

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2. Students work on problems alone and check work with a partner.

Review fractions with a mini warm-up. Sample reasoning is shown. Drawing a diagram might help.

1. Write a fraction that is equal to the mixed number $3\frac{4}{5}$. (We can think of $3\frac{4}{5}$ as $\frac{15}{5} + \frac{4}{5}$, so it is equal to $\frac{19}{5}$.)
2. Write a mixed number that is equal to this fraction: $\frac{11}{4}$. (This is equal to $\frac{8}{4} + \frac{3}{4}$, so it is equal to $2\frac{3}{4}$.)

It is quicker to sketch the rectangles on blank paper, but some students may benefit from using graph paper to support entry into these problems.

If students did not do the previous activity the same day as this activity, remind them of the earlier work:

- Draw a rectangle that is 16 units by 5 units.
 - a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been decomposed into squares.
 - b. How many squares of each size are there? (Three 5-by-5 squares and five 1-by-1 squares.)
 - c. What is the side length of the smallest square? (1)

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about the connection between fractions and decomposition of rectangles. Some students may benefit from a physical demonstration of how to draw line segments to decompose rectangles into squares. Invite students to engage in the process by offering suggested directions as you demonstrate.

Supports accessibility for: Visual-spatial processing; Organization

Anticipated Misconceptions

Students may not see any connections between the decomposition of the rectangles and the fraction problems. Encourage them to look at the number of squares in the decomposition.

Student Task Statement

1. Draw a rectangle that is 21 units by 6 units.
 - a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been entirely decomposed into squares.
 - b. How many squares of each size are in your diagram?
 - c. What is the side length of the smallest square?
2. Draw a rectangle that is 28 units by 12 units.
 - a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been decomposed into squares.
 - b. How many squares of each size are in your diagram?
 - c. What is the side length of the smallest square?
3. Write each of these fractions as a mixed number with the smallest possible numerator and denominator:
 - a. $\frac{16}{5}$
 - b. $\frac{21}{6}$
 - c. $\frac{28}{12}$

4. What do the fraction problems have to do with the previous rectangle decomposition problems?

Student Response

- 21-by-6 rectangle
 - Diagrams vary.
 - Three 6-by-6 squares and two 3-by-3 squares
 - 3
- 28-by-12 rectangle
 - Diagrams vary.
 - Two 12-by-12 squares and three 4-by-4 squares
 - 4
- Fractions as mixed numbers:
 - $\frac{16}{5} = 3\frac{1}{5}$
 - $\frac{21}{6} = 3\frac{3}{6} = 3\frac{1}{2}$
 - $\frac{28}{12} = 2\frac{4}{12} = 2\frac{1}{3}$
- Answers vary. Sample responses: The given fractions had the same numbers as the side lengths of the given rectangles. The mixed numbers included the numbers of squares. For example, the 28-by-12 rectangle was partitioned into 2 large squares and 3 small squares, and the associated mixed number was $2\frac{1}{3}$. The size of the smallest square is related to the fraction used to make the mixed number. For example, in the 28-by-12 rectangle, the smallest square had sides of length 4. To make the mixed number, $\frac{4}{12}$ is rewritten as $\frac{1}{3}$.

Activity Synthesis

Invite students to share some of their observations. Ask, "What connections do you see between the rectangle drawings and the fractions?"

At this point, it is sufficient for students to notice *some* connections between a partitioned rectangle and its associated fraction. They will have more opportunities to explore, so there's no need to make sure they notice *all* of the connections right now. Here are examples of things that it is possible to notice using the rectangles in this activity but might not be noticed until students see more examples:

- The given fractions had the same numbers as the side lengths of the given rectangles.
- The mixed numbers included the numbers of squares. For example, the 28-by-12 rectangle was partitioned into 2 large squares and 3 small squares, and the associated mixed number was $2\frac{1}{3}$.

- The size of the smallest square is related to the fraction used to make the mixed number. For example, in the 28-by-12 rectangle, the smallest square had sides of length 4. To make the mixed number, $\frac{4}{12}$ is rewritten as $\frac{1}{3}$.

Access for English Language Learners

Speaking: MLR8 Discussion Supports. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or make explicit connections between the two representations (numerical and geometric). Encourage students to use the terms partition and decomposition in their explanations.

Design Principle(s): Support sense-making; Optimize output (for explanation)

3.3 Finding Equivalent Fractions

Optional: 30 minutes

In this activity, students make the connection between the fraction determined by the original rectangle and the resulting fraction more precise (MP6). The two rectangles taken together are designed to help students notice that decomposing rectangles is a geometric way to determine the greatest common factor of two numbers. (This is a geometric version of Euclid's algorithm for finding the greatest common factor.)

Building On

- 5.NF.B.3
- 6.NS.B.4

Launch

Arrange students in groups of 2. Provide access to graph paper. Students work on problems alone and compare work with a partner.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to the two large rectangles prepared on graph paper. Encourage students to annotate diagrams with details to show how each value is represented. For example, number of squares in total, side length of the smallest square.

Supports accessibility for: Visual-spatial processing; Organization

Student Task Statement

1. Accurately draw a rectangle that is 9 units by 4 units.

- a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
 - b. How many squares of each size are there?
 - c. What are the side lengths of the last square you drew?
 - d. Write $\frac{9}{4}$ as a mixed number.
2. Accurately draw a rectangle that is 27 units by 12 units.
- a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
 - b. How many squares of each size are there?
 - c. What are the side lengths of the last square you drew?
 - d. Write $\frac{27}{12}$ as a mixed number.
 - e. Compare the diagram you drew for this problem and the one for the previous problem. How are they the same? How are they different?
3. What is the greatest common factor of 9 and 4? What is the greatest common factor of 27 and 12? What does this have to do with your diagrams of decomposed rectangles?

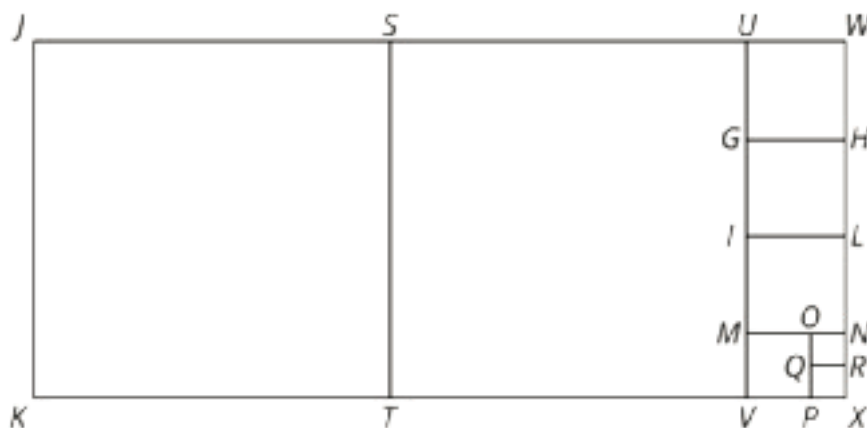
Student Response

1. 9-by-4 rectangle.
 - a. Diagrams vary.
 - b. Two 4-by-4 squares and four 1-by-1 squares
 - c. 1 by 1
 - d. $\frac{9}{4} = 2\frac{1}{4}$
2. 27-by-12 rectangle:
 - a. Diagrams vary.
 - b. Two 12-by-12 squares and four 3-by-3 squares
 - c. 3 by 3
 - d. $\frac{27}{12} = 2\frac{3}{12} = 2\frac{1}{4}$
 - e. It is similar because there are 2 large squares and 4 small squares. It is different because the squares are not the same size. The larger rectangle partitioned into larger squares is a scaled up version of the smaller rectangle partitioned into smaller squares.

3. The greatest common factor of 9 and 4 is 1. The greatest common factor of 27 and 12 is 3. The greatest common factor is the same as the side length of the smallest square.

Are You Ready for More?

We have seen some examples of rectangle tilings. A *tiling* means a way to completely cover a shape with other shapes, without any gaps or overlaps. For example, here is a tiling of rectangle $KXWJ$ with 2 large squares, 3 medium squares, 1 small square, and 2 tiny squares.



Some of the squares used to tile this rectangle have the same size.

Might it be possible to tile a rectangle with squares where the squares are *all different sizes*?

If you think it is possible, find such a rectangle and such a tiling. If you think it is not possible, explain why it is not possible.

Student Response

Such tilings do exist, but they are hard to find! In fact, people thought that it was impossible for a long time. An example is a 32-by-33 rectangle that can be tiled with squares of side length 18, 15, 8, 7, 4, 14, 10, 9, and 1. (For more examples of solutions and more history on the matter, research Martin Gardner's November 1958 column in *Scientific American*.)

When presented with this problem, people usually think that there are no rectangles that can be tiled with squares that are all different sizes. Productive and enjoyable conversations can ensue from trying to explain why.

Activity Synthesis

It is not necessary for students to understand a general argument for why chopping rectangles can help you know the greatest common factor of two numbers. However, for this particular example, students may notice that:

- All of the segments in the larger partitioned rectangle are three times longer than their corresponding segments in the smaller partitioned rectangle.
- 27 and 12 are each 3 times larger than 9 and 4, respectively.

3.4 It's All About Fractions

Optional: 30 minutes

This activity extends the work with rectangles and fractions to continued fractions. Continued fractions are not a part of grade-level work, but they can be reasoned about and rewritten using grade-level skills for operating on fractions (MP8). In particular, the insight that $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ (a special case of invert and multiply) is helpful.

In this activity, students consolidate their understanding about how the greatest common factor of the numerator and denominator of a fraction can help them write an equivalent fraction whose numerator and denominator have greatest common factor 1—sometimes called “lowest terms.”

Building On

- 6.NS.B.4

Addressing

- 6.NS.A

Launch

Arrange students in groups of 2. Students work alone and compare their work with a partner.

Consider a quick warm-up like this:

Write a fraction that is equal to each expression:

1. $3 + \frac{1}{5}$

2. $\frac{1}{3 + \frac{1}{5}}$

3. $2 + \frac{1}{3 + \frac{1}{5}}$

Access for Students with Disabilities

Representation: Develop Language and Symbols. Eliminate barriers and provide concrete manipulatives to connect symbols to concrete objects or values. For example, provide students with access to the four large rectangles prepared on graph paper.

Supports accessibility for: Visual-spatial processing; Fine-motor skills

Student Task Statement

1. Accurately draw a 37-by-16 rectangle. (Use graph paper, if possible.)

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. How many squares of each size are there?

c. What are the dimensions of the last square you drew?

d. What does this have to do with $2 + \frac{1}{3 + \frac{1}{5}}$?

2. Consider a 52-by-15 rectangle.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $3 + \frac{1}{2 + \frac{1}{7}}$.

c. Notice some connections between the rectangle and the fraction.

d. What is the greatest common factor of 52 and 15?

3. Consider a 98-by-21 rectangle.

a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.

b. Write a fraction equal to this expression: $4 + \frac{1}{1 + \frac{7}{14}}$.

c. Notice some connections between the rectangle and the fraction.

d. What is the greatest common factor of 98 and 21?

4. Consider a 121-by-38 rectangle.

a. Use the decomposition-into-squares process to write a continued fraction for $\frac{121}{38}$. Verify that it works.

b. What is the greatest common factor of 121 and 38?

Student Response

1. 37-by-16 rectangle

a. Diagrams vary.

b. Two 16-by-16 squares, three 5-by-5 squares, five 1-by-1 squares

c. 1

d. 2, 3, and 5 are numbers that appear in the continued fraction.

2. 52-by-15 rectangle

a. Diagrams vary.

b. $\frac{52}{15}$

c. When the continued fraction is rewritten, the numerator and denominator equal the side lengths of the rectangle. When the rectangle was partitioned into squares, there were 3, 2, and 7 squares of different sizes. These match the numbers in the continued fraction that was given.

d. 1

3. 98-by-21 rectangle

a. Diagrams vary.

b. $\frac{98}{21}$ or $\frac{14}{3}$

c. The side lengths of the rectangle were 98 and 21, and the fraction is $\frac{98}{21}$ (or equivalent). The fraction written with the smallest possible numbers is $\frac{14}{3}$, and both 14 and 3 are multiplied by 7, the result is $\frac{98}{21}$.

d. 7

4. 121-by-38 rectangle

a. $3 + \frac{1}{5 + \frac{1}{2 + \frac{1}{3}}}$

b. 1

Section: Voting

Lesson 4: How Do We Choose?

Goals

- Apply reasoning about ratios and percentages to analyze (orally and in writing) voting situations involving two choices.
- Comprehend the terms “majority” and “supermajority” (in spoken and written language).
- Critique (using words and other representations) a statement reporting the results of a vote.

Lesson Narrative

This lesson is optional. This is the first of three lessons that explore the mathematics of voting: democratic processes for making decisions. The activities in these lesson build on each other. Doing all of the activities in the three lessons would take more than three class periods—possibly as many as five. It is not necessary to do the entire set of activities to get some benefit from them, although more connections are made the farther one gets. As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an optional opportunity to go more deeply and make connections between concepts.

The activities in this lesson are about voting on issues where there are two choices. Students use equivalent ratio concepts and skills developed in grade 6 to compare voting results of two groups, to determine whether an issue wins an election with a supermajority rule, and discover that a few people can determine the results of an election when very few people vote.

Most of the activities use students’ skills from earlier units to reason about ratios (MP2) in the context of real-world problems (MP4). While some of the activities do not involve much computation, they all require serious thinking. In many activities, students have to make choices about how to assign votes and justify their methods (MP3).

Most importantly, this lesson addresses topics that are important for citizens in a democracy to understand. Teachers may wish to collaborate with a civics or government teacher to learn how the fictional middle-school situations in this lesson relate to real-world elections.

Alignments

Addressing

- 6.RP.A.1: Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, “The ratio of wings to beaks in the bird house at the zoo was 2 : 1, because for every 2 wings there was 1 beak.” “For every vote candidate A received, candidate C received nearly three votes.”

- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
- 6.RP.A.3.c: Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

Instructional Routines

- MLR2: Collect and Display
- MLR6: Three Reads
- MLR7: Compare and Connect

Required Materials

Colored pencils

Four-function calculators

Graph paper

Scissors

Student Learning Goals

Let's vote and choose a winner!

4.1 Which Was “Yessier”?

Optional: 10 minutes

This activity gives students a chance to recall and use various ratio strategies in the context of a voting problem. Two classes voted on a yes or no question. Both classes voted yes. Students are asked to determine which class was more in favor (“yessier”). Students need to make sense of the invented word “yessier” by thinking about how it might be quantified (MP1, MP2).

Monitor for students who use different ways to make sense of the problem.

Addressing

- 6.RP.A.3

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 2–4. Ask students to use the mathematical tools they know to answer the question.

Anticipated Misconceptions

Students may not understand the question. The word “yessy” was invented by other students solving a similar problem. It is not standard English.

If students do not understand what comparing the two classes means, give a more extreme example, such as comparing 20 to 1 with 11 to 10. The 20 to 1 class is much more yessy because almost everyone said yes. The other class had almost equal yesses and nos.

Student Task Statement

Two sixth-grade classes, A and B, voted on whether to give the answers to their math problems in poetry. The “yes” choice was more popular in both classes.

	yes	no
class A	24	16
class B	18	9

Was one class more in favor of math poetry, or were they equally in favor? Find three or more ways to answer the question.

Student Response

Answers vary. Sample response:

One strategy is to find simpler equivalent ratios, then compare them by finding a common amount.

Class B’s ratio was 2 : 1, since 18 is two groups of 9. This means there were twice as many yesses as nos. If Class A was the same “yessiness” as Class B, it would have 24 yesses and 12 nos. There are more than 12 nos, so that means that Class A is more “no-y,” which is less “yessy.” Alternatively, equally “yessy” could be 32 yesses and 16 nos. There are fewer than 32 yesses, so Class A is less “yessy” than Class B.

The same approach can work using the ratio 3 : 2 for Class A. A 3 : 2 ratio with 18 yesses would have 12 nos, more than the 9 nos in Class B.

Here are some equivalent ratios shown in a table for each class. Many others are possible.

	yes	no
class A	24	16
	12	8
	3	2
	6	4
	18	12

	yes	no
class B	18	9
	6	3
	2	1
	4	2
	24	12

Another strategy involves finding the totals, then comparing percentages. The ratios could also be shown on double number lines, one pair for each class.

	yes	no	total	fraction yesses	percent yesses
class A	24	16	40	$\frac{24}{40} = \frac{3}{5}$	60%
class B	18	9	27	$\frac{18}{27} = \frac{2}{3}$	66.7%

Activity Synthesis

The situation is mathematically the same as other rate comparison problems, such as comparing the tastes or colors of two mixtures.

Invite several students to present different methods, at least one who used a ratio of yesses to nos, and another who used a ratio of yesses to all students. Make sure to present a solution using percentages.

If more than a few students did not use multiplicative techniques (for example, if they compared only the yesses, or subtracted to find how many more yesses than nos) remind them of the rate comparisons they did earlier in the year, in which they could check by tasting a drink mixture or looking at the color of a paint mixture.

Access for English Language Learners

Speaking, Listening, Representing: MLR2 Collect and Display. Use this routine to display examples of the different methods students used to determine which class was “yessier.” Circulate and collect examples of formal and informal language, calculations, tables, or other writing. Ask students to compare and contrast percentage calculations with fractions and equivalent ratios.
Design Principle(s): Cultivate conversation; Maximize meta-awareness

4.2 Which Class Voted Purpler?

Optional: 10 minutes

This activity is the same type of situation as the previous one: comparing the voting of two groups on a yes or no issue. However, the numbers make it more difficult to use "part to part" ratios. Again, students need to be thinking about how to make sense of (MP1) and quantify the class voting decisions (MP2).

Addressing

- 6.RP.A.3

Launch

Arrange students in groups of 2–4. Provide access to four-function calculators.

Anticipated Misconceptions

If students do not understand what comparing the two classes means, give a more extreme example, such as comparing 20 to 1 with 11 to 10. The 20 to 1 class is much more purple because almost everyone said yes. The other class had almost equal yesses and nos.

Students may be stuck with the difficult-looking numbers, expecting to be able to do calculations to create equivalent ratios mentally. Suggest that they find the total number of votes in each class.

Students may compute percentages incorrectly, forgetting that percentages are rates out of 100. So it is *not* correct to say that room A has 54% yesses. However, you can carefully make sense of this percent as a comparison: Class A had 54% as many nos as yesses; Class B had 61% as many nos as yesses. This means that Class B was more no-y, so it was less yessy than Class A.

Student Task Statement

The school will be painted over the summer. Students get to vote on whether to change the color to purple (a "yes" vote), or keep it a beige color (a "no" vote).

The principal of the school decided to analyze voting results by class. The table shows some results.

In both classes, a majority voted for changing the paint color to purple. Which class was more in favor of changing?

	yes	no
class A	26	14
class B	31	19

Student Response

Answers vary, Possible response:

Room A was "yessier," with 65% yesses. Room B had a smaller percentage of yesses, 62%.

	yes	no	total	fraction yesses	percent yesses
room A	26	14	40	$\frac{26}{40} = \frac{13}{20}$	65%
room B	31	19	50	$\frac{31}{50}$	62%

Ratios could be used, but calculations are more difficult. Here are some equivalent ratios that could be used. Lines marked with a * have a common number of nos. The line with a ** has a common number of yesses.

	yes	no
room A	26	14
	13	7
*	$13 \cdot 19 = 247$	$7 \cdot 19 = 133$
*	$26 \cdot 19 = 494$	$14 \cdot 19 = 266$
**	$13 \cdot 31 = 403$	$7 \cdot 31 = 217$

	yes	no
room B	31	19
*	$31 \cdot 7 = 217$	$19 \cdot 7 = 133$
*	$31 \cdot 14 = 434$	$19 \cdot 14 = 266$
**	$31 \cdot 13 = 403$	$19 \cdot 13 = 247$

A rate of nos per yes also makes sense: Room A has about 0.54 nos per yes, and Room B has about 0.61 nos per yes. This makes Room B more no-y, which means less yessy. Unit rate for each ratio: Room A has about 1.9 yesses per no, while Room B has about 1.6 yesses per no. Therefore, Room A is yessier.

Activity Synthesis

Invite students to show different ways of solving the problem, including using equivalent ratios and percentages. Ask students to explain their thinking. Correct ideas with incorrect calculations are still worth sharing.

Access for Students with Disabilities

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. Once students have determined which class was more in favor of changing, pause the class. Invite students to share their strategies for finding each fraction using percents and equivalent ratios to justify their reasoning. Create a display that includes each strategy labeled with the name and the fraction they represent of a total. Keep this display visible as students move on to the next problems.

Supports accessibility for: Conceptual processing; Memory

4.3 Supermajorities

Optional: 10 minutes

This activity introduces the idea of requiring a supermajority. A supermajority is a voting rule that is used for issues where it is important to have more than just barely above half of the voters agreeing. To win, a choice must have more than the given fraction of the votes. In this activity, two supermajority rules are given: one as a fraction, one as a percent. Students find a fraction of the total votes and a percentage of the total votes. They then compare the fraction and the percent.

This activity encourages students to think about votes in ratios and percents. Students make sense of problems and reason quantitatively (MP1 and MP2).

Addressing

- 6.RP.A.3.c

Instructional Routines

- MLR7: Compare and Connect

Launch

Arrange students in groups of 2–4. Provide access to a four-function calculator.

Explain the difference between a majority and supermajority: "In many voting situations, a choice that wins a *majority* of the votes wins. A majority is more than half the votes. So if 1,000 votes were cast, a majority is any number over 500; 501 is the smallest number of votes that can win.

Many groups have special election rules for very important issues. Sometimes they require a supermajority: to win, you need more than a certain fraction that is more than half. For raising taxes, some governments require a $\frac{2}{3}$ supermajority. To change (amend) the U.S. constitution, an amendment must get a $\frac{2}{3}$ supermajority of both the Senate and House of Representatives, and be ratified by $\frac{3}{4}$ of the states. Sometimes supermajorities are described as percents, such as 60%."

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about reasoning quantitatively about operations involving decimals and percentages. Allow students to use calculators to ensure inclusive participation in the activity.

Supports accessibility for: Memory; Conceptual processing

Student Task Statement

1. Another school is also voting on whether to change their school's color to purple. Their rules require a $\frac{2}{3}$ supermajority to change the colors. A total of 240 people voted, and 153 voted to change to purple. Were there enough votes to make the change?
2. This school also is thinking of changing their mascot to an armadillo. To change mascots, a 55% supermajority is needed. How many of the 240 students need to vote "yes" for the mascot to change?
3. At this school, which requires more votes to pass: a change of mascot or a change of color?

Student Response

1. 153 votes are not enough to win.
One method: $\frac{2}{3}$ of 240 is twice as much as $\frac{1}{3}$ of 240. $\frac{1}{3}$ of 240 is 80, so $\frac{2}{3}$ of 240 is 160. Since you need one vote more than $\frac{2}{3}$ of 240, 161 votes are needed to win.
A method with more calculation: $\frac{153}{240}$ of the votes were for changing the color. We need to check that this fraction is greater than $\frac{2}{3}$. Long division or a calculator would do this, by converting to a decimal: $\frac{153}{240} \approx 0.64$, and $\frac{2}{3} \approx 0.67$. So 153 is not enough votes to change.
2. 133 votes are needed to change the mascot. 55% of 240 is $\frac{55}{100} \cdot 240$. This can be computed with decimal multiplication: $(0.55)(240) = 132$ or with fraction multiplication:
 $\frac{55}{100} \cdot 240 = \frac{11}{20} \cdot 240 = 11 \cdot \frac{240}{20} = 11 \cdot 12 = 132$. Since a supermajority is more votes than the fraction of the total, 133 votes are needed to change the mascot.
3. A $\frac{2}{3}$ supermajority needs more votes than a 55% supermajority. The question is asking: which is more, $\frac{2}{3}$ of a number, or 55% of the same number? Since we don't know what the number is in advance, we can compare the fractions $\frac{2}{3}$ and $55\% = \frac{55}{100}$. Writing both as decimals, we see that $\frac{2}{3} \approx 0.67$ and $55\% = 0.55$, so it takes more votes to change the color. This question can also be answered by comparing fractions using common denominators or common numerators. Here is one way: $\frac{2}{3} = \frac{200}{300}$ and $\frac{55}{100} = \frac{165}{300}$.

Reasoning about fractions can give a quicker answer: since $\frac{2}{3}$ of 240 is more than 55% of 240, $\frac{2}{3}$ of any number is more than 55% of the same number, 133.

Activity Synthesis

Choose one or more students' methods to present.

Access for English Language Learners

Conversing, Speaking, Listening: MLR7 Compare and Connect. Before the whole-class discussion, invite students to share their work with a partner or small group. Display the following questions to support conversation: "What was your strategy?", "Does anyone want to add on to ___'s strategy?", "Did anyone solve the problem in a similar way, but would explain it differently?", and "Does anyone have a completely different method for solving?" Listen for and amplify observations that include mathematical language and reasoning.

Design Principle(s): Cultivate conversation; Maximize meta-awareness

4.4 Best Restaurant

Optional: 20 minutes

This activity shows how a few people can make a decision if many people don't vote. The mathematics involves repeated "percent of" operations, each percent giving a smaller amount than the previous step. The main issue in this problem is to identify "percent of what?" for each percentage. The first percentage is 25% of the people in town subscribe to the newspaper. The second percentage is 20% of the result of the previous number, and the third is 80% of the second result.

Students need to give a written explanation, clearly show their calculations by writing expressions and equations, and make a diagram that accurately shows the sizes of all the groups in the problem (MP3). The diagram might be on a 10×10 grid, or a tape diagram. Graph paper is a good way to make sure the sizes are right. Tape diagrams can also be made with a folded strip of paper, if students are accustomed to folding fractions.

Addressing

- 6.RP.A.1
- 6.RP.A.3.c

Instructional Routines

- MLR6: Three Reads

Launch

Arrange students in groups of 2–4. Tell students that sometimes local newspapers or magazines ask their readers to vote for their favorite businesses. In this activity, they think about whether this is a good way to decide which businesses are the best or are most popular. (In other words, is this a scientific survey?) Make graph paper, colored pencils or markers, and scissors available.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to blank square grid and tape diagrams to support information processing. Encourage students to annotate diagrams with details to show how each value is represented. For example, number of all people in town, number of newspaper subscribers in town, number of newspaper subscribers who voted.

Supports accessibility for: Visual-spatial processing; Organization

Access for English Language Learners

Reading, Speaking: MLR6 Three Reads. Use this routine to support reading comprehension, without solving, for students. Use the first read to orient students to the situation. Ask students to describe what the situation is about (A town's newspaper ask subscribers to vote on the best restaurant in town). Use the second read to identify quantities and relationships. Ask students what can be counted or measured without focusing on the values (number of people in town, number of newspaper subscribers, number of subscribers who voted, and number of votes for Darnell's). After the third read, ask students to brainstorm possible strategies to solve the problem.

Design Principle(s): Support sense-making; Cultivate conversation; Maximize meta-awareness

Anticipated Misconceptions

Students may wonder how they can answer the question without knowing how many people are in the town. Encourage them to invent a total number of people (such as 100 or 1,000) or to show the percents as parts of a 10-by-10 square. Remind them that the answer is a percentage, not a number of people. Make sure to discuss the fact that, no matter what the number of people in the town is, the percentage at the end is still the same.

Diagrams may still be too abstract for some. Demonstrate with a large 10-by-10 square: Cut a 5-by-5 square out; this is 25% of the total, representing the subscribers. Put the other 75% aside. Now find 20% of the 5-by-5 square. This is $\frac{1}{5}$ of the square, so is 5 squares. Cut off a strip of 5 to represent the subscribers who voted. Put the rest aside. Now find 80% of 5 squares, which is 4 squares. Cut off and put aside one square. What is left represents 80% of the subscribers who voted. It's only 4% of the people in the town!

Student Task Statement

A town's newspaper held a contest to decide the best restaurant in town. Only people who subscribe to the newspaper can vote. 25% of the people in town subscribe to the newspaper. 20% of the subscribers voted. 80% of the people who voted liked Darnell's BBQ Pit best.

Darnell put a big sign in his restaurant's window that said, "80% say Darnell's is the best!"

Do you think Darnell's sign is making an accurate statement? Support your answer with:

- Some calculations
- An explanation in words
- A diagram that accurately represents the people in town, the newspaper subscribers, the voters, and the people who liked Darnell's best

Student Response

Darnell's sign is very misleading; only 4% of the people in town actually voted for Darnell's.

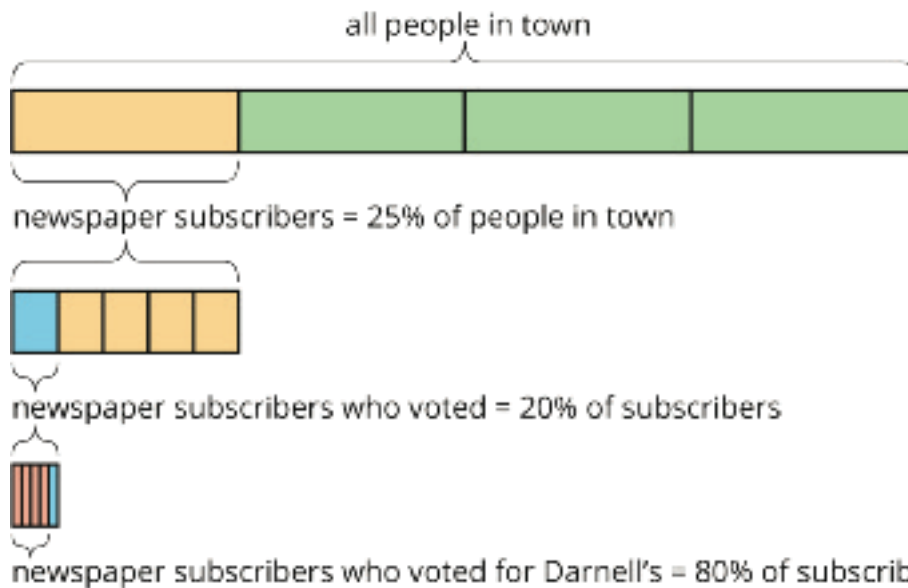
If there are 100 people in the town, then 25 of them subscribe to the paper. 20% of 25 people is 5 subscribers who voted. 80% of 5 is 4 subscribers who voted for Darnell's. So 4% of the people in town thought Darnell's is best. Darnell's sign is misleading. Some correct statements:

- 80% of people who voted for Best Restaurant liked Darnell's best.
- There were 4 times as many people who thought Darnell's was best, compared to all the other restaurants.

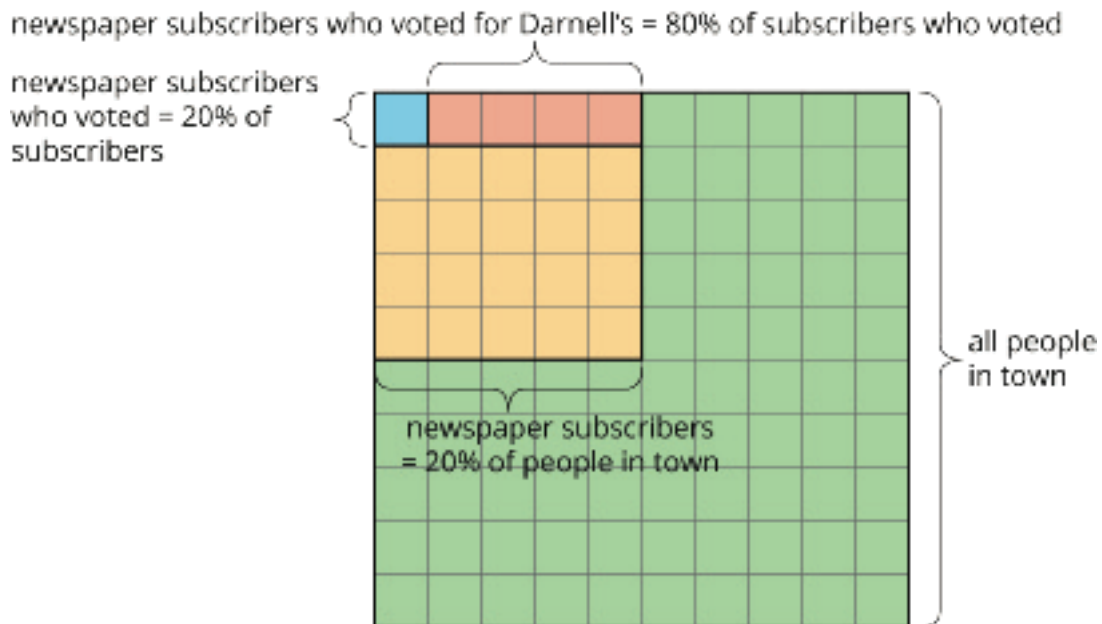
If there are some other number of people in the town, this reasoning still works. One person in the previous calculation now represents a group of $\frac{1}{100}$ of the people in town.

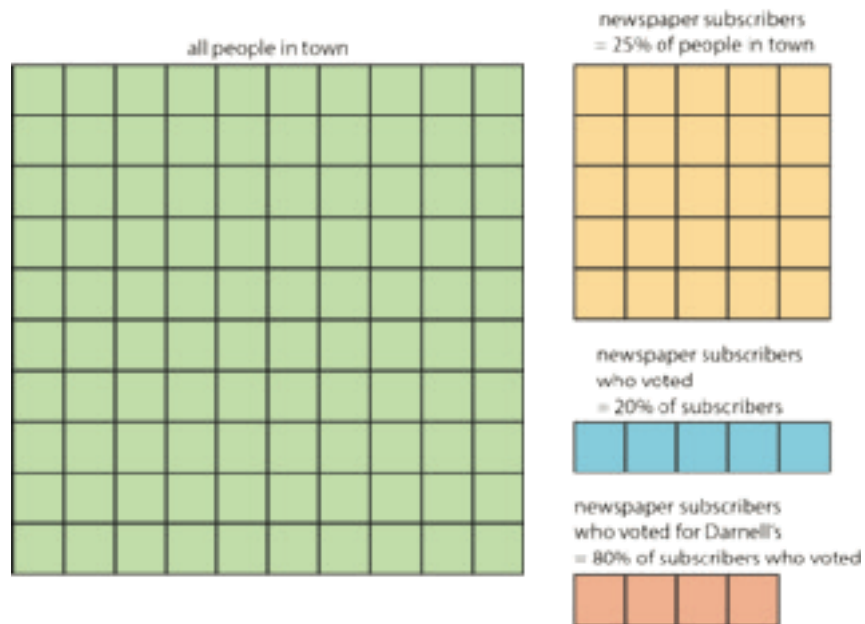
If any students have internalized the fact that to find $n\%$ of something, multiply by $\frac{n}{100}$, then they could reason that 80% of 20% of 25% of the people in town is $(0.80)(0.20)(0.25)$ times the number of people in town. This product is 0.04, or 4%, of the people in town.

A tape diagram that shows this reasoning should show the fraction or percentages of the whole town and also the percentage of the previous set. An accurate diagram can be drawn on graph paper, or folded from a strip of paper. A sequence of diagrams, such as the pictures shown, are more effective to show the steps of reasoning, as opposed to a single tape or area diagram.



An area diagram might be on a 10×10 grid, each square representing 1% of the town's people.





Activity Synthesis

This problem requires students to revise their idea of what is “the whole” three times: initially, it’s the number of people in the town. Then it’s the number of subscribers, then the number of voters. “Percent of what?” is a useful question to ask.

Ask students who chose a specific number of people in the town what percentage they got. Try to find students who chose different numbers of people. All should get 4%.

Choose several diagrams to display and discuss. Ask what part of the diagram represents each quantity:

- all the people in town
- 1% of the people in town
- the people who subscribe to the newspaper (25% of the people in town)
- the people who voted (20% of the subscribers, which is 5% of the people in town)
- the people who voted for Darnell’s (80% of those who voted, 4% of the people in town)

Discuss the fact that Darnell’s sign is misleading, asking students whom you noticed had interesting or well-stated answers. Students may want to discuss the unfairness of the results when only a few people vote. You might want to consult with a civics/social studies teacher about actual numbers in a recent election, and whether this is likely to be too controversial to discuss in your class.

Lesson 5: More than Two Choices

Goals

- Apply reasoning about ratios and percentages to analyze (orally and in writing) voting situations involving more than two choices.
- Choose and justify (orally) which voting system seems the fairest for dealing with more than two choices.
- Compare and contrast (orally and in writing) different voting systems for dealing with more than two choices, i.e., plurality, runoff, and instant runoff.

Lesson Narrative

This lesson is optional. It is the second of three lessons that explores the mathematics of voting. The activities in this lesson build on each other and on the previous lesson. As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

The five activities in this lesson deal with elections in which there are more than two choices. For example, if there are three choices, then the top vote getter might be approved by only 34% of the voters. Students explore several different rules for determining the winner: plurality, runoff, and instant runoff, and discover that the rules can give different results from the same set of voter preferences. They think about which voting rule more fairly represents the opinions of the voters. The mathematics in these activities emphasizes quantitative reasoning in a real-world situation (MP2 and MP4).

Most of the activities use students' skills from earlier units to reason about ratios and proportional relationships (MP2) in the context of real-world problems (MP4). While some of the activities do not involve much computation, they all require serious thinking.

Most importantly, this lesson addresses topics that are important for citizens in a democracy to understand. Teachers may wish to collaborate with a civics/government teacher to learn how the fictional middle-school situations in this lesson relate to real-world elections.

Alignments

Building On

- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

Addressing

- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

Instructional Routines

- MLR8: Discussion Supports

Student Learning Goals

Let's explore different ways to determine a winner.

5.1 Field Day

Optional: 5 minutes

This is the first of five activities about elections where there are more than two choices. This introductory activity gets students thinking about the fairness of a voting rule. If the choice with the most votes wins, it's possible that the winning choice was preferred by only a small percentage of the voters.

Addressing

- 6.RP.A.3

Launch

Students work alone and share solutions with whole class.

Student Task Statement

Students in a sixth-grade class were asked, "What activity would you most like to do for field day?" The results are shown in the table.

activity	number of votes
softball game	16
scavenger hunt	10
dancing talent show	8
marshmallow throw	4
no preference	2

1. What percentage of the class voted for softball?
2. What percentage did not vote for softball as their first choice?

Student Response

1. 40% voted for softball since $\frac{16}{40} = \frac{2}{5} = 40\%$.
2. 60% did not vote for softball, at least as their first choice.

Activity Synthesis

Poll the class about the answers. (40% of the class voted for softball, so a majority of the class did not vote for softball as their first choice.) Ask if it is possible to determine whether softball was a highly rated choice by those who voted for another field day activity. (Not without holding another vote.)

In this voting system the *plurality* wins, the choice with the most votes, even if it is less than 50%.

5.2 School Lunches (Part 1)

Optional: 30 minutes

This activity presents a method for deciding the winner of an election with more than two choices: runoff voting. If no choice has a majority of votes, then one or more choices with the fewest votes are eliminated and another vote is held between the remaining choices. Repeat until one choice gets a majority of the votes.

Students learn the technique of analyzing the results by holding their own vote. A fictitious story (choosing a company to supply school lunches) is provided for students to vote on a situation with four choices, each of which may have some positive and negative aspects. They follow two different systems of voting rules to see how results can differ depending on the rule system used. Students use quantitative reasoning (MP2) to analyze and compare two different voting rules.

Note: This activity includes a lot of teacher-directed voting activity. Students periodically stop to record information and determine the winner, or the need to do another round of voting, or reflect on the results.

Here is the situation to vote on: Imagine the kitchen that usually prepares our school lunches is closed for repairs for a week. We get to choose which of four catering companies to feed everyone for a week. You can choose only one caterer. The school has found four catering companies that will supply a week of lunches for everyone. No changes in the menus are possible.

Make sure students understand the situation. Students vote by drawing symbols next to the four menu choices or on pre-cut voting slips of paper.

choice	symbol
A. Meat Lovers	
B. Vegetarian	
C. Something for Everyone	
D. Concession Stand	

Voting System #1. Plurality: Conduct a vote using the plurality wins voting system:

If there is an even number of students in the class, vote yourself to prevent a tie at the end. Ask students to raise their hand if the lunch plan was their first choice. Record the votes in a table for all to see.

lunch plan	number of votes
A. Meat Lovers	
B. Vegetarian	
C. Something for Everyone	
D. Concession Stand	

Students work through questions 1 and 2 in the activity in groups. Then they discuss question 3 as a whole class.

"How could we measure how satisfied people are with the result? For example, people whose top choice was the winner will be very satisfied. People whose last choice was the winner will be very dissatisfied."

Students vote with a show of hands, and record the votes.

what choice did you rank the winner?	number of people	% of people
top choice (star)		
second choice (smiley)		
third choice (square face)		
last choice (X)		

Voting system #2: Runoff

Students work through questions 4–6 alternating between conducting the next round of voting as a whole class and analyzing the results in their groups. "Use your same choices that you recorded. We'll count the votes in a different, more complicated way. If one choice did not get a majority, we hold a runoff vote. Eliminate the choice that got the fewest votes. Then we vote again. If your first choice is out, vote for your second choice."

- Record the votes in a table like the first one, except that one of the choices will be gone.
- "Did the same choice get the most votes both times?" (Sometimes no. Results may vary in your class.)

- "Did one of the choices get a majority?" (If so, that choice wins. If not, eliminate the choice with the fewest votes and vote again. Repeat until one choice gets a majority of the votes.)

Again ask for satisfaction with the results of the voting. Record numbers in column 2.

What choice did you rank the winner?	Number of people	% of people
top choice (star)		
second choice (smiley)		
third choice (square face)		
last choice (X)		

Students compute percentages in the last column and work on question 7 in groups.

Building On

- 6.RP.A.3

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2–4.

Introduce the situation: "When there are more than two choices, it's often hard to decide which choice should win. For example, in the field day question, softball got the most votes, but only 40% of the votes were for softball and 60% were not for softball. But were these really votes against softball, or did some of those people like softball, but just liked another choice more?"

In this lesson, we'll try two voting systems. We'll vote on an imaginary situation: choosing a caterer to supply student lunches."

See the Activity Narrative for instructions for conducting the activity.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about the fairness of a voting rule. Some students may benefit from watching a physical demonstration of the runoff voting process. Invite students to engage in the process by offering suggested directions as you demonstrate.

Supports accessibility for: Visual-spatial processing; Organization

Access for English Language Learners

Conversing: MLR8 Discussion Supports. Prior to voting and calculating results, invite discussion about the four menus as part of the democratic process. Display images of any foods that are unfamiliar to students such as, hummus, liver, pork cutlets, pita, beef stew, meat loaf. Provide students with the following questions to ask each other: “What are the pros and cons of this menu?” “What would you like or dislike?” Students may have adverse feelings toward certain foods due to personal preferences or beliefs. Allow students time in small group to share ideas in order to better connect with the idea of making personal decisions and the purpose of voting.

Design Principle(s): Cultivate conversation; Support sense-making

Anticipated Misconceptions

Students may not know what some of the foods are. You can either explain, or tell them that it’s up to them to vote for unknown foods or not.

- Liver is an internal organ, not muscle meat. Many people don’t like it.
- Hummus is a bean dip made of chickpeas. Pita is middle eastern flatbread.

The voting rules are somewhat complicated. Acting out the voting process should make things more clear.

Student Task Statement

Suppose students at our school are voting for the lunch menu over the course of one week. The following is a list of options provided by the caterer.

Menu 1: Meat Lovers

- Meat loaf
- Hot dogs
- Pork cutlets
- Beef stew
- Liver and onions

Menu 2: Vegetarian

- Vegetable soup and peanut butter sandwich
- Hummus, pita, and veggie sticks
- Veggie burgers and fries
- Chef’s salad
- Cheese pizza every day
- Double desserts every day

Menu 3: Something for Everyone

- Chicken nuggets
- Burgers and fries
- Pizza
- Tacos
- Leftover day (all the week's leftovers made into a casserole)
- Bonus side dish: pea jello (green gelatin with canned peas)

Menu 4: Concession Stand

- Choice of hamburger or hot dog, with fries, every day

To vote, draw one of the following symbols next to each menu option to show your first, second, third, and last choices. If you use the slips of paper from your teacher, use only the column that says "symbol."



1. Meat Lovers _____
2. Vegetarian _____
3. Something for Everyone _____
4. Concession Stand _____

Here are two voting systems that can be used to determine the winner.

- Voting System #1. *Plurality*: The option with the most first-choice votes (stars) wins.
- Voting System #2. *Runoff*: If no choice received a majority of the votes, leave out the choice that received the fewest first-choice votes (stars). Then have another vote.

If your first vote is still a choice, vote for that. If not, vote for your second choice that you wrote down.

If there is still no majority, leave out the choice that got the fewest votes, and then vote again. Vote for your first choice if it's still in, and if not, vote for your second choice. If your second choice is also out, vote for your third choice.

1. How many people in our class are voting? How many votes does it take to win a majority?
2. How many votes did the top option receive? Was this a majority of the votes?

3. People tend to be more satisfied with election results if their top choices win. For how many, and what percentage, of people was the winning option:
 - a. their first choice?
 - b. their second choice?
 - c. their third choice?
 - d. their last choice?
4. After the second round of voting, did any choice get a majority? If so, is it the same choice that got a plurality in Voting System #1?
5. Which choice won?
6. How satisfied were the voters by the election results? For how many, and what percentage, of people was the winning option:
 - a. their first choice?
 - b. their second choice?
 - c. their third choice?
 - d. their last choice?
7. Compare the satisfaction results for the plurality voting rule and the runoff rule. Did one produce satisfactory results for more people than the other?

Student Response

For the actual class vote, numbers will vary.

Here is a scenario for a class with 50 students (for details see next activity):

Voting system #1. Plurality:

First choice vote: Meat: 21 votes, 42%; Veg: 13 votes, 26%; Ev: 9 votes, 18%; Con: 7 votes, 14%

1. 50 students are voting. It takes 26 votes to win a majority.
 2. Meat won a plurality, but not a majority.
 3. 21 students got their first choice, 29 students got their last choice.
- Voting System #2. Runoff:
4. No majority after second round.
 5. So Something for Everyone wins.

6. How satisfied voters were:

- 9 voters got their first choice.
- 7 voters got their second choice.
- 34 voters got their third choice.
- Nobody had to put up with their last choice winning.

7. With the runoff voting fewer students got their first choice but nobody got their last choice.

Activity Synthesis

Ask students which system seems more fair, plurality or runoff. (The plurality system doesn't take second, third, etc., choices into account, while the runoff system does.)

In an election in Oakland, California, a candidate won by campaigning to ask voters to vote for her for first choice, and if she was not their first choice, then put her as their second choice. Is this like what happened with one of the votes we analyzed?

5.3 School Lunch (Part 2)

Optional: 20 minutes

In this activity students revisit the situation from the previous activity but they analyze the votes of a different class. In this case the members of different student clubs all voted for the same lunch option. Students repeat the process of the run-off election on the provided data and compare it to a plurality vote. They use quantitative reasoning (MP2) to analyze and compare the two different voting rules.

Students must think through the voting process and determine which choice is eliminated at each round, and what votes the club presidents will turn in at every round of voting.

Building On

- 6.RP.A.3

Launch

Arrange students in groups of 4. Tell students that they analyze the results of the vote from a different class for the same lunch caterer situation from the previous activity. Tell them, "There are four clubs in this other class, and everyone in each club agrees to vote exactly the same way, as shown in the table."

Have each group of four act out the voting for this class: each person is the president of a club, and delivers the votes for all the club members. Demonstrate with a group, "This person is the president of the barbecue club. Tell us how many votes you are turning in, and for which choice."

Give students 10 minutes to work through the questions with their group, followed up with whole-class discussion.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about reasoning quantitatively. Allow students to use calculators to ensure inclusive participation in the activity.

Supports accessibility for: Memory; Conceptual processing

















Anticipated Misconceptions

The voting rules are somewhat complicated. Acting out the voting process should make things more clear.

Student Task Statement

Let's analyze a different election.

In another class, there are four clubs. Everyone in each club agrees to vote for the lunch menu exactly the same way, as shown in this table.

	Barbecue Club (21 members)	Garden Club (13 members)	Sports Boosters (7 members)	Film Club (9 members)
A. Meat Lovers				
B. Vegetarian				
C. Something for Everyone				
D. Concession Stand				

1. Figure out which option won the election by answering these questions.
 - a. On the first vote, when everyone voted for their first choice, how many votes did each option get? Did any choice get a majority?
 - b. Which option is removed from the next vote?
 - c. On the second vote, how many votes did each of the remaining three menu options get? Did any option get a majority?
 - d. Which menu option is removed from the next vote?
 - e. On the third vote, how many votes did each of the remaining two options get? Which option won?

2. Estimate how satisfied all the voters were.
 - a. For how many people was the winner their first choice?
 - b. For how many people was the winner their second choice?
 - c. For how many people was the winner their third choice?
 - d. For how many people was the winner their last choice?
3. Compare the satisfaction results for the plurality voting rule and the runoff rule. Did one produce satisfactory results for more people than the other?

Student Response

1.
 - a. Meat: 21 votes, 42%; Veg: 13 votes, 26%; Ev: 9 votes, 18%; Con: 7 votes, 14%. Meat won a plurality, but not a majority. You don't really need to compute a percent for this. There were 50 votes in all, so 26 votes are needed to win a majority.
 - b. Eliminate Concession Stand, since it only got 7 votes.
 - c. In the new vote, everyone still votes for their first choice, except the 7 people in the Sports Booster Club. They originally voted for Concession Stand, so now their votes go to their second choice, which is Something for Everyone. Meat: 21 votes, 42%; Veg: 13 votes, 26%; Ev: $9+7=16$ votes, 32%. Meat still has the most votes, but still not a majority.
 - d. Second runoff vote: eliminate Vegetarian. The Garden Club members' first choice is gone. But their second choice, Concession Stand, is also gone. So their 13 votes go to their third choice, Something for Everyone. Meat: 21 votes, 42%; Ev: $9+7+13=29$ votes, 58%.
 - e. So Something for Everyone wins.
2.
 - a. 9 voters in the Film Club got their first choice.
 - b. 7 voters in the Sports Booster Club got their second choice.
 - c. 21 members of the Barbecue Club and 13 members of the Garden Club got their third choice.
 - d. Nobody had to put up with their last choice winning.
3. With the plurality voting rule, 21 students would have gotten their first choice and 29 students would have gotten their last choice. So even though fewer students got their first choice with the runoff voting rule, more students got a higher choice than with the plurality voting rule.

Activity Synthesis

Ask for results from the fictitious class.

Notice that after the first round, Meat seemed to be winning. After the second round, Vegetarian seemed to be winning. But the actual winner was Something for Everyone.

Ask students:

- How did the results of this class compare to our own class?
- What are some advantages and disadvantages of plurality and runoff voting? (Plurality takes less effort but could be less fair.)
- Which system seems more fair, plurality or runoff? (The plurality system doesn't take second, third, etc., choices into account, while the runoff system does.)

5.4 Just Vote Once

Optional: 30 minutes

This activity presents another method for choosing among three or more choices when none wins a majority, instant runoff voting. Voters again rank their choices. Each choice is given points, with 0 for the last choice, 1 for the next to last, and so on. The choice with the most total points wins, and no runoff elections are needed. Students use quantitative reasoning (MP2) to compare two models of fairness in voting (MP4).

Building On

- 6.RP.A.3

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2–4.

Introduce another method of voting: "The runoff system sometimes needs more than one election if no choice got a majority. If we are all here together, that's not a big problem. But if it were a vote with everyone in a city, or county, or state, it would be too complicated and expensive to have more than one vote. The instant runoff system gives each vote points 0 for the last choice, 1 for the next to last, and so on. The choice with the most total points wins, and no runoff elections are needed. Let's redo our election using this system, and then see what the other class's votes would choose. Remember what your choices were for the first time we voted for school lunch providers. Write down the points for each of the choices."

Then either ask for votes by hand-raising or ask students to come to the board to record their choices.

Raising hands: "Raise your hand if you gave Meat 3 points (count and record in table). Raise your hand if you gave Meat 2 points, etc." There will be 16 categories. Everyone should raise their hand 4 times.

choice	number of votes for top choice (star)	number of votes for second choice (smiley)	number of votes for third choice (square face)	number of votes for last choice (X)
A				
B				
C				
D				

Come to the board: Have students come up and record their numbers. Each student should have a 0, a 1, a 2, and a 3, one in each category. The table below shows the points for two students' choices.

	points
A. Meat Lovers	3, 0, ...
B. Vegetarian	2, 3, ...
C. Something for Everyone	1, 1, ...
D. Concession stand	0, 2, ...

After the results are recorded for all to see and students understand the presented information, students work in groups and answer the questions in the activity statement.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts. After students have solved the first 2–3 problems, check in with either select groups of students or the whole class. Invite students to share the strategies they have used so far, as well as inviting them to ask any questions they have before continuing.

Supports accessibility for: Organization; Attention

Anticipated Misconceptions

Students may not remember how to do the satisfaction survey. Remind them of the work done in the previous activity. Ask them to fill out a similar table:

what choice did you rank the winner?	number of people	% of people
top choice (star)		
second choice (smiley)		
third choice (square face)		
last choice (X)		

Student Task Statement

Your class just voted using the *instant runoff* system. Use the class data for following questions.

1. For our class, which choice received the most points?
2. Does this result agree with that from the runoff election in an earlier activity?
3. For the other class, which choice received the most points?
4. Does this result agree with that from the runoff election in an earlier activity?
5. The runoff method uses information about people's first, second, third, and last choices when it is not clear that there is a winner from everyone's first choices. How does the instant runoff method include the same information?
6. After comparing the results for the three voting rules (plurality, runoff, instant runoff) and the satisfaction surveys, which method do you think is fairest? Explain.

Student Response

1. Results vary for the actual voting in class.
2. Answer vary.
3. The Vegetarian option won with 97 points. A got 21 first place votes and 29 last place votes: $21 \cdot 3 + 29 \cdot 0 = 63$ points. B got 21 second place votes, 13 first place votes, and 7+9 third place votes: $21 \cdot 2 + 13 \cdot 3 + 7 \cdot 1 + 9 \cdot 1 = 97$ points. C got 21+13 third place votes, 7 second place votes, and 9 first place votes: $(21 + 13) \cdot 1 + 7 \cdot 2 + 9 \cdot 3 = 75$ points. D got 21 last place votes, 13+9 second place votes, and 7 first place votes: $21 \cdot 0 + (13 + 9) \cdot 2 + 7 \cdot 3 = 65$ points.
4. This is a different result than we got from the other two voting systems. Meat (A) won under the plurality system, and Something for Everyone (C) won under the runoff system. Even though A had the largest group vote for it, it was last choice for everyone else, so it could not pick up any extra points for being second or third choice. Vegetarian (B) won by having a biggish number of first place points, and a bigger number of second place points, and some more third place points.

5. The instant runoff method includes the information about people's choices in the number of points that are assigned to each vote.
6. Answers vary. In the plurality method, most people get their first choice. In the runoff method, nobody got their last choice. In the instant runoff method, more people got a higher choice, even if it was not their first one.

Are You Ready for More?

Numbering your choices 0 through 3 might not really describe your opinions. For example, what if you really liked A and C a lot, and you really hated B and D? You might want to give A and C both a 3, and B and D both a 0.

1. Design a numbering system where the size of the number accurately shows how much you like a choice. Some ideas:
 - The same 0 to 3 scale, but you can choose more than one of each number, or even decimals between 0 and 3.
 - A scale of 1 to 10, with 10 for the best and 1 for the worst.
2. Try out your system with the people in your group, using the same school lunch options for the election.
3. Do you think your system gives a more fair way to make choices? Explain your reasoning.

Student Response

Answers vary.

Activity Synthesis

Poll students about the results of the instant runoff vote. Ask:

- How do the three voting methods we have seen compare?
- Which method should we use the next time our class has to make a decision? Why?

We have seen several methods for fairly deciding between more than two choices. There is no single fairest method. Some methods give one winner, others a different winner with the same vote.

Access for English Language Learners

Conversing: MLR8 Discussion Supports. Use this routine to support small-group discussion. Display the following prompts: "I think the ____ method is most fair because . . .", "I agree/ disagree because . . .", "Does anyone else have something to add to this explanation?", "How can we justify that more students were represented in the final results?" These prompts will help students summarize the results of each type of voting system.

Design Principle(s): Cultivate conversation; Maximize meta-awareness

5.5 Weekend Choices

Optional: 10 minutes

This voting activity helps students summarize the voting systems for more than two choices that were discussed in the previous lessons. Five students vote on three choices of weekend activities. In this activity, students engage in quantitative reasoning (MP2) to compare two mathematical models for fairness in voting (MP4).

Building On

- 6.RP.A.3

Launch


Arrange students in groups of 2–4.

Student Task Statement

Clare, Han, Mai, Tyler, and Noah are deciding what to do on the weekend. Their options are cooking, hiking, and bowling. Here are the points for their instant runoff vote. Each first choice gets 2 points, the second choice gets 1 point, and the last choice gets 0 points.

	cooking	hiking	bowling
Clare	2	1	0
Han	2	1	0
Mai	2	1	0
Tyler	0	2	1
Noah	0	2	1

1. Which activity won using the instant runoff method? Show your calculations and use expressions or equations.

- 
2. Which activity would have won if there was just a vote for their top choice, with a majority or plurality winning?
 3. Which activity would have won if there was a runoff election?
 4. Explain why this happened.

Student Response

1. Hiking won under the instant runoff (points) system. Cooking got three first place votes. Hiking got two first place votes and three second place votes. Bowling got two second place votes.

The calculation of points for cooking is $3 \cdot 2 = 6$.

The calculation of points for hiking is $2 \cdot 2 + 3 \cdot 1 = 7$.

The calculation of points for bowling is $2 \cdot 1 = 2$.

2. With a plurality wins system, cooking would win, since it got a majority of the first place votes.
3. With a runoff system cooking would win since it already got the majority of first place votes in one round of voting.
4. Hiking won by getting several first place votes and more second place votes. The 3 points from second place votes gave more points than the third first place vote that cooking got.

Lesson 6: Picking Representatives

Goals

- Compare and contrast different ways to distribute representatives, and recognize that changing the way the votes are grouped can affect the outcome.
- Critique (orally and in writing) whether a method for distributing representatives is fair.
- Suggest a method for distributing representatives and justify (orally) why it is fair.

Lesson Narrative

This lesson is optional. The five activities in this third lesson on the mathematics on voting return to the situation of an election with two choices. However, rather than directly choosing the result, voters elect representatives, each of whom then casts a single vote for all the people they represent. The activities explore ways to “share” the representatives fairly between groups of people. In the first activity, numbers have been designed so that representatives (or computers) can be shared exactly proportionally between several groups. In later activities, it’s impossible to share representatives fairly; students may use division with decimal quotients or with remainders to try to find the least unfair way. The final activity asks students to *gerrymander* several districts: to divide it into sections in two ways to influence the final voting result in opposite ways. The mathematics here involves geometric properties of shapes on maps: area and connectedness, as well as some proportional reasoning.

Most of the activities use students’ skills from earlier units to reason about ratios and proportional relationships (MP2) in the context of real-world problems (MP4). While some of the activities do not involve much computation, they all require serious thinking and decision making (MP3).

Most importantly, this lesson addresses topics that are important for citizens in a democracy to understand. Teachers may wish to collaborate with a civics or government teacher to learn how the fictional middle-school situations in this lesson relate to real-world elections.

Alignments

Addressing

- 6.NS.B.3: Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.
- 6.RP.A.2: Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. \$Expectations for unit rates in this grade are limited to non-complex fractions.
- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

- 6.RP.A.3.c: Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

Instructional Routines

- MLR2: Collect and Display
- MLR3: Clarify, Critique, Correct

Required Materials

Four-function calculators

Student Learning Goals

Let's think about fair representation.

6.1 Computers for Kids

10 minutes

This activity introduces the question for the last five activities in the voting unit: How can we fairly share a small number of representatives between several groups of people?

The first question of the activity asks students to distribute computers to families with children. In this question, computers can be shared so that the same number of children share a computer in each family. In the second question, fair sharing is not possible, so students need to decide and explain which alternative is the fairest, or the least unfair (MP3). Monitor for students who chose different ways to distribute the computers.

Addressing

- 6.NS.B.3
- 6.RP.A.3

Launch

Arrange students in groups of 2. Give students 5 minutes quiet think time and then ask them to compare their work with a partner.

Anticipated Misconceptions

Division has two roles in this activity, as discussed in the unit on dividing fractions. Dividing 16 children by 8 computers is like putting 16 pounds of almonds into 8 bags, giving 2 pounds of almonds per bag. Bagging 6 pounds of almonds at 2 pounds of almonds per bag fills 3 bags.

Student Task Statement

A program gives computers to families with school-aged children. They have a certain number of computers to distribute fairly between several families. How many computers should each family get?

1. One month the program has 8 computers. The families have these numbers of school-aged children: 4, 2, 6, 2, 2.
 - a. How many children are there in all?
 - b. Counting all the children in all the families, how many children would use each computer? This is the number of children per computer. Call this number A .
 - c. Fill in the third column of the table. Decide how many computers to give to each family if we use A as the basis for distributing the computers.

family	number of children	number of computers, using A
Baum	4	
Chu	2	
Davila	6	
Eno	2	
Farouz	2	

- d. Check that 8 computers have been given out in all.
2. The next month they again have 8 computers. There are different families with these numbers of children: 3, 1, 2, 5, 1, 8.
 - a. How many children are there in all?
 - b. Counting all the children in all the families, how many children would use each computer? This is the number of children per computer. Call this number B .
 - c. Does it make sense that B is not a whole number? Why?
 - d. Fill in the third column of the table. Decide how many computers to give to each family if we use B as the basis for distributing the computers.

family	number of children	number of computers, using B	number of computers, your way	children per computer, your way
Gray	3			
Hernandez	1			
Ito	2			
Jones	5			
Krantz	1			
Lo	8			

- e. Check that 8 computers have been given out in all.
- f. Does it make sense that the number of computers for one family is not a whole number? Explain your reasoning.
- g. Find and describe a way to distribute computers to the families so that each family gets a whole number of computers. Fill in the fourth column of the table.
- h. Compute the number of children per computer in each family and fill in the last column of the table.
- i. Do you think your way of distributing the computers is fair? Explain your reasoning.

Student Response

1. The computers can be fairly shared by two children in each family. There are 16 children and 8 computers. Details are shown in this table.
 - a. 16 children
 - b. $A = 2$ children per computer

c.

family	number of children	number of computers, using A
Baum	4	2
Chu	2	1
Davila	6	3
Eno	2	1
Farouz	2	1

d. No response required.

2. It isn't possible to fairly distribute the computers, since you can't split computers between families. There are 20 children and 8 computers, so $B = 2.5$ children per computer would be fair. In practice, this means two computers for every 5 children. But this isn't possible unless there is a multiple of 5 children in each family. Other less fair solutions are possible.

a. 20 children

b. $B = 2.5$ children per computer. Written as a fraction, $B = \frac{5}{2}$.

c. $B = 2.5$ makes sense. It's an average, not an actual amount for any children or families.

d. Divide number of children by 2.5 or $\frac{5}{2}$ children per computer for each family to get number of computers.

family	number of children	number of computers, using B	number of computers, your way	children per computer, your way
Gray	3	1.2 or $\frac{6}{5}$	1	3
Hernandez	1	0.4 or $\frac{2}{5}$	1	1
Ito	2	0.8 or $\frac{4}{5}$	1	2
Jones	5	2	2	2.5
Krantz	1	0.4 or $\frac{2}{5}$	1	1
Lo	8	3.2 or $\frac{16}{5}$	2	4

e. The sum of entries in column 3 is 8.

- f. It doesn't make sense for a family to get a fractional or decimal amount of a computer; they only work when they are whole. A half of a computer is a broken computer.
- g. Responses vary. One solution is given in column 4 of the table. This distribution gives one computer to each family, which uses 6 computers. The two last computers are given to the two largest families. A student could argue that this distribution is somewhat fair because all children at least have access to a computer.
- h. See table.
- i. It's not completely fair because the Hernandez and Krantz children get the computer all to themselves, while the Lo children need to share with 3 others.

Activity Synthesis

Invite students to share their answers.

For question 1, there should be an agreement that every two children share a computer.

For question 2, find students who chose different ways to distribute the computers.

Ask students which distribution they think is more fair and explain why. There may be no answer that everyone agrees on.

6.2 School Mascot (Part 1)

15 minutes

This lesson explores mathematical difficulties that arise in a representative democracy, where people do not vote individually, but vote for representatives who vote for all their constituents. This is in part a sharing problem. If all the people in a town are to be represented by a few people, the representatives should be shared as equally as possible. However, sometimes the groups to be represented are predetermined, such as classrooms or states. It's not always possible to have the same numbers of constituents per representative. This part of the activity is mathematically the same as sharing computers among families, as in the previous activity.

A further difficulty is that different people in the same group will usually have different opinions. The representative has only one vote, so it's impossible to fairly represent the opinions of all the constituents.

The mathematical issues involve unit rates and division, usually resulting in decimals.

This activity uses a voting situation with one vote per class. Similarly to the families with computers, three classrooms need to share three votes. The vote for the class is whichever choice wins a majority in the class election. Students discover that this system is unfair, since a class voting heavily for one choice counts for the same as a class barely voting for the choice ("yessiness"). They use division to try to devise a more fair system. Monitor for different ways to assign votes.

Note: a banana slug is a bright yellow snail without a shell that lives in redwood forests. Students at the University of California Santa Cruz voted for the banana slug as their mascot; the administration thought sea lions were more dignified.

Addressing

- 6.NS.B.3
- 6.RP.A.3

Launch

Arrange students in groups of 2. Students start with 5 minutes quiet think time, followed by comparing work with a partner. Provide access to four-function calculators.

Access for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about calculations involving fractions and percentages. Allow students to use calculators to ensure inclusive participation in the activity.

Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Students may be confused about the two-step process: each class votes, then the representative votes for the winner of the class vote. There are only three people voting at the second (representative) level. If necessary, act out the vote by dividing the class into three groups, not necessarily equal, and appointing a representative for each.

Students may object that this system is not fair. They are right: there is the potential for a minority choice to win the election.

Student Task Statement

A school is deciding on a school mascot. They have narrowed the choices down to the Banana Slugs or the Sea Lions.

The principal decided that each class gets one vote. Each class held an election, and the winning choice was the one vote for the whole class. The table shows how three classes voted.

	banana slugs	sea lions	class vote
class A	9	3	banana slug
class B	14	10	
class C	6	30	



1. Which mascot won, according to the principal's plan? What percentage of the votes did the winner get under this plan?
2. Which mascot received the most student votes in all? What percentage of the votes did this mascot receive?
3. The students thought this plan was not very fair. They suggested that bigger classes should have more votes to send to the principal. Make up a proposal for the principal where there are as few votes as possible, but the votes proportionally represent the number of students in each class.
4. Decide how to assign the votes for the results in the class. (Do they all go to the winner? Or should the loser still get some votes?)
5. In your system, which mascot is the winner?
6. In your system, how many representative votes are there? How many students does each vote represent?

Student Response

1. The banana slugs win with $\frac{2}{3}$, or 67% of the representatives' vote (rounded), since Classes A and B voted for banana slugs. Banana slugs got a total of 29 student votes and sea lions got 43.
2. So sea lions should have won with about 60% of the vote because they got 43 votes out of a total of 72 student votes.
3. The smallest number of representatives to give proportional representation has Class A with 1 vote, Class B with 2 votes, and Class C with 3 votes.
Answers vary. Possible response: Use fractions. Class A has 12 students, or $\frac{12}{72} = \frac{1}{6}$ of all the

students, Class B has 24 students for $\frac{24}{72} = \frac{2}{6}$ of the students, and Class C has 36 students for $\frac{36}{72} = \frac{3}{6}$ of the students. So 6 representatives can be shared fairly among the three classes.

Another method: The greatest common divisor of 12, 24, and 36 is 12. A proportional system of votes would give one vote to every 12 students, so Class A would get 1 vote, Class B 2 votes, and Class C 3 votes.

Other choices are possible, but they will need more than 6 representatives. For representation to be exactly proportional, there must be a multiple of 6 representatives.

4. If the choice winning a majority in a class determines all the votes for the class, then A gives its one vote to banana slugs, B gives 2 votes for banana slugs, and all 3 of C's votes go to sea lions. It's a tie! If you try to assign votes proportionally within classes, then Class B should probably give one vote for banana slugs and one for sea lions, since the numbers are fairly close. This would give 2 total class votes for banana slugs and 4 for sea lions. Now sea lions win.

There should be a criterion for how to split the class votes; is 14 to 10 closer to 1 to 1, or 2 to 0?

5. Sea lions win if B's votes are split between banana slugs and sea lions. It's a tie if the winner of the class election gets all the votes for that class.
6. There are 6 representatives who vote. Each represents 12 students.

Activity Synthesis

Invite students to share their ways to assign votes and the reasons for their system. Select students who have different methods.

Students should recognize, after discussion, that the system the principal proposed is unfair. A majority in a small class voting for banana slugs can overwhelm a larger number voting for sea lions in a larger class.

A more fair system should take the sizes of the classes into account.

6.3 Advising the School Board

30 minutes

This activity includes two problems of assigning representatives proportionally, with schools sending students to advise the school board. In the first problem, school sizes have been carefully planned so that each school has the same number of students per representative as the district as a whole. In the second, this is not possible, in part because of a very large and a very small school.

The mathematics includes finding rates by division, quotients and divisors that are decimals, and rounding, as well as quantitative reasoning (MP2).

The mathematics is the same as the previous activity distributing computers to families. It is also the same as the problem of assigning congressional representatives to states. Some states have very large populations, and others have very small populations.

As students work, monitor for different ways that they assign advisors.

Addressing

- 6.NS.B.3
- 6.RP.A.2

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 2–4. Provide access to four-function calculators if desired.

Explain the situation: “The school board (the elected people who make major decisions about all the schools) wants students from the schools to help them decide, and to give the board advice about what the students at each school think. They would like 10 students to be chosen to come to school board meetings. These students will be called advisors. Big schools should send more advisors than small schools, but even the tiniest school should send at least one advisor. If possible, the number of advisors should be proportional to the number of students at the school.”

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organization and problem solving. After students have solved the first 2–3 problems, check in with either select groups of students or the whole class. Invite students to share the strategies they have used so far as well as inviting them to ask any questions they have before continuing.

Supports accessibility for: Organization; Attention

Access for English Language Learners

Speaking, Writing, Representing: MLR2 Collect and Display. Use this routine to record language that students use to explain their strategies for discovering fair ratios of advisors. While students work through the problems, circulate and record language they use in explaining their methods for achieving fairness in the ratios. There will be multiple ways to calculate or compare. Encourage students to compare numbers, explain how they decided what is fair, and to describe the ideal situation. Pay attention to what they find problematic about working with partial numbers (fractions or decimals) concerning advisor count. Expect students to use specific language like “proportional,” “rates,” “ ____ per ____.”

Design Principle(s): Support sense-making; Maximize meta-awareness

Student Task Statement

1. In a very small school district, there are four schools, D, E, F, and G. The district wants a total of 10 advisors for the students. Each school should have at least one advisor.

school	number of students	number of advisors, using A
D	48	
E	12	
F	24	
G	36	

- a. How many students are in this district in all?
- b. If the advisors could represent students at different schools, how many students per advisor should there be? Call this number A .
- c. Using A students per advisor, how many advisors should each school have?
Complete the table with this information for schools D, E, F, and G.
2. Another district has four schools; some are large, others are small. The district wants 10 advisors in all. Each school should have at least one advisor.

school	number of students	number of advisors, using B	number of advisors, your way	students per advisor, your way
Dr. King School	500			
O'Connor School	200			
Science Magnet School	140			
Trombone Academy	10			

- a. How many students are in this district in all?
- b. If the advisors didn't have to represent students at the same school, how many students per advisor should there be? Call this number B .

- c. Using B students per advisor, how many advisors should each school have? Give your quotients to the tenths place. Fill in the first “number of advisors” column of the table. Does it make sense to have a tenth of an advisor?
- d. Decide on a consistent way to assign advisors to schools so that there are only whole numbers of advisors for each school, and there is a total of 10 advisors among the schools. Fill in the “your way” column of the table.
- e. How many students per advisor are there at each school? Fill in the last row of the table.
- f. Do you think this is a fair way to assign advisors? Explain your reasoning.

Student Response

1.

school	number of students	number of advisors, using A
D	48	4
E	12	1
F	24	2
G	36	3

- a. 120 students in the district
- b. A is 12 students per advisor
- c. See table.

2.

school	number of students	number of advisors, using B	number of advisors, your way	students per advisor, your way
Dr. King School	500	5.9	6	83.3
O'Connor School	200	2.4	2	100
Science Magnet School	140	1.6	1	140
Trombone Academy	10	0.1	1	10

- a. 850 students in the district
- b. B is 85 students per advisor
- c. See table. It doesn't make sense to have a tenth of an advisor; you can't have a fraction of a person.
- d. Responses vary. The lesson plan shows a strategy: round to the nearest whole number, then adjust. The first attempt gives the Trombone Academy no advisors. The second attempt gives them one of the two from the Science Magnet.
- e. Responses vary. See table.
- f. Responses vary. The example is unfair because the Trombone Academy is very small but still gets one advisor, and there are 10 students per advisor, compared to the Science Magnet, with 140 students per advisor. Dr. King School gets almost the ideal number of students per advisor.

Activity Synthesis

Invite several students to present their work, especially their attempts to assign advisors fairly. Choose students who chose different ways to assign advisors.

The ideal number of students per representative is 85, an average. King School is very close to this ideal. The other schools have much higher or lower numbers.

The big idea here is that it's impossible to be completely fair. The Trombone Academy will have more clout than the bigger schools because their advisor is representing only 10 students. On the other hand, if the Trombone Academy gets no advisors, then their views aren't represented at all, so this isn't fair either.

6.4 School Mascot (Part 2)

10 minutes

In the previous activities, representatives (“advisors”) were assigned to groups that couldn’t be changed: schools. Sometimes the groups or districts for representatives can be changed, as in districts for the U.S. House of Representatives, and for state legislatures, wards in cities, and so on. Often, the people in an area have similar opinions, so it’s possible to design districts where you can reliably predict the outcomes.

In this lesson, students use geometric reasoning about areas and connectedness to experiment with drawing districts in a way that predict the outcome of elections. This is often called gerrymandering.

Addressing

- 6.RP.A.3
- 6.RP.A.3.c

Instructional Routines

- MLR3: Clarify, Critique, Correct

Launch

Arrange students in groups of 2–4.

Access for Students with Disabilities

Representation: Develop Language and Symbols. Activate or supply background knowledge about calculating area. Share examples of expressions for area in a few different forms to illustrate how area can be expressed to represent district blocks' votes for seal lions and banana slugs mascots. Allow continued access to concrete manipulatives such as snap cubes for students to view or manipulate.

Supports accessibility for: Visual-spatial processing; Conceptual processing

Student Task Statement

The whole town gets interested in choosing a mascot. The mayor of the town decides to choose representatives to vote.

There are 50 blocks in the town, and the people on each block tend to have the same opinion about which mascot is best. Green blocks like sea lions, and gold blocks like banana slugs. The mayor decides to have 5 representatives, each representing a district of 10 blocks.

Here is a map of the town, with preferences shown.



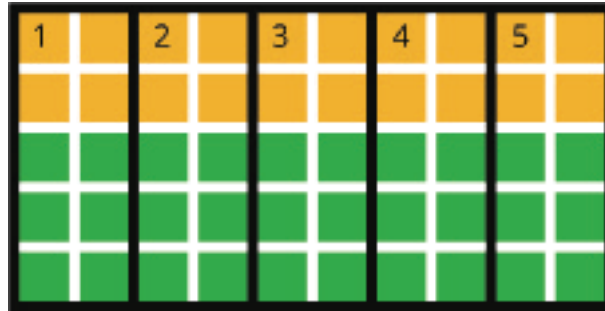
- Suppose there were an election with each block getting one vote. How many votes would be for banana slugs? For sea lions? What percentage of the vote would be for banana slugs?
- Suppose the districts are shown in the next map. What did the people in each district prefer? What did their representative vote? Which mascot would win the election?



Complete the table with this election's results.

district	number of blocks for banana slugs	number of blocks for sea lions	percentage of blocks for banana slugs	representative's vote
1	10	0		banana slugs
2				
3				
4				
5				

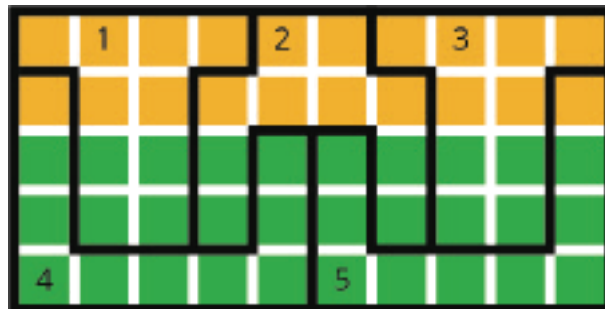
- Suppose, instead, that the districts are shown in the new map below. What did the people in each district prefer? What did their representative vote? Which mascot would win the election?



Complete the table with this election's results.

district	number of blocks for banana slugs	number of blocks for sea lions	percentage of blocks for banana slugs	representative's vote
1				
2				
3				
4				
5				

4. Suppose the districts are designed in yet another way, as shown in the next map. What did the people in each district prefer? What did their representative vote? Which mascot would win the election?



Complete the table with this election's results.

district	number of blocks for banana slugs	number of blocks for sea lions	percentage of blocks for banana slugs	representative's vote
1				
2				
3				
4				
5				

- Write a headline for the local newspaper for each of the ways of splitting the town into districts.
- Which systems on the three maps of districts do you think are more fair? Are any totally unfair?

Student Response

- 20 votes for banana slugs, 30 votes for sea lions, so sea lions win with 60% of the vote.
- Sea lions win with 3 of 5 representatives.

district	number of blocks choosing banana slugs	number of blocks choosing sea lions	percentage of blocks choosing banana slugs	representative's vote
1	10	0	100%	banana slugs
2	10	0	100%	banana slugs
3	0	10	0%	sea lions
4	0	10	0%	sea lions
5	0	10	0%	sea lions

- Sea lions win with all 5 representatives.

district	number of blocks choosing banana slugs	number of blocks choosing sea lions	percentage of blocks choosing banana slugs	representative's vote
1	4	6	40%	sea lions
2	4	6	40%	sea lions
3	4	6	40%	sea lions
4	4	6	40%	sea lions
5	4	6	40%	sea lions

4. Banana slugs win with 3 of 5 representatives.

district	number of blocks choosing banana slugs	number of blocks choosing sea lions	percentage of blocks choosing banana slugs	representative's vote
1	6	4	60%	banana slugs
2	6	4	60%	banana slugs
3	6	4	60%	banana slugs
4	1	9	10%	sea lions
5	1	9	10%	sea lions

5. Responses vary. Examples:

First map: 60% of Districts and 60% of People Vote for Sea Lions

Second map: All Districts, but only 60% of People Vote for Sea Lions

Third map: Banana Slugs Win with 60% of Districts, but Only 40% of People

6. Responses vary. The first map seems fairest since the percentages of the people and the representatives match. The second map has the same winner as the vote of the people but different percentages. The third map seems totally unfair: the percentages are reversed. More than half the people voted for sea lions, but banana slugs won.

Activity Synthesis

Ask students to share some of their headlines. Discuss the fairness of the different district arrangements.

Access for English Language Learners

Reading, Writing, Speaking: MLR3 Clarify, Critique, Correct. Use this routine to help students identify an error in calculating a percent and to justify their own calculations and “fairness” reasoning. Display the following statement before discussing the final problem: “Both map 1 and 2 are equally fair since 60% of people prefer sea lions. That means they show the same results.” Give students 1–2 minutes to improve on the statement in writing. Look for students to notice that the error is in the total number of districts that voted, even though out of that number, 60% did choose sea lions. Give students 2–3 minutes to discuss the original statement and their improvement with a partner. Ask, “Is there an error in the reasoning behind this statement? If so, what is it?” For extra support, encourage students to discuss the “total number of districts.” Encourage students to explain what was not “equal” and how the 60% interpretation did not address “equal total values.” Select 1–2 students to share their rewritten statements with the class.

Design Principle(s): Optimize output (for explanation); Cultivate conversation; Maximize meta-awareness

6.5 Fair and Unfair Districts

30 minutes

Students design districts in three towns to “gerrymander” the results of elections. In two of the towns, the election results can be skewed to either color. In the third, it isn’t possible to skew the results.

Addressing

- 6.RP.A.3
- 6.RP.A.3.c

Launch

Arrange students in groups of 2–4.

Explain the history of gerrymandering: Sometimes people in charge of designing districts make them in strange shapes to give the election results they want. One of the first was Elbridge Gerry (governor of Massachusetts in 1812), whose party designed a district that many people thought looked like a salamander. They called a Gerrymander, and the name stuck. It means a very strangely shaped, spread-out district designed to produce a certain result.

Usually districts are required to be connected: a person traveling to all parts of the district should be able to stay inside the district. There should be no “islands” that are separated by parts of other districts.

Access for Students with Disabilities

Engagement: Internalize Self-Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to choose and respond to four out of six task statements.

Supports accessibility for: Organization; Attention

Anticipated Misconceptions

The white streets on the maps do not disconnect the districts.

However, probably squares that are only connected at their corners should not be considered connected to each other. (But you or the students can make their own rule, since they are in charge of making districts.)

Student Task Statement

- Smallville's map is shown, with opinions shown by block in green and gold. Decompose the map to create three connected, equal-area districts in two ways:

- Design three districts where *green* will win at least two of the three districts. Record results in Table 1.



Table 1:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

- Design three districts where *gold* will win at least two of the three districts. Record results in Table 2.



Table 2:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

2. Squaretown's map is shown, with opinions by block shown in green and gold. Decompose the map to create five connected, equal-area districts in two ways:
- a. Design five districts where *green* will win at least three of the five districts. Record the results in Table 3.

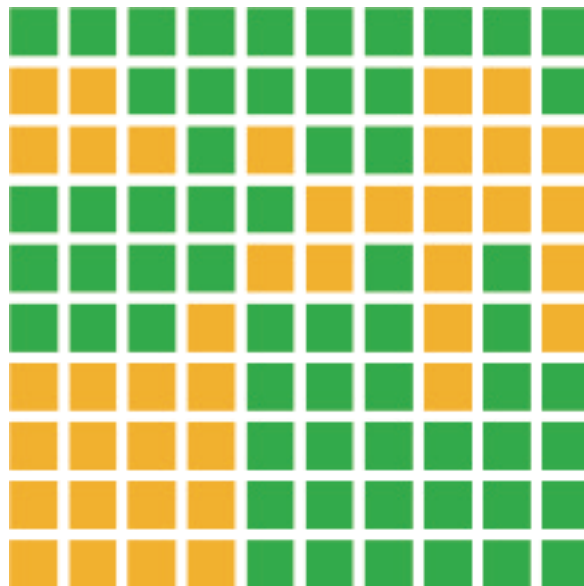


Table 3:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				
4				
5				

b. Design five districts where *gold* will win at least three of the five districts. Record the results in Table 4.

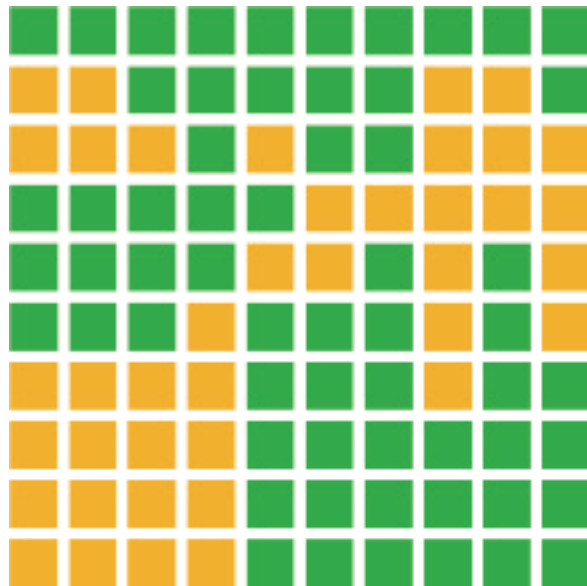


Table 4:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				
4				
5				

3. Mountain Valley's map is shown, with opinions by block shown in green and gold. (This is a town in a narrow valley in the mountains.) Can you decompose the map to create three connected, equal-area districts in the two ways described here?

- a. Design three districts where *green* will win at least 2 of the 3 districts. Record the results in Table 5.



Table 5:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

- b. Design three districts where *gold* will win at least 2 of the 3 districts. Record the results in Table 6.



Table 6:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

Student Response

1. Answers vary. Districts can be designed so that either color wins the election. Two examples are shown.



Table 1 shows that green wins.

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1	9	1	90%	green
2	6	4	60%	green
3	3	7	30%	gold



Table 2 shows that gold wins.

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1	10	0	100%	green
2	4	6	40%	gold
3	4	6	40%	gold

2. Answers vary. Districts can be designed so that either color wins the election. Two examples are shown.

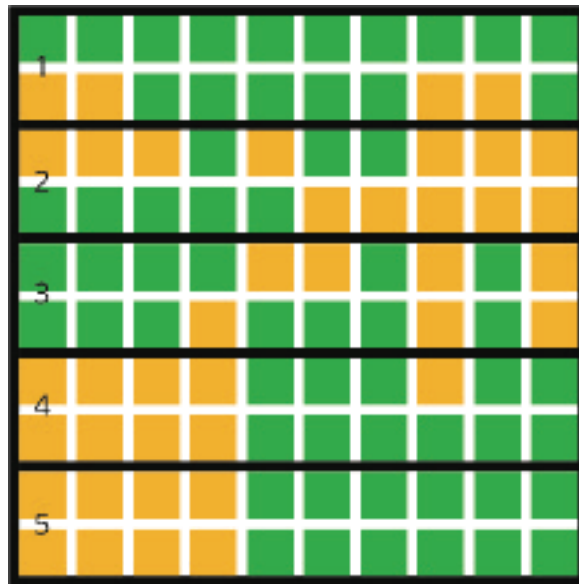


Table 3 shows that green wins.

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1	16	4	80%	green
2	8	12	40%	gold
3	13	7	65%	green
4	11	9	55%	green
5	12	8	60%	green

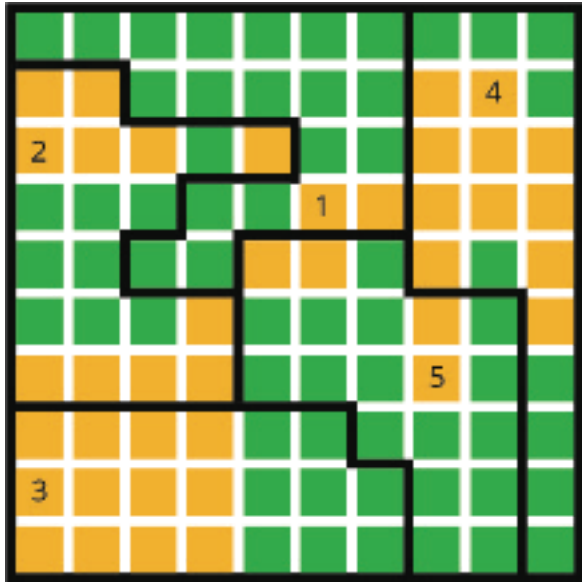


Table 4 shows that gold wins.

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1	18	2	90%	green
2	9	11	45%	gold
3	8	12	40%	gold
4	9	11	45%	gold
5	16	4	80%	green

3. Gold wins the election. There is only one way to draw connected districts because the town is so narrow.

It is impossible to complete table 5.

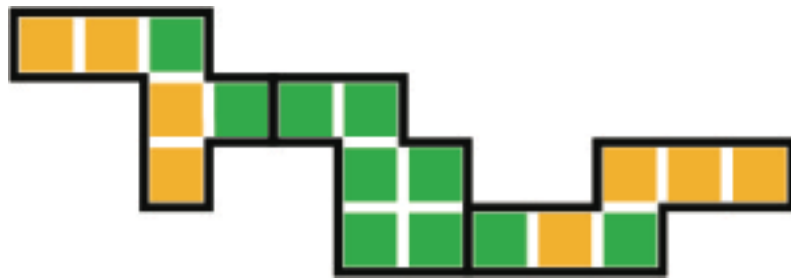


Table 6:

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1	2	4	33%	gold
2	6	6	100%	green
3	2	4	33%	gold

Activity Synthesis

This set of questions is similar to the previous activity, except that the patterns are irregular, so more planning and computation is needed to design districts.

Ask several students with different maps to show and explain their work.

The important result for this problem is that it's possible to design both fair districts (where the result of the vote is similar to the vote if all individual votes were counted) and unfair districts. According to the rules (equal area and connected), these sets of districts are legal. Encourage students to try to devise some rules for drawing districts that would be more fair.

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